Probability theory 1

26/10/21

We have known the notion of random variables and we saw some examples.

<u>Definition</u>: Events A_n increase to A, $(A_n \uparrow A)$, if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \quad \text{and} \quad A = \bigcup_{n=1}^{\infty} A_n . \qquad \big(A_n = [0,n) \Rightarrow A = [0,\infty) \big)$$

<u>Definition</u>: Events A_n decrease to A, $(A_n \downarrow A)$, if

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \quad \text{and} \quad A = \bigcap_{n=1}^{\infty} A_n . \qquad \big(A_n = (0,\tfrac{1}{n}) \Rightarrow A = \emptyset \big)$$

$$\big(A_n = [0,\tfrac{1}{n}] \Rightarrow A = \{0\} \big)$$

$$\big(A_n = [0,\tfrac{1}{n}) \Rightarrow A = \{0\} \big).$$

Theorem: If $A_\infty, A_1, A_2, \cdots$ are events such that $A_n \uparrow A_\infty$, then $\mathbb{P}(A_n) \uparrow \mathbb{P}(A_\infty)$.

Proof: We know that if $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$.

We have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots$

$$\Rightarrow \mathbb{P}(A_1) \leq \mathbb{P}(A_2) \leq \cdots \leq \mathbb{P}(A_n) \leq \mathbb{P}(A_{n+1}) \leq \cdots$$

This shows that $\mathbb{P}(A_n)$ increases as n increases.

(in other words $\{\mathbb{P}(A_n)\}$ is an increasing sequence)

Let $C_1 = A_1$, $C_2 = A_2 \setminus A_1$, $C_3 = A_3 \setminus A_2$, $\cdots$, $C_n = A_n \setminus A_{n-1}, \cdots \; \forall \; n \geq 2$.

Then $\{C_n\}_{n=1}^{\infty}$ is a disjoint sequence of events. Also, for any $K \geq 1$, $\bigcup_{n=1}^{K} C_n = \bigcup_{n=1}^{K} A_n$ & $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$ —— (*)

$P(C_n) = P(A_n \setminus A_{n-1}) = P(A_n) - P(A_{n-1})$ (because A_{n-1} & $A_n \setminus A_{n-1}$ are disjoint events and their union is A_n)

Now,

$$0 \leq |P(A_\infty) - P(A_n)| = \left| P\left(\bigcup_{K=1}^{\infty} A_K\right) - P\left(\bigcup_{K=1}^{n} A_K\right) \right|$$

$$= \left| P\left(\bigcup_{K=1}^{\infty} C_K\right) - P\left(\bigcup_{K=1}^{n} C_K\right) \right|$$

$$= \left| \sum_{K=1}^{\infty} P(C_K) - \sum_{K=1}^{n} P(C_K) \right| = \left| \sum_{K=n+1}^{\infty} P(C_K) \right|$$

$$\leq \sum_{k=n+1}^{\infty} |P(C_k)| \longrightarrow 0 \text{ as } n \to \infty$$ (∵ it is a tail of a convergent series).

$\xrightarrow{\quad\quad} (**)$

Since C_k's are disjoint we have $P\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} P(C_k)$ &

$$P\left(\bigcup_{k=1}^{n} C_k\right) = \sum_{k=1}^{n} P(C_k)$$

$$0 \leq \underline{P\left(\bigcup_{k=1}^{\infty} C_k\right)} \leq 1 \Rightarrow 0 \leq \sum_{k=1}^{\infty} P(C_k) \leq 1$$

$\left\{P\left(\bigcup_{k=1}^{n} C_k\right)\right\}_{n=1}^{\infty}$

$\downarrow$

convergent

Another way to write the line (**)

$$0 \leq |P(A_\infty) - P(A_n)| = \sum_{k=n+1}^{\infty} |P(C_k)| < \varepsilon \quad \forall\, n \geq n_0 \text{ for some } n_0 \in \mathbb{N}.$$

Therefore, $\{P(A_n)\}_{n=1}^{\infty}$ converges to $P(A_\infty)$.

<u>Theorem</u>: If $A_\infty, A_1, A_2, A_3, \cdots$ are events such that $A_n \downarrow A_\infty$.

