Probability theory 1

2/11/21

- We have known about random variables, their pmfs, their CDFs and their expectations.
- We consider the Poisson distribution.
- Poisson distribution:

Let X be a random variable taking values in $0,1,2,\cdots$. Let $\lambda \in (0,\infty)$ such that

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for} \quad k \in \{0,1,2,\ldots\}.$$

The random variable X is called a Poisson random variable with parameter λ.

<u>Theorem:</u> (<u>Poisson limit theorem</u>)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Binomial random variables such that $X_n \sim \text{Binomial}(n, p_n)$ for $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} n\,p_n = \lambda \in (0,\infty), \qquad \cancel{\left(n p_n \neq \lambda \Rightarrow p_n = \frac{\lambda}{n} \right)}$$

then the probability mass function of X_n converges to the same of the Poisson (λ) random variable i.e.,

$$\lim_{n\to\infty} P(X_n = k) = P(\text{Poisson}(\lambda) = k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k\in\{0,1,2,\ldots\}.$$

<u>Proof:</u> Fix a $k \geq 0$, $(k \in \{0,1,2,\cdots\})$

$$P(X_n = k) = \binom{n}{k} p_n^{\,k} (1-p_n)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}\, p_n^{\,k}\, (1-p_n)^{n-k}$$

$$= \left(\frac{1}{k!}(1-p_n)^{-k}\right)\left(\prod_{i=0}^{k-1}(n-i)\,p_n\right)(1-p_n)^{n} \text{ —— } (*)$$

Note that $\lim\limits_{n\to\infty} (1-p_n)^{-k} = 1$ —① $\left(\text{Since, } \lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} np_n \cdot \frac{1}{n} = 0 \right)$

$np_n \to \lambda \in (0,\infty)$

For every fixed $i = 0, 1, \cdots, k-1.$

$$\lim_{n\to\infty} (n-i)p_n = \lim_{n\to\infty} np_n - ip_n = \lambda \quad \text{—②}$$

Furthermore, for any $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ such that

$|np_n - \lambda| \leq \varepsilon$, for all $n \geq N$

$$\Rightarrow -\varepsilon \leq np_n - \lambda \leq \varepsilon$$

$$\Rightarrow \frac{\lambda-\varepsilon}{n} \leq p_n \leq \frac{\lambda+\varepsilon}{n}.$$

Hence
$$\underline{\left(1-\frac{\lambda+\varepsilon}{n}\right)^n \le \left(1-p_n\right)^n} \le \left(1-\frac{\lambda-\varepsilon}{n}\right)^n$$

Therefore
$$e^{-(\lambda+\varepsilon)} \le \liminf_{n\to\infty} (1-p_n)^n \le \limsup_{n\to\infty} (1-p_n)^n \le e^{-(\lambda-\varepsilon)}$$

Letting $\varepsilon \downarrow 0$, it follows that
$$e^{-\lambda} \le \liminf_{n\to\infty} (1-p_n)^n \le \limsup_{n\to\infty} (1-p_n)^n \le e^{-\lambda}$$

Hence,
$$\lim_{n\to\infty} (1-p_n)^n = e^{-\lambda}. \quad \text{------} \quad (3)$$

Putting the values from ①, ② and ③ in ⊛, we get

$$\lim_{n\to\infty} P(X_n = k) = \frac{1}{k!}\lambda^k e^{-\lambda} = P(\text{Poisson}(\lambda) = k), \quad k \in \{0,1,2,\cdots\}.$$

