Probability theory 1

29/10/21

- We have known random variables, probability mass functions, cumulative distribution functions.

- Recall pmf $f(k) := \mathbb{P}(X=k)$ for $k \in A \subseteq \mathbb{Z}$

  CDF $F(x) := \mathbb{P}(X \leq x)$ for $x \in \mathbb{R}$.

- We proceed with calculation for the CDF's for some random variables.

Example:

1) Suppose $X \sim \text{Geometric}(p)$ random variable. Then, for all $n \in \mathbb{N}$

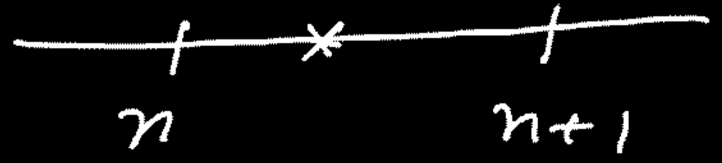
$$P(X \le n) = \sum_{i=1}^{n} P(X=i)$$

$$P(X=n) = (1-p)^{n-1} p$$

$$F(n) = \sum_{i=1}^{n} (1-p)^{i-1} p$$

$$= p \, \frac{\left(1-(1-p)^n\right)}{1-(1-p)} = 1-(1-p)^n .$$

$x \in [n, n+1)$



$$P(X \le x)$$
$$= P(X \le n)$$

Therefore, the CDF of X is,

$$F(x) = \mathbb{P}(X \le x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor} & x > 0 \\ 0 & x \le 0 \end{cases}$$

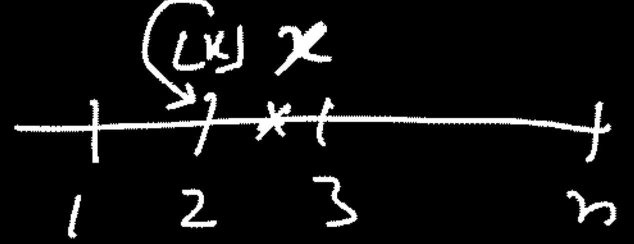
where, $\lfloor x \rfloor$ is the greatest integer not greater than x.

<u>Example 2)</u> Let X be a number chosen at random from $\{1, 2, \cdots, n\}$. This distribution is known as Uniform distribution on $\{1, 2, \cdots, n\}$. Then

$$f(k) = \mathbb{P}(X = k) = \frac{1}{n} \qquad \text{for} \quad k \in \{1, 2, \cdots, n\}.$$

So, the CDF of X is

$$F(x) = \mathbb{P}(X \le x) = \begin{cases} \frac{1}{n} \lfloor x \rfloor & \text{if } 1 \le x \le n \\ 0 & \text{if } x < 1 \\ 1 & \text{if } x > n \end{cases}$$



$$\mathbb{P}(X \le x) = \mathbb{P}(X \le \lfloor x \rfloor)$$

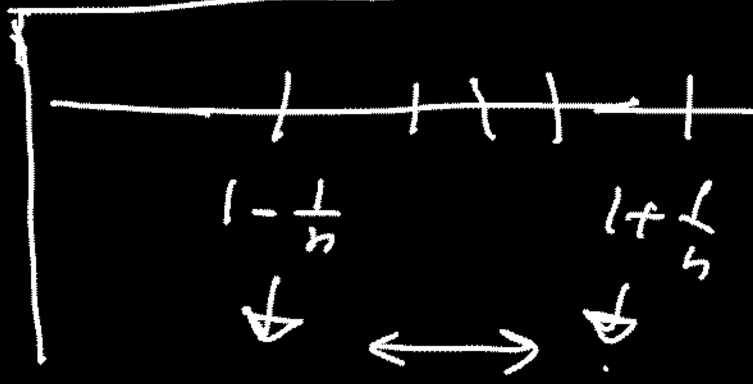
<u>HW:</u> Calculate the CDF of Binomial (n, p) distribution.

$$F(x) = \sum_{i=1}^{\lfloor x \rfloor} f(x)$$

<u>Theorem:</u> Suppose that X has a CDF F. Then for all $x \in \mathbb{R}$,

$$F(x-) = \mathbb{P}(X < x)$$

and $F(x) - F(x-) = \mathbb{P}(X = x)$.



<u>Proof:</u> Take $\{x_n\}_{n=1}^{\infty}$ be a monotone increasing sequence which increases to x.

Then the events $\{X \le x_n\} \uparrow \{X < x\}$. Therefore, applying a previous theorem we have already done, we get,

$F(x-) = \lim_{y \to x^-} F(y)$

"any sequence $\{y_n\} \to x$." such that $y_n < x$ $\forall n$

$$F(x_n) = P(X \le x_n) \uparrow P(X < x)$$

$$\Rightarrow \quad F(x^-) = P(X < x)$$

$$F(x^-) = \lim_{y_n \uparrow x} F(y_n)$$

Now, $F(x) - F(x-) = P(X \le x) - P(X < x) = P(X = x)$.

