Probability theory 1

3/11/21

<u>Theorem</u>: For a random variable X taking values $x_1, x_2, \dots$ and a function $f: \mathbb{R} \to \mathbb{R}$ such that $f(X)$ has an expectation, it holds that

$$E(f(X)) = \sum_{n \geq 1} f(x_n)\, \mathbb{P}(X = x_n).$$

$E(X) = \sum_n x_n p_n$

$E(X^2) = \sum_n x_n^2 p_n$

$E(X^n) = \sum_n x_n^n p_n$

<u>Proof</u>: $$E(f(X)) = \sum_{n \geq 1} y_n\, \mathbb{P}(f(X) = y_n)$$

$$\begin{array}{ccccc} \Omega & \xrightarrow{X} & \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ \{\omega\} & & \{x_1, x_2, \dots\} & & \{y_1, y_2, \dots\} \end{array}$$

where $y_1, y_2, \dots$ are the distinct values that $f(X)$ takes.

Also observe that $f^{-1}(y_n) = \{x_{n_1}, x_{n_2}, \dots\}$ (say) for all $n \in \mathbb{N}$.

and $\{f^{-1}(y_n)\}_{n=1}^{\infty}$ is a sequence of disjoint sets. Note that

$$\underline{\bigcup_{n=1}^{\infty} f^{-1}(y_n) = \{x_1, x_2, \dots\}}$$

So,
$$E(f(X)) = \sum_{n \geq 1} y_n P(X = f^{-1}(y_n))$$

$$= \sum_{n \geq 1} y_n \sum_{j \geq 1} P(X = x_{n_j})$$

$$= \sum_{n \geq 1} \sum_{j \geq 1} f(x_{n_j}) P(X = x_{n_j})$$

$E(f(X))$ exists.
$\sum |y_n| P(X = f^{-1}(y_n)) < \infty$.

$$= \sum_{n \geq 1} f(x_n)\, P(X = x_n) \qquad \square$$

<u>Corollary:</u> $E(X^k) = \sum_{n\geq 1} x_n^k p_n$ and $E(\sin X) = \sum_{n \geq 1} (\sin x_n)\, p_n$.

<u>Theorem:</u> If X has a variance, then $E((X-a)^2)$ is defined for any $a \in \mathbb{R}$.

<u>Proof:</u>

$$E(X^2) = E\left((X-\mu+\mu)^2\right) \quad \text{where } \mu = E(X)$$

$$= E\left((X-\mu)^2\right) - 2\mu E(X-\mu) + \mu^2$$

$$= \operatorname{Var}(X) + \mu^2 < \infty.$$

$\cancel{(E(X))^2 \leq E(X^2)}$.

$$\xcancel{\begin{array}{c} E(X^2) - (E(X))^2 < \infty \\ \Downarrow \\ E(X^2) < \infty. \end{array}}$$

Fix $a \in \mathbb{R}$, $\quad E((X-a)^2) = E(X^2) - 2a\,E(X) + a^2 < \infty \quad \square$

Theorem: If $E(X^2)$ is defined, then prove that

$$\min_{a \in \mathbb{R}} E((X-a)^2) = Var(X) = E((X-\mu)^2) \qquad \mu = E(X).$$

$$\| $$

$$\min\{E((X-a)^2) \mid a \in \mathbb{R}\}$$

$$\min_n \left\{\frac{1}{n}\right\}$$

$$= 0$$

Proof: Fix an $a \in \mathbb{R}$

$$E((X-a)^2) = E((X-\mu+\mu-a)^2)$$

$$= E((X-\mu)^2) + 2(\mu-a)E(X-\mu) + (\mu-a)^2$$

$E(X^2) < \infty \Rightarrow E((X-\mu)^2) < \infty$

$(E(X))^2 \leq E(X^2)$

$E((X-E(X))^2) \geq 0$

$$= \mathrm{Var}(X) + (\mu - a)^2 \geq \mathrm{Var}(X)$$

$$(\because (\mu - a)^2 \geq 0).$$

and the equality holds if $\mu = a$. $\square$

Example: Let $X \sim \text{Binomial}(n, p)$. Recall $E(X) = np$.