Then $P(A_n) \downarrow P(A_\infty)$.

<u>Proof</u>: <u>Hint</u>: Consider the sequence of events

$$B_n = A_n^c \quad \text{then} \quad B_n \uparrow A_\infty^c.$$

<u>Remark</u>: Find a counter example when the events are not monotone. (or prove that monotonicity is not needed).

$$\left\{\sum_{n=1}^{m} a_n\right\} = \{a_m\}$$

$$\downarrow$$

$$a$$

$$\left\{\sum_{n=1}^{m} a_n - a\right\} \to 0$$

$$\sum_{n=1}^{m} a_m - \sum_{n=1}^{\infty} a_m$$

$$\overset{\|}{\sum_{n=m+1}^{\infty} a_m}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\Downarrow$$

$$\sum_{n=m}^{\infty} \frac{1}{2^n} \to 0 \text{ as } m \to \infty.$$

Example: A fair die is rolled repeatedly independently. Let N be the random variable which denote the number of ones observed. Show that $P(N=\infty)=1$.

$$\{N=\infty\} = \bigcup_{k=1}^{\infty}\{N<k\}$$

Answer: Enough to show that $P(N<\infty)=0$.

Observe that $\{N<\infty\} = \underbrace{\bigcup_{n=1}^{\infty} B_n}$ where $B_n = \{\text{no one's after } n \text{ tosses}\}$

$$= \underbrace{\bigcup_{k=1}^{\infty}\{N<k\}}$$

$$\bigcap_{k=1}^{\infty}\{N<k\} = \{N<1\}$$

Suppose there is a 1 in the $k+10$th roll and total no. of 1s is $<k$ then that occurance is in $\{N<k\}$

Claim: $P(B_n)=0 \ \forall n \geq 1$. Observe that this claim proves our need.

Let us write $B_n = \bigcap_{m=1}^{\infty} C_{n,m}$ where

$$C_{n,m} = \{ \text{no one's in the tosses } n+1, n+2, \dots, n+m \}$$

Now, $\mathbb{P}(C_{n,m}) = \left(\frac{5}{6}\right)^m \longrightarrow 0$ as $m \to \infty$ —— ①

Observe that $C_{n,m} \downarrow B_n$ & $C_{n,m}$ decreases as $m \to \infty$.

$$\mathbb{P}(C_{n,m}) \downarrow \mathbb{P}(B_n) \text{ as } m \to \infty. \quad \text{—— ②}$$

① + ② $\Rightarrow$ $\mathbb{P}(B_n) = 0$. $\square$

but not in B_K.

$$B_n = \bigcup_{m=1}^{\infty} C_{n,m}$$

$C_{n,1} \overset{\text{or}}{\wedge} C_{n,2} \overset{\text{or}}{\wedge} C_{n,3}$

$A_n = \{\text{occurrence of } H\}$ for n odd

$\quad = \{ \text{"} \quad \text{"} \ T \}$ for n even

A_n's are not monotone

$\bigcup_{n=1}^{\infty} A_n = \{\text{occurrence of } H \text{ or } T\}$

$P(\quad \checkmark \quad) = 1$

$P(A_n) = \frac{1}{2} \ \forall n$. $\mathbb{P}(A_n) \to ?$.

Definition:

Suppose that X is a random variable. Define a function

$F_X : \mathbb{R} \to [0,1]$ by

$$F_X(x) = \mathbb{P}(X \le x), \quad x \in \mathbb{R} \quad (\text{Recall } X: \Omega \to \mathbb{R})$$

$\mathbb{P}(X=k)$
$\parallel$
$f_X(k)$

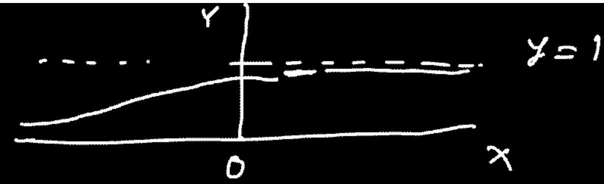
The function F is called the cumulative distribution function, abbreviated as C.D.F. of X.

Theorem: If F is the CDF of a random variable X, then,

i) F is non decreasing

ii) $\lim\limits_{x \to -\infty} F(x) = 0$

iii) $\lim_{x \to \infty} F(x) = 1$ and



iv) F is right continuous.