$(\{X = x\}$ & $\{X < x\}$ are disjoint and their union is $\{X \le x\})$.

Corollary: The CDF of X is continuous at the point $x \in \mathbb{R}$ if and only if

$\mathbb{P}(X=x) = 0$.

Pf: We had from last day $F(x+) = F(x)$.

From today (previous theorem) $F(x-) = -\mathbb{P}(X=x) + F(x)$
$$= F(x).$$

Definition: (Expectation):

Suppose that the random variable X takes values $x_1, x_2, \ldots$ with respective probabilities $p_1, p_2, \ldots$. The expectation of X is

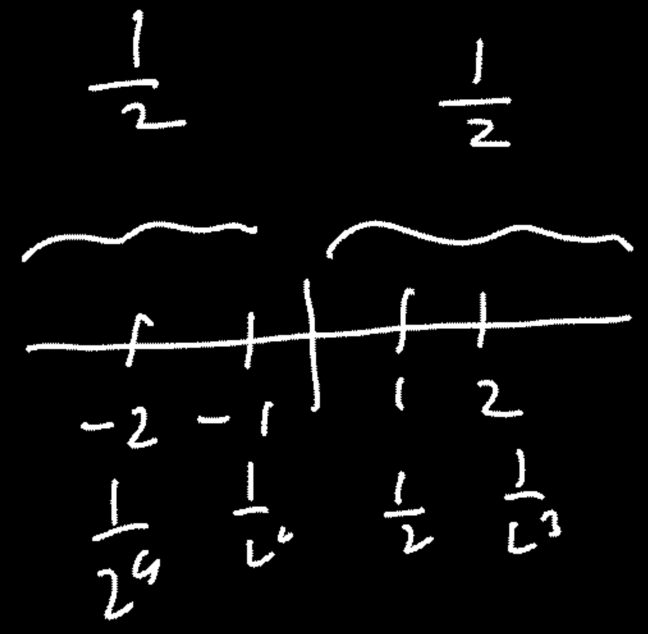
$$E(X) := \sum_{i=1}^{\infty} x_i p_i$$

which is defined only when $\sum_{i \geq 1} |x_i| p_i < \infty$.

** $$\cancel{E(X)} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2$$

$$\sum_{n=1}^{\infty} x_n p_n = \frac{1}{\ln 2} - \frac{1}{2\ln 2} + \frac{1}{3\ln 2} - \cdots = 1$$

put probabilities $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\}$ to the points $\{\frac{2}{\ln 2}, -\frac{2}{\ln 2}, \frac{2^3}{3\ln 2}, \cdots\}$



$\sum x_i p_i$

$=$

the expectation here does not exist.

respectively. Then $E(X)$ is ⊛. Since in this example $\sum |x_n| p_n$ is not convergent we call $E(X)$ to be undefined.

Examples:

1) Bernoulli(p): $E(X) = \sum_{n \geq 1} x_n p_n = 1 \cdot p + 0 \, (1-p)$
$= p.$

2) Binomial (n,p):

$$E(X) = \sum_{k \geq 0} x_k p_k = \sum_{k=0}^{n} k \, P(X=k) = \sum_{k=1}^{n} k \, P(X=k) \quad \text{——} \; ⓐ$$

Identity:

$$k\binom{n}{k} = \frac{k\,n!}{(n-k)!\,k!} = n\,\frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} = n\binom{n-1}{k-1}$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

From ⓐ $\displaystyle\sum_{k=1}^{n} \underbrace{k\binom{n}{k}} p^k (1-p)^{n-k} = \sum_{k=1}^{n} p\,n\binom{n-1}{k-1} p^{k-1} (1-p)^{\{(n-1)-(k-1)\}}$

$$= p\,n\,(p+(1-p))^{n-1}$$

$$= p\,n$$

Example: Expectation of Geometric (p): $\quad X \sim \text{Geom}(p) \quad X \in \{0, 1, 2, \dots\}$

Case I)

$$E(X) = \sum_{k=0}^{\infty} k \, P(X=k) = \sum_{k=0}^{\infty} k \, (1-p)^k \, p$$

$$= p(1-p) \sum_{k=0}^{\infty} k (1-p)^{k-1}$$

$$= p(1-p) \left[\frac{d}{dp} \left(- \sum_{k=0}^{\infty} (1-p)^k \right) \right]$$

$$= p(1-p) \left[\frac{d}{dp} \left(- \frac{1}{1-(1-p)} \right) \right]$$

$$= p(1-p) \left[\frac{1}{p^2} \right] = \frac{1-p}{p}.$$

case 2) $E(X) = \sum_{k=1}^{\infty} k\,(1-p)^{k-1}\,p = p\left[\frac{d}{dp}\left(-\sum_{k=1}^{\infty}(1-p)^{k}\right)\right]$ $\quad X \in \{1, 2, \cdots\}$

$$= p\left[\frac{d}{dp}\left(-\frac{1-p}{p}\right)\right] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Example 4) $X \sim \text{Uniform}\{1, 2, \cdots, n\}$.