Theorem: Let X be a random variable defined on the probability space (Ω, P). If X has an expectation then,

$$E(X) = \sum_{\omega\in\Omega} X(\omega)\, P(\{\omega\})$$

$$E(X) = \sum_{n\geq 1} x_n p_n$$

Proof: Suppose $X(\Omega) = \{x_1, x_2, \cdots\}$. Then,

$$E(X) = \sum_{n\geq 1} x_n \, \mathbb{P}(X = x_n) = \sum_{n\geq 1} x_n \sum_{\omega\in\Omega : X(\omega) = x_n} \mathbb{P}(\{\omega\})$$

$$= \sum_{n\geq 1} \sum_{\omega\in\Omega : X(\omega) = x_n} X(\omega)\, \mathbb{P}(\{\omega\}) = \sum_{\omega\in\Omega} X(\omega)\, \mathbb{P}(\{\omega\})$$

$$\sum_i \sum_j x_i x_j$$

$$= \sum_i x_i \sum_j x_j$$

Since, $\Omega = \bigcup_{n\geq 1} \underbrace{\{\omega\in\Omega : X(\omega) = x_n\}}_{= X^{-1}(x_n)}$

$$\boxed{\sum_{\omega'\in\Omega} = \sum_{\omega'\in \bigcup_{n\geq 1}\{\omega\in\Omega : X(\omega) = x_n\}}}$$

$$= \sum_{\omega'\in\{\omega\in\Omega : X(\omega) = x_1\}} + \sum_{\omega'\in\{\omega\in\Omega : X(\omega) = x_2\}} + \cdots$$

$$= \sum_{n\geq 1} \sum_{\omega'\in\{\omega\in\omega : X_n(\omega) = x_n\}}$$

Theorem: Let X and Y be two random variables (with finite expectations) defined on the same probability space. Then $X+Y$ has expectation and

$$E(X+Y) = E(X) + E(Y).$$

Pf: $$E(X+Y) = \sum_{\omega \in \Omega} (X+Y)(\omega) P(\{\omega\}) = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\}) + \sum_{\omega \in \Omega} Y(\omega) P(\{\omega\})$$

$$= E(X) + E(Y).$$

Hw: Try to prove the above theorem using $E(X) = \sum_{n \geq 1} x_n p_n$.

Theorem: If X is a random variable with expection then for all $\alpha \in \mathbb{R}$,

$$E(\alpha X) = \alpha E(X).$$

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\{\omega\})$$

Proof: Complete the proof yourselves using the previous theorem.

$$\left(V = \{ E(X) : X \text{ is a random variable } \Omega \text{ to } \mathbb{R} \}, +, \cdot \right)$$

$$E(X_1) + E(X_2) = E(X_1 + X_2).$$

$$\boxed{E(X) = 0 \not\Rightarrow X \equiv 0.}$$

rough^n.

$$X \equiv 0$$

$$E : V \to \mathbb{R}$$

$$X \equiv 0$$

$$X \not\equiv 0$$

$$E(X) = 0$$

$X = 1$ w.p. $\frac{1}{2}$

$= -1$ w.p. $\frac{1}{2}$.

$S \leq V$

$E^c = S \perp E$

Definition: If X is a random variable with expectation μ, its variance is defined as

$$Var(X) := E\left((X-\mu)^2\right),$$

whenever the expectation on the right hand side is defined.

HW: Find a discrete random variable such that $E(X) < \infty$ but $Var(X)$ does not exists.

Theorem: If X is a random variable with variance defined. Then,

$$Var(X) = E(X^2) - (E(X))^2.$$

Pf: $\mathrm{Var}(X) = E((X-\mu)^2) = E(X^2 - 2\mu X + \mu^2)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - (E(X))^2$ (since $E(X) = \mu$)

$X = 1$ w.p. $\frac{1}{2}$

$= -1$ w.p. $\frac{1}{2}$

$E(X) = 1 \cdot \frac{1}{2} - \frac{1}{2} 1$

$= 0$.

$E((E(X)^2))$

Corollary: For a random variable with variance defined, $(E(X))^2 \leq E(X^2)$.

~~$E(X) =$~~

~~$X = (-1)^n n$ w.p. $\frac{1}{n^2} \frac{6}{\pi^2}$~~

~~$E(X) = \frac{(-1)^n}{n} \frac{6}{\pi^2}$~~

Proof: Since $(X-\mu)^2$ is a non-negative random variable.