Proof: i) Observe that if $x \leq y$ then $\{X \leq x\} \subseteq \{X \leq y\}$

$$\Rightarrow P\{X \leq x\} \leq P\{X \leq y\}$$

$$\Rightarrow F(x) \leq F(y)$$

$\therefore$ F is monotone non decreasing.

ii) Consider the sequence of events $\{(X \leq -n)\}_{n=1}^{\infty}$

$\{(X \leq -n)\} \downarrow \phi$ as $n \to \infty$.

replace small 'n''s by any monotone decreasing sequence $\{a_n\}_{n=1}^{\infty} \downarrow -\infty$.

$\Rightarrow P(X \le -n) \downarrow P(\emptyset) = 0$

$\quad\quad\; \parallel$

$\quad\; F(-n)$

Therefore $\quad F(-n) \to 0$ as $n \to \infty$ through natural numbers.

$\Rightarrow \lim_{x \to -\infty} F(x) = 0.$ because F is non decreasing.

and use the fact that, $\exists\, n \in \mathbb{Z}$ such that, $n < x \le n+1$ for any real number x.

$$F(n) \le F(x) \le F(n+1)$$

iii) Hint: Consider $\{X \le n\} \uparrow \{X < \infty\} = \{X \in \mathbb{R}\}$ as $n \to \infty$.

iv) Let $g : \mathbb{R} \to \mathbb{R}$ be a function. Then g is continuous at a point $x_0 \in \mathbb{R}$ if

$$\lim_{x \to x_0^-} g(x) = \lim_{x \to x_0^+} g(x) = g(x_0) .$$

g is continuous on $\mathbb{R}$ if g is continuous at any $x \in \mathbb{R}$.

g is left continuous at $x_0 \in \mathbb{R}$ if $\displaystyle\lim_{x \to x_0^-} g(x) = g(x_0)$.

g is right continuous at $x_0 \in \mathbb{R}$ if $\displaystyle\lim_{x \to x_0^+} g(x) = g(x_0)$.

$\displaystyle\lim_{x \to x_0^-} g(x) = l$ if for any (monotone increasing) sequence $x_n \to x_0$ such that $x_n < x_0$ we have $g(x_n) \to l$. $(l, m \in \mathbb{R})$.

$\displaystyle\lim_{x \to x_0^+} g(x) = m$ if for any sequence $x_n \to x_0$ such that $x_n > x$, $\Rightarrow g(x_n) \to m$.

Without loss of generality we may take the sequences to be monotone.

iv) Let $\{x_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence converging to x such that

$x_n > x \quad \forall\, n \geq 1$.

Then the event $\quad \{X \leq x_n\} \downarrow \{X \leq x\}$

$$\Rightarrow \underset{\substack{\| \\ F(x_n)}}{P(X \leq x_n)} \downarrow \underset{\substack{\downarrow\!/ \\ F(x).}}{P(X \leq x)}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. For any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, there is a subsequence $\{x_{n_{k_l}}\}_{l=1}^{\infty}$ which converges to l. Then $x_n \to l$.

$\therefore \lim\limits_{x_n \to x} F(x_n) = F(x)$. Hence F is right continuous.

* Check that $\{X \leq x_n\} \uparrow \{X < x\}$ when $x_n \uparrow x$ & $x_n < x$.

$[0, 1-\frac{1}{n}) \uparrow$
$[0, 1)$

Hypergeometric (N, K, n) distribution:

Let $N \in \mathbb{N}\cup\{0\}$, $K \in \{0,1,\dots,N\}$ and $n \in \{0,1,\dots,N\}$. Let X be a random variable from Ω to $\{\max(0, n+K-N), \dots, \min(n,K)\}$

$$= \{0 \vee n+K-n, \dots, n \wedge K\}$$

$$\boxed{\begin{matrix} K \\ k \end{matrix}}$$

such that the probability mass function is

$$f(k) = \mathbb{P}(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \qquad k \in \{0 \vee n+K-N, \dots, n \wedge K\}$$

HW Find the CDF of the above distribution.

Tomorrows class will be taken by Arijit Sir.