Probability theory 1

16/11/21

<u>Q</u>: Find random variables X and Y such that $E(XY)$ exists but both $E(X)$ and $E(Y)$ do not exist.

<u>Ans</u>: Consider $T = n$ w.p. $\frac{c}{n^2}$ where $\frac{1}{c} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $X = T\mathbb{1}(T \text{ is odd})$ and $Y = T\mathbb{1}(T \text{ is even})$

$$E(X) = \sum_{n \text{ odd}} \frac{c}{n} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{3n} = \infty \quad (\text{because } 3n \geq 2n+1)$$

$n - 1 \geq 0 \ \forall n \geq 1)$

$$E(Y) = \sum_{n \text{ even}} \frac{c}{n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

$$\geq \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty.$$

$E(X)$ and $E(Y)$ do not exist.

$$\boxed{\begin{aligned} &\mathbb{1}(A) \cdot \mathbb{1}(B) \\ &= \mathbb{1}(A \cap B) \\ &\underline{\underline{\text{HW.}}} \end{aligned}}$$

$$XY = T^2 \, \mathbb{1}(T \text{ is even}) \, \mathbb{1}(T \text{ is odd})$$

$$= T^2 \, \mathbb{1}(T \text{ is even and } T \text{ is odd}) = 0.$$

$E(XY) = 0.$

H.W.

Choose X and Y such that XY is not identically equal to 0.

Remark: If X and Y are independent then $\operatorname{Cov}(X,Y) = 0$. However, the converse is not always true.

For example:

Let us consider the joint distribution of X and Y as

$$\mathbb{P}(X=-1, Y=1) = \mathbb{P}(X=0, Y=-2) = \mathbb{P}(X=1, Y=1) = \frac{1}{3}.$$

$$\mathbb{P}(X=-1, Y=-2) = \mathbb{P}(X=0, Y=1) = \mathbb{P}(X=1, Y=-2) = 0$$

i.e. $X \in \{-1, 0, 1\}$ and $Y = \{1, -2\}$.

Now we find the marginal distributions of X and Y.

$$P(X=-1) = P(X=-1, Y=1) + P(X=-1, Y=-2) = \frac{1}{3}$$

$$P(X=0) = P(X=0, Y=1) + P(X=0, Y=-2) = \frac{1}{3}$$

$$P(X=1) = \qquad\qquad = \frac{1}{3}$$

$$E(X) = 0.$$

$$P(Y=1) = P(X=-1, Y=1) + P(X=0, Y=1) + P(X=-1, Y=1) = \frac{2}{3}$$

$$P(Y=-2) = \qquad\qquad = \frac{1}{3}$$

$$E(Y) = 0.$$

$$E(XY) = (-1) \times \frac{1}{3} + 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) = 0.$$

Therefore, $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$

$$P(X=-1, Y=1) = \frac{1}{3}$$

$$\neq$$

$$P(X=-1)P(Y=1) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

Hence X and Y are not independent but $\mathrm{cov}(X,Y) = 0$.

Last day's exercise: Let X be a random variable such that $E(X^2)$ is defined.

Suppose that Y takes the values ± 1 with probability $\frac{1}{2}$, independently of X.

Define $Z = XY$. Show that $\mathrm{Cov}(X, Z) = 0$. Are X and Z independent?

Ans: $E(X^2) < \infty$. $E(Y) = 0 \overset{?}{\Rightarrow} E(XY) = 0$ (Try to find X, Y s.t. $E(X) = 0$ but $E(XY) \neq 0$).

$$E(XY) = \sum_{n=1}^{\infty} x_n P(X = x_n, Y = 1) + \left(\sum_{n=1}^{\infty} -x_n P(X = x_n, Y = -1) \right)$$

$$= \sum_{n=1}^{\infty} x_n P(X = x_n) P(Y = 1) - \sum_{n=1}^{\infty} x_n P(X = x_n) P(Y = -1)$$

$$= \frac{1}{2} \left[E(X) - E(X) \right] = 0$$

(where $X \in \{x_1, x_2, \ldots\}$ w.p. $\{p_1, p_2, \ldots\}$ respectively).