$\therefore E((X-\mu)^2) \geq 0 \Rightarrow \mathrm{Var}(X) \geq 0 \Rightarrow E(X^2) - (E(X))^2 \geq 0$ $\square$

$X = c$ w.p. 1 $\quad E(X) = c$

- $X \in \{1, 2, \dots, n, \}$
$\downarrow \downarrow \quad \downarrow$

$P(X=n) = \frac{1}{\alpha}\frac{1}{n^3} \ \forall n \in \{1, 2, \dots\}$

$\alpha = \zeta(3)$ $\quad \sum_{n=1}^{\infty} \frac{1}{\alpha}\frac{1}{n^3} = 1$

$$E(X) = \sum n \frac{1}{\alpha}\frac{1}{n^3} = \frac{1}{\alpha}\sum \frac{1}{n^2} < \infty$$

$$E(X^2) = \sum n^2 \frac{1}{\alpha}\frac{1}{n^3} = \frac{1}{\alpha}\sum \frac{1}{n} = \infty.$$

$\therefore \ Var(X) = \infty.$ $\qquad Var(X) = E(X^2) - (E(X))^2$

Example: Let $X \sim \text{Binomial}(n, p)$. Recall $E(X) = np$.

We have, $Var(X) = E(X^2) - (E(X))^2 = E(X^2) - n^2p^2$.

$E(X^2) = \underline{\qquad\qquad}$ $(**)$

$$E(X) = \sum_{n=1}^{\infty} x_n p_n \qquad E(X^2) = \sum_{n=1}^{\infty} x_n^2 p_n$$

Theorem: Suppose that X is a random variable defined on $(\Omega, \mathbb{P})$. Let $f: \mathbb{R} \to \mathbb{R}$ be a function and define $Y = f(X)$.

If Y has an expectation, then

$$E(Y) = \sum_{n \geq 1} f(x_n) \mathbb{P}(X = x_n).$$

$$\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$\underline{f(X) = f \circ X}$$

Pf: We have $Y = f(X)$ takes the values $f(x_1), f(x_2), \cdots$. (may or may not be distinct).

Now $E(Y) = \sum_{n \geq 1} f(x_n) \mathbb{P}(Y = f(x_n))$

$$= \sum_{n \geq 1} f(x_n) \mathbb{P}(X = x_n). \quad \square$$

$f = 1$

$X = 1$

$= 2$

$1 \, \mathbb{P}(f(x) = 1)$

$+ 2 \, \mathbb{P}(f(x) = 1)$

Remark: $E(X^n) = \sum_{n \geq 1} x^n p_n$ $\qquad E(\sin X) = \sum_{n \geq 1} (\sin x_n) p_n$

X is a random variable.

$X = 1$ w.p. $\frac{1}{2}$
$\quad = 2$ w.p. $\frac{1}{2}$

$f \equiv 1$

$f(X) \equiv 1 \longrightarrow f(x_n) \equiv 1$

$$E(f(X)) = 1 \times P(f(X) = 1)$$
$$= 1 \; P(X=1) + 1 \; P(X=2)$$

$$E(Y) = \sum_{\omega \in \Omega} Y(\omega) \, P(\{\omega\}) = \sum_{\omega \in \Omega} f(X)(\omega) \, P(\{\omega\})$$

$\{\omega : X(\omega) = x_n\}$
$\{\omega : f(X)(\omega) = f(x_n)\}$

$$= \sum_{n \geq 1} \sum_{\omega \in \Omega : f(X)(\omega) = f(x_n)} f(X)(\omega) \, \mathbb{P}(\{\omega\}) = \sum_{n \geq 1} f(x)(\omega) \sum_{\omega \in \Omega : f(X)(\omega) = f(x_n)} \mathbb{P}(\{\omega\})$$

$$= \sum_{n \geq 1} x_n \sum$$

$\boxed{\text{Postpone for now.}}$