$$E(X) = \sum_{k=1}^{n} k \cdot P(X=k) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n}\,\frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Theorem: Let X be random variable taking values in $\{0, 1, 2, \cdots\}$.

If X has an expectation, then,

$$E(X) = \sum_{n=0}^{\infty} P(X > n) = \sum_{n=0}^{\infty} \left(1 - F(n)\right)$$

<u>Proof:</u> By definition, $E(X) = \sum_{n=0}^{\infty} x_n p_n$

$$= \sum_{n=0}^{\infty} x_n P(X=n) = \sum_{n=1}^{\infty} n P(X=n)$$

$$= P(X=1) +$$
$$P(X=2) + P(X=2) +$$
$$P(X=3) + P(X=3) + P(X=3) + \cdot$$
$$\cdots$$
$$\underbrace{P(X=n) + P(X=n) + \cdots + P(X=n)}_{n \text{ times}} +$$
$$\cdots$$

$$= P(X>0) + P(X>1) + \cdots = \sum_{n=0}^{\infty} P(X>n).$$

<u>HW:</u> Balls are drawn at random with replacement from an urn containing 10 black balls, 5 red balls, 5 white balls. Let X denote the number of draws needed to see balls of all colours. Calculate $E(X)$. (You may use $E(X) = \sum_{n=1}^{\infty} \underbrace{P(X>n)}$)

<u>Hint:</u>

$B = \{$ no black balls in the first n draws $\}$

$W = \{$ " " white " " " " " $\}$

$R = \{$ " " red " " " " " $\}$

$\mathbb{P}(X>n) = \mathbb{P}(B \cup W \cup R) = \cdots$ use principle of inclusion-exclusion.

<u>HW:</u> Find $E(X)$ when $X \sim \text{Geom}(p)$ using $E(X) = \sum_{n=0}^{\infty} \mathbb{P}(X>n)$.

<u>Counterexample:</u> Let $A_n = \{\text{Tail occurs in a toss}\}$ for n odd.

$= \{\text{Head occurs in a toss}\}$ for n even.

$\mathbb{P}(A_n) = \frac{1}{2} \ \forall n \in \mathbb{N}$. $\quad \bigcup_{n=1}^{\infty} A_n = \{\text{Either Head or tail occurs}\}$

$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$

$\{\mathbb{P}(A_n)\}_{n=1}^{\infty} = \left\{\frac{1}{2}\right\}_{n=1}^{\infty} \to \frac{1}{2} \neq 1 = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right)$.

$A_n \uparrow A_\infty = \bigcup_{n=1}^{\infty} A_n$

$\mathbb{P}(A_n) \uparrow \mathbb{P}(A_\infty)$

* ① f is right continuous at $x \in \mathbb{R}$ if for any sequence x_n converges to x such that $x_n > x \; \forall n$, then $f(x_n) \to f(x)$.

② f is right continuous at $x \in \mathbb{R}$ if for any monotone decreasing sequence $x_n \downarrow x$ such that $x_n > x \; \forall n$ then $f(x_n) \to f(x)$.

Proof of equivalence of ① & ②:

① $\Rightarrow$ ② is obvious.

② $\Rightarrow$ ① We need to show that for any sequence $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Theorem ⑥: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. If for any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ (of $\{x_n\}_{n=1}^{\infty}$)

has a further subsequence $\{x_{n_{k_l}}\}_{l=1}^{\infty}$ such that $x_{n_{k_l}} \to x$ as $l \to \infty$.

Then $x_n \to x$ as $n \to \infty$.

We have $\{x_n\} \to x$. Consider the sequence $\{f(x_n)\}_{n=1}^{\infty}$. Let $\{f(x_{n_k})\}_{k=1}^{\infty}$ be any sequence of $\{f(x_n)\}_{n=1}^{\infty}$. $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and we may choose a monotone decreasing subsequence of $\{x_{n_k}\}_{k=1}^{\infty}$ (call it $\{x_{n_{k_l}}\}_{l=1}^{\infty}$) by default $x_{n_{k_l}} \to x$ as $l \to \infty$. Using ② $f(x_{n_{k_l}}) \to f(x)$. Thus Applying Theorem ⓑ we conclude the $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$.