We write $Z = XY$.

$$\mathrm{Cov}(X, Z) = E(XZ) - E(X)E(Z)$$

$$= E(XZ) = E(X^2 Y) = E(X^2)E(Y) = 0 \quad (E(X^2) < \infty \text{ needed here})$$

$$X = x_1 \text{ w.p. } p_1 \qquad Y = 1 \text{ w.p. } \tfrac{1}{2} \qquad XY = x_1 \text{ w.p. } \tfrac{1}{2}p_1$$

$$= x_2 \text{ w.p. } p_2 \qquad = -1 \text{ w.p. } \tfrac{1}{2} \qquad = -x_1 \text{ w.p. } \tfrac{1}{2}p_1$$

$$= \ldots \qquad = x_2 \text{ w.p. } \tfrac{1}{2}p_2$$

$$\ldots \qquad = -x_2 \text{ w.p. } \tfrac{1}{2}p_2$$

$$\vdots$$

$$-x_1 \to -1$$

$$P(X = x_1, Z = x_1) = P(X = x_1, XY = x_1) = P(X = x_1, Y = 1) = p_1 \tfrac{1}{2} = \underline{\tfrac{p_1}{2}}$$

$$P(X = x_1) P(Z = x_1) = p_1 \left(\tfrac{1}{2} p_1 + \tfrac{1}{2} p_5\right) = \underline{\tfrac{1}{2} p_1^2}$$

$$p_1\left(\tfrac{1}{2}(p_1 + p_5)\right) = \frac{p_1}{2}$$

$$p_1 + p_5 \neq 1$$

~~If $\frac{p_1}{2} = \frac{p_1^2}{2} \Rightarrow p_1(1-p_1) = 0 \Rightarrow$ either $p_1 = 0$ or $p_1 = 1$.~~

~~Now X and Z are not independent if X does not achieve the value $-x_1$ with positive probability and $P(X=x_1) = p_1 \in \{0,1\}$.~~

$P(X=x_1, Z=x_1) = \frac{p_1}{2}$ & $P(X=x_1)P(Z=x_1) = \frac{1}{2}p_1^2$ or

$P(X=x_1)P(Z=x_1) = p_1\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right)$ (when $X=-x_1$ w.p. p_2)

If $P(X=x_1, Z=x_1) = P(X=x_1)P(Z=x_1) \Rightarrow \frac{p_1}{2} = \frac{p_1}{2}(p_1+p_2) \Rightarrow p_1+p_2 = 1$.

but $|X|$ is not degenerate

Whenever $|X|$ is not degenerate the X and Z are not independent.

<u>Recall:</u> X and Y are random variables. $\mathrm{Var}(X), \mathrm{Var}(Y) < \infty$. Then

$$\mathrm{Corr}(X,Y) = \mathrm{Cov}(X,Y)\,(\mathrm{Var}(X))^{-1/2}\,(\mathrm{Var}(Y))^{-1/2}$$

<u>Theorem:</u> (<u>Already proved</u>) If $\mathrm{Corr}(X,Y) = 1$. Then,

$$Y = E(Y) + \sqrt{\frac{\mathrm{Var}(Y)}{\mathrm{Var}(X)}}\,(X - E(X)) \quad \text{almost surely,}$$

and if $\mathrm{Corr}(X,Y) = -1$, then,

$$Y = E(Y) - \sqrt{\frac{\mathrm{Var}(Y)}{\mathrm{Var}(X)}}\,(X - E(X)) \quad \text{almost surely.}$$

$X = Y$ a.s.

$P(X=Y) = 1$.

In Ber(1)

$P(X=1) = 1$

$P(X=0) = 0$.

<u>Exercise</u>: Suppose that A_1, A_2, A_3 are pairwise independent events, with $P(A_i) = \frac{1}{2}$ $\forall\ i = 1,2,3$. Assume that $P(A_1 \cap A_2 \cap A_3) = \frac{1}{4}$. Show that

$$P(\text{the number of events in } A_1, A_2, A_3 \text{ which occur is odd}) = 1.$$

<u>Solution</u>: Define $X_i = \mathbb{1}_{A_i}$ for all $i = 1,2,3$.

$$P\left(\sum_{i=1}^{3} \mathbb{1}_{A_i} \text{ is odd}\right) = 1$$

All that needs to be shown is that $\underline{X_1 + X_2 + X_3 \text{ is odd}}$, almost surely, which is same as

$$X_3 = \begin{cases} 1 & \text{if } X_1 = X_2 \\ 0 & \text{if } X_1 \neq X_2 \end{cases}$$

almost surely.

