Probability theory 1

22/10/21

- We came to know the notion of independence and saw some examples. We had a postponed theorem.

- <u>Theorem</u>: Suppose that $A_1, A_2, \dots, A_n$ are independent events. If $i_1, \dots, i_k, j_1, \dots, j_l \in \{1, 2, \dots, n\}$ are distinct $(k + l \leq n)$. Then,

$$(*) \quad \mathbb{P}\left(A_{i_1} \cap \dots \cap A_{i_k} \cap A_{j_1}^c \cap \dots \cap A_{j_l}^c\right) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k})\, \mathbb{P}(A_{j_1}^c) \cdots \mathbb{P}(A_{j_l}^c).$$

<u>Proof</u>: <u>Step 1</u>: Enough to prove the following claim:

Claim 1: If $A_1, A_2, \ldots, A_n$ are independent then for any $k (\le n)$

$$P(A_1 \cap \cdots \cap A_{k-1} \cap A_k^c \cap \cdots \cap A_n^c) = P(A_1) \cdots P(A_{k-1}) P(A_k^c) \cdots P(A_n^c)$$

(due to rearrangements and needful events).

Step 2: Enough to prove the following claim to prove the above claim:

Claim 2: If $A_1, \ldots, A_n$ are independent then $A_1, A_2, \ldots, A_{n-1}, A_n^c$ are independent.

If claim 2 is true then

Choose $B_1 = A_1$, $B_2 = A_2$, $\ldots$, $B_{n-1} = A_n^c$ and $B_n = A_{n-1}$

Applying claim 2 we get $A_1, A_2, \ldots, A_{n-1}^c, A_n^c$ are independent.

We proved step 2 (claim 2) in the last class.

Random variables:

Definition: A function defined on a sample space to the real line is called a random variable.

Till today we have been concerned with random variables without using the term.

Tossing a coin: $X : \Omega \to \{0, 1\}$

$$X(H) = 1 \quad \text{and} \quad X(T) = 0.$$

$P(X=1) = \frac{1}{2} = P(X=0)$ for a fair coin.

Total number obtained in two rolls of a fair die.

$X : \Omega \to \{2, 3, \ldots, 7, \ldots, 12\}$ $\quad X((i,j)) = i+j$.

$\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$

$P(X=3) = \frac{2}{36}$ and so on.

Distribution of a random variable:

Let X be a random variable (i.e. $X : \Omega \to \mathbb{R}$) and $x_1, x_2, \ldots$

$f: A \to B$

$f^{-1}(\{0\})$

$= \{x \in A : f(x) = 0\}$

$= \{f(\cdot) = 0\}$

$= \{f(\cdot) = 0\}$

$f^{-1}(x)$

be the values which it takes (i.e. the range of the function X is $\{x_1, x_2, \cdots\}$).

The event $\{X = x_j\}$ means that, the set consists of all the sample points on which X takes the value x_j; (for any $j \in \mathbb{N}$) its probability is denoted by

$$f_X(x_j) := P(X = x_j) \qquad \text{for } j \in \mathbb{N}. \qquad \text{“} f_X : \mathrm{Im}(X) \to [0,1] \text{”}.$$

is called the 'probability distribution'/'probability mass function' of the random variable X.

Clearly, $0 \le f_X(x_j) \le 1 \quad \forall j \in \mathbb{N}$ and

$$\sum_{j=1}^{\infty} f_X(x_j) = 1 .$$

(We are in a countable sample space).

Examples:

1) Bernoulli (p) random variable

Let $X: \Omega \to \{0,1\}$ be a random variable. The probability mass function is

$$f(1) = P(X=1) = p \quad \text{and} \quad f(0) = P(X=0) = 1-p .$$

We saw tossing a coin is a particular example of Bernoulli random variable.

2) <u>Binomial random variable</u> : (<u>Binomial (n, p)</u>)

Let $X : \Omega \to \{0, 1, 2, \cdots, n\}$ be a random variable such that the probability mass function is

$$f(k) = \underline{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } 1 \leq k \leq n.$$

Suppose there are $10 (=n)$ true false questions and picking the option 'true' with probability $\frac{1}{3} (=p)$ and 'false' with probability $\frac{2}{3} (=1-p)$.

Then choosing exactly k many 'true' options is a Binomial $\left(10, \frac{1}{3}\right)$ random variable.

