Probability theory 1

30/11/21

- We have gone through random experiments, probabilities, independence, random variables, PMF, CDF, expectation, variance, joint distributions, covariance and correlation.
- We shall continue with joint conditional distribution and conditional expectation.
- <u>Definition:</u> Let X and Y be two random variables defined on a sample space Ω. Suppose that $A \subset \Omega$, is such that $\mathbb{P}(A) > 0$. Then conditional distribution of (X, Y) given A is

$$p_A(x,y) = P(X=x, Y=y \mid A) = \frac{P(\{X=x\} \cap \{Y=y\} \cap A)}{P(A)} \quad \text{for } x,y \in \mathbb{R}.$$

Example: Consider an urn containing 5 balls of three colours – white, black and red. Ten balls are drawn with replacement from the urn. Let X and Y denote the number of black and white balls respectively.

① Given the information that the first drawn ball was red, find the conditional distribution of X and Y.

Ans: $X \in \{0,1,2,\cdots,9\}$ $\quad Y \in \{0,1,\cdots,9\}$

$A = \{$ the first drawn ball is red $\}$. Let $x, y \in \{0,1,\cdots,9\}$

$$p_A(x,y) = P(X=x, Y=y \mid A) = \frac{P(X=x, Y=y, A)}{P(A)}$$

$$= \frac{\left(\frac{5}{15}\right) \frac{9!}{x!\,y!\,(9-x-y)!} \left(\frac{5}{15}\right)^{x} \left(\frac{5}{15}\right)^{y} \left(\frac{5}{15}\right)^{9-x-y}}{\left(\frac{5}{15}\right)}$$

$$= \frac{9!}{x!\,y!\,(9-x-y)!} \left(\frac{1}{3}\right)^{9}$$

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1st R: 1 | x B | y W | $9-x-y$ R

H.W. draw without replacement check independence among the draws.

② Find the conditional distribution of X and Y given $X+Y=7$.

Ans: Here $A = \{X+Y=7\}$. $X, Y \in \{0,1,2,\dots,7\}$ s.t. $x+y=7$. When $x+y \neq 7$ we have $p_A(x,y) = 0$.

When $\underset{\Uparrow}{\underline{x+Y=7}}$,

$$p_A(x,y) = P\left(X=x, Y=y \mid X+Y=7\right) = \frac{P(X=x, Y=y, X+Y=7)}{P(X+Y=7)}$$

$$= \frac{P(X=x, \widetilde{Y=y,\ x+y=7})}{P(X+Y=7)} = \frac{P(X=x, Y=7-x)}{\sum_{k=0}^{7} P(X=k, Y=7-k)}$$

$$= \frac{\frac{10!}{x!(7-x)!\,3!}\left(\frac{5}{15}\right)^x\left(\frac{5}{15}\right)^{7-x}\left(\frac{5}{15}\right)^3}{\sum_{k=0}^{7}\frac{10!}{k!(7-k)!\,3!}\left(\frac{5}{15}\right)^{10}}$$

③ Given that $X=3$, find the conditional distribution of Y.

<u>Ans:</u> $A=\{X=3\}$. $Y\in\{0,1,\dots,7\}$. For $y\in\{0,1,\dots,7\}$.

$$p_A(y) = P(Y=y \mid A) = \frac{P(Y=y, X=3)}{P(X=3)} = \frac{P(Y=y, X=3)}{\sum_{k=0}^{7} P(X=3, Y=k)}$$

$$= \frac{\frac{10!}{3!\, y!\, (7-y)!} \left(\frac{5}{15}\right)^3 \left(\frac{5}{15}\right)^y \left(\frac{5}{10}\right)^{10-y-3}}{\sum_{k=0}^{7} \frac{10!}{3!\, k!\, (7-k)!} \left(\frac{1}{3}\right)^{10}} \qquad \square$$

<u>Example:</u> Two numbers are chosen without replacement from $\{1, 2, \dots, N\}$. The smaller number is denoted by S and the larger one by L.

① It is given that a number k has been drawn where $1 \le k \le N$. Calculate the conditional distribution of S and L given this information.

Ans: (S, L) is supported on $T = \{ (i,j) :$ if $i = k$ then $k < j \le N$; if $j = k$ then $1 \le i < k \}$

$[r] := \{1, 2, \dots, r\}$

$\{[k-1] \times \{k\}\} \cup \{\{k\} \times \{[N] \setminus [k]\}\}$

Define D_1 = the number in the first draw

D_2 = the number in the second draw.