That is, we need to show,

$$\underline{\underline{X_3 = 1-(X_1-X_2)^2}} \quad \text{almost surely.}$$

We know that X_1 and X_3 are independent random variables ($\because$ A_1 & A_3 are independent) and so are X_2 and X_3. Hence,

$$\mathrm{Cov}\left((X_1-X_2)^2, X_3\right) = \mathrm{Cov}(X_1^2, X_3) - 2\,\mathrm{Cov}(X_1X_2, X_3) + \mathrm{Cov}(X_2^2, X_3)$$

$$= -2\left[E(X_1X_2X_3) - E(X_1X_2)E(X_3)\right]$$

$$= -2\left[P(A_1\cap A_2\cap A_3) - P(A_1\cap A_2)P(A_3)\right]$$

$$= -2\left[\frac{1}{4} - \left(\frac{1}{2}\right)^3\right] = -\frac{1}{4}.$$

$$\boxed{\begin{aligned} &E(X_1X_2X_3) \\ &= E[\mathbb{1}_{A_1}\mathbb{1}_{A_2}\mathbb{1}_{A_3}] \\ &= E[\mathbb{1}(A_1\cap A_2\cap A_3)] \\ &= P(A_1\cap A_2\cap A_3) \end{aligned}}$$

$$\mathrm{Var}(X_3) = E(X_3^2) - (E(X_3))^2 = E(X_3) - (E(X_3))^2$$

$$= P(A_3) - (P(A_3))^2 = \frac{1}{2} - \frac{1}{4} = +\frac{1}{4} \quad \text{and}$$

$$\mathrm{Var}((X_1-X_2)^2) = E((X_1-X_2)^4) - (E((X_1-X_2)^2))^2 = \dots\dots = \frac{1}{4}.$$

$$\begin{cases} \underline{(X_1-X_2)}^2 = 1 & \text{w.p. } P(X_1 \neq X_2) = \frac{1}{2} \\ = 0 & \text{w.p. } P(X_1 = X_2) = \frac{1}{2} \end{cases}$$

$X_1 \neq X_2$

$\mathbb{1}_{A_1} \neq \mathbb{1}_{A_2}$

$\mathbb{1}_{A_1}(\omega) = 0 \quad \text{w.p. } P(A^c) = 1/2$

$\phantom{\mathbb{1}_{A_1}(\omega)} = 1 \quad \text{w.p. } P(A) = 1/2$

since. $\mathbb{1}_{A_1} \neq \mathbb{1}_{A_2} \Rightarrow P(\mathbb{1}_{A_1} = 1, \mathbb{1}_{A_2} = 0) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \rightarrow (+) = \frac{1}{2}$

or $P(\mathbb{1}_{A_1} = 0, \mathbb{1}_{A_2} = 1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

So, $(X_1-X_2)^2 \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ $\quad Var\left((X_1-X_2)^2\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$.

Hence, $$Corr\left((X_1-X_2)^2, X_3\right) = \frac{Cov\left((X_1-X_2)^2, X_3\right)}{\sqrt{Var\left((X_1-X_2)^2\right) Var(X_3)}} = \frac{-1/4}{\sqrt{\frac{1}{4}\cdot\frac{1}{4}}} = -1$$

$$X_3 = \frac{1}{2} - \sqrt{\frac{1/4}{1/4}}\left((X_1-X_2)^2 - \frac{1}{2}\right) \quad \text{a.s.}$$

$$X_3 = -(X_1-X_2)^2 + 1 \quad \text{a.s.} \quad \square$$

$Corr(X,Y) = -1$

$Y = E(Y) - \sqrt{\frac{Var(Y)}{Var(X)}}\,(X - E(X))$

almost surely.