3) Geometric (p) random variable :

Case 1) Let $X : \Omega \to \underset{\mathbb{N} \cup \{0\}}{\underset{\shortparallel}{\{0,1,2,\dots\}}}$ be a random variable such that

X_1

$$\boxed{X_2^{(\cdot)} = X_1^{(\cdot)} + 1}$$

the probability mass function is

$$f(k) = P(X=k) = (1-p)^k p \quad \text{for } k \geq 0.$$

Case 2) Let $X : \Omega \to \{1,2,\dots\} = \mathbb{N}$ be a random variable such that

the probability mass function is

X_2

$$f(k) = P(X=k) = (1-p)^{k-1} p \quad \text{for } k \geq 1.$$

4) Suppose that n balls are drawn without replacement from an urn containing w white and b black balls. Let X be the number of white balls drawn. So,

$$X : \Omega \to \{0, 1, 2, \ldots, w\} \quad \text{if } w \leq n \quad \left(\text{or } X : \Omega \to \{0, 1, \ldots, w \wedge n\} \right)$$

with probability mass function.

$$f(k) = P(X=k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$$

$$\left\{ \begin{array}{c} k \leq w \\ n-k \leq b \end{array} \right\}$$

$k \in \{0, 1, \ldots, w \wedge n\}$.

$w \wedge n$ = minimum of w and n

$w \vee n$ = maximum of w and n.

5) From $\{1,2,\dots,N\}$, $N \in \mathbb{N}$, n numbers are chosen without replacement. Let X denote the largest number. $(n \leq N)$

In this case $X : \Omega \to \{n, n+1, \dots, N\}$ such that the probability mass function is

$$f(k) = P(X=k) = \frac{\binom{k-1}{n-1}}{\binom{N}{n}} \quad \text{for } n \leq k \leq N$$

$$= \frac{n \times 1\,(k-1)(k-2)\cdots(k-(n-1)) \times (\ )}{N(N-1)\cdots(N-(n-1)) \times (\ )}$$

$$= \frac{(k-1)!\; n!\,(N-n)!}{(n-1)!\,(k-1-n+1)!\; N!}$$

$$=$$

x_3

$(x_1, x_2, x_3, \dots x_3)$

$(x_1, x_2, x_3, \dots x_n)$

$(x_1, x_1, x_3; \to x_3$

$(1,2,3)$

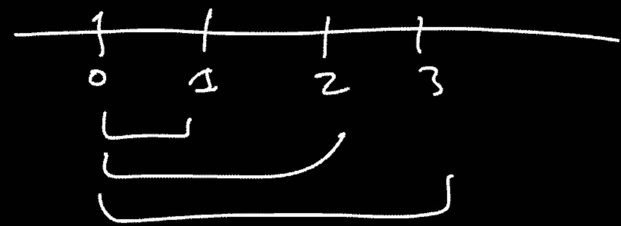
$(2,1,3)$

Definition: Events A_n increases to A, we say $A_n \uparrow A$ if

$$A_1 \subseteq A_2 \subseteq \cdots \quad (A_k \subseteq A_{k+1} \ \forall\, k \geq 1) \text{ and}$$

$$A = \bigcup_{n\geq 1}^{\infty} A_n$$

Example: Let $A_n = [0, n)$, $A_k \subseteq A_{k+1} \ \forall\, k \geq 1$



$A_n \uparrow [0, \infty)$ because $\bigcup_{n=1}^{\infty} [0, n) = [0, \infty)$.

Definition: Events A_n decreases to A, we say $A_n \downarrow A$ if

$$A_1 \supseteq A_2 \supseteq \cdots \quad (A_k \supseteq A_{k+1} \ \forall\, k \geq 1) \text{ and } A = \bigcap_{n=1}^{\infty} A_n.$$

Example: $A_n = [0, \frac{1}{n})$ and $A = \{0\}$. ($A_n = (0, \frac{1}{n})$ then $A = \emptyset$).

Theorem: If $A_1, A_2, \cdots$ are disjoint events then,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Pf: $$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\omega \in \bigcup_{n=1}^{\infty} A_n} p(\omega) \qquad \text{(by definition)}$$

$$= \sum_{n=1}^{\infty} \underbrace{\sum_{\omega \in A_n} p(\omega)} \qquad (0 \le p(\omega) \le 1).$$

$$= \sum_{n=1}^{\infty} P(A_n) \qquad \text{(by definition)}$$

$\square$