We know that $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$$= E(X^2) - n^2 p^2$$

$$E(X^2) = \sum_{k=0}^{n} k^2 P(X=k) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{\left(E(|X|^{\frac{1}{r}})\right)^r \leq \left(E(|X|^{\frac{1}{s}})\right)^s \quad \forall\ r \geq s}$$

$$r = 1, \quad s = 1/2$$

$$\underline{E(X) \leq \left(E(X^2)\right)^{1/2}}$$

$$\Downarrow$$

$$\left(\sum x_n^{\frac{1}{r}} p_n\right)^r \leq \left(\sum x_n^{1/s} p_n\right)^s \quad \forall\ r \geq s$$

Lyapunov's inequality.

$$\underline{E(X) \leq E(|X|).}$$

$$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \qquad \left(k\binom{n}{k} = n\binom{n-1}{k-1} \right)$$

$$= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^{j} (1-p)^{m-j} \qquad \left(\begin{matrix} j = k-1 \\ m = n-1 \end{matrix} \right)$$

$$= np \left[\sum_{j=1}^{m} j \binom{m}{j} p^{j} (1-p)^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^{j} (1-p)^{m-j} \right]$$

$$= np \left[\sum_{j=1}^{m} m \binom{m-1}{j-1} p^{j} (1-p)^{m-j} + (p + (1-p))^{m} \right] \qquad \left(j\binom{m}{j} = m\binom{m-1}{j-1} \right)$$

$$= np\left[mp \underbrace{\sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} (1-p)^{m-1-(j-1)}}_{=1} + 1 \right]$$

$$= np\left[mp + 1 \right] = np\left(np - p + 1 \right) = n^2p^2 - np^2 + np$$

$$\therefore \mathrm{Var}(X) = E(X^2) - n^2p^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p).$$

<u>Example:</u> Let $X \sim \mathrm{Geom}(p)$. Recall $\underline{E(X) = \frac{1-p}{p}}$.

(case 1)

$X \in \{0, 1, 2, \cdots\}$

Now, $E(X^2) = \sum_{k=0}^{\infty} k^2 \, P(X=k) = \sum_{k\geq 0} k^2 (1-p)^k p$

$$\left| \begin{aligned} &\frac{d^2}{dp^2} (1-p)^k \\ &= k(k-1)(1-p)^{k-2} \end{aligned} \right.$$

$$= p(1-p)^2 \sum_{k\geq 0} k \cdot k \, (1-p)^{k-2}$$

$$= p(1-p)^2 \left[\sum_{k\geq 2} k(k-1)(1-p)^{k-2} + \underline{k(1-p)^{k-2}} \right]$$

$$= p(1-p)^2 \left[\frac{d^2}{dp^2} \left(\sum_{k=0}^{\infty} (1-p)^k \right) \right] + p(1-p) \sum_{k=0}^{\infty} k(1-p)^{k-1}$$

$$= p(1-p)^2 \left[\frac{d^2}{dp^2} \left(\frac{1}{p} \right) \right] + p(1-p) \left[\frac{d}{dp} \left(-\frac{1}{p} \right) \right]$$

$$= p(1-p)^2 \frac{2}{p^3} + p(1-p)\frac{1}{p^2}$$

$$= \frac{2(1-p)^2}{p^2} + \frac{1-p}{p}.$$

Hence $\mathrm{Var}(X) = E(X^2) - \left(\frac{1-p}{p}\right)^2 = \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p} - \frac{(1-p)^2}{p^2}$

$$= \frac{(1-p)}{p}\left[\frac{2(1-p)}{p} + 1 - \frac{1-p}{p}\right] = \frac{(1-p)}{p}\left[\frac{1-p}{p} + 1\right]$$

$$= \frac{(1-p)}{p^2} \qquad \square$$

Example: Let $X \sim$ Uniform $\{1, \cdots, n\}$. $E(X) = \frac{n+1}{2}$.

Now, $E(X^2) = \sum_{k=1}^{n} k^2 \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6}$

$$Var(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \frac{n+1}{2} \frac{n-1}{6}$$

$$= \frac{n^2-1}{12} \quad \square.$$

Example: Let $X \sim$ Poisson (λ). $\mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k \in \{0, 1, 2, \cdots\}$.

$$E(X) = \sum_{k \geq 0} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k \geq 1} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

H.W: Calculate $E(X^2) = \lambda^2 + \lambda$ (Check!!)

So, $\quad \mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda. \qquad \square.$