Then, $S = \min\{D_1, D_2\}$ and $L = \max\{D_1, D_2\}$. Now for $(i,j) \in T$, and $i = k$, $j > k$

$$p_A(k,j) = \mathbb{P}(S = k, L = j \mid A) = \frac{\mathbb{P}(S = k, L = j, A)}{\mathbb{P}(A)}$$

where,

$A = \{D_1 = k\} \cup \{D_2 = k\}$

$$= \frac{\mathbb{P}\left(\min\{D_1, D_2\} = k, \max\{D_1, D_2\} = j, \overbrace{\{D_1 = k\} \cup \{D_2 = k\}}\right)}{\mathbb{P}(\{D_1 = k\} \cup \{D_2 = k\})}$$

$p_A(x,y) = (\ )\ x,y \dots$

$= (\ )\ x,y \dots$

$=$

$$= \frac{\mathbb{P}(\min\{D_1, D_2\} = k, \max\{D_1, D_2\} = j, \{D_1 = k\}) + \mathbb{P}(\min^{\{D_1,D_2\}} = k, \max^{\{D_1,D_2\}} = j, D_2 = k)}{\mathbb{P}(D_1 = k \lor D_2 = k)}$$

$\mathbb{P}(A \cap B \cap (C \cup D))$
$= \mathbb{P}(A \cap B \cap C)$
$+ \mathbb{P}(A \cap B \cap D)$

$$= \frac{\mathbb{P}(D_1 = k, D_2 = j) + \mathbb{P}(D_2 = k, D_1 = j)}{\mathbb{P}(D_1 = k) + \mathbb{P}(D_2 = k)} = \frac{\frac{1}{N(N-1)} + \frac{1}{N(N-1)}}{\frac{1(N-1)}{N(N-1)} + \frac{1}{N}}$$

$$= \frac{2N}{2N(N-1)} = \frac{1}{N-1}$$

So, $p_A(k, j) = \frac{1}{N-1}$ for all $j \in \{k+1, \ldots, N\}$

Similarly $p_A(i, k) = \frac{1}{N-1}$ for all $i \in \{1, \ldots, k-1\}$

② Given that $S = \frac{N}{2}$, Calculate the conditional distribution of L. (HW).

$N \in 2\mathbb{N}$.

<u>Definition:</u>

Let X be a random variable taking values $x_1, x_2, \cdots$ and having an expectation and A is an event with $\mathbb{P}(A) > 0$. The conditional expectation of X given A is

$$\underline{\underline{E(X|A)}} = \sum_{n=1}^{\infty} x_n \, \mathbb{P}(X = x_n | A). \quad \text{—— } ⊛$$

<u>Remark:</u> If X has an expectation then the sum on the right hand side of ⊛ is absolutely convergent.

$\cdot \sum_{n\geq 1} |x_n| \, \mathbb{P}(X = x_n | A) = \sum_{n\geq 1} |x_n| \frac{\mathbb{P}(X = x_n, A)}{\mathbb{P}(A)} \leq \frac{1}{\mathbb{P}(A)} \sum_{n\geq 1} |x_n| \, \mathbb{P}(X = x_n)$ (monotonicity of probability)

$$= \frac{1}{\mathbb{P}(A)} E(|X|) < \infty$$

Hence, $\sum_{n\geq 1} |x_n| \, \mathbb{P}(X = x_n | A)$ is convergent.

<u>Theorem</u>: If X and Y are independent and X has an expectation, then

$$E(X | Y = y) = E(X).$$

for every y such that $\mathbb{P}(Y = y) > 0$.

<u>Proof</u>: $E(X | Y = y) = \sum_{n\geq 1} x_n \, \mathbb{P}(X = x_n | Y = y) = \sum_{n\geq 1} x_n \, \mathbb{P}(X = x_n) = E(X).$

Remark: $E(X \mid Y \in B) = E(X)$ if $B \subseteq$ the values Y takes & $P(Y \in B) > 0$.

Back to the example 1:

5W
5B
5R

- 10 balls are drawn with replacement
- $X \sim$ # black balls

  $Y \sim$ # white balls
- $A = \{$ the first drawn ball is red $\}$

Want to find $E(X \mid A)$: $X \in \{0, 1, 2, \dots, 9\}$. For $x \in \{0, 1, \dots, 9\}$

$$p_A(x) = P(X = x \mid A) = \frac{P(X = x, A)}{P(A)} = \frac{\sum_{y=0}^{10-(x+1)} P(X = x, Y = y, A)}{\left(\frac{1}{3}\right)}$$

$$= \frac{\displaystyle\sum_{y=0}^{10-x-1} \frac{9!}{x!\,y!\,(9-x-y)!} \left(\frac{1}{3}\right)^{10}}{\left(\frac{1}{3}\right)} = \sum_{y=0}^{10-x-1} \frac{9!}{x!\,y!\,(9-x-y)!} \left(\frac{1}{3}\right)^{9}$$

$$= \left(\frac{1}{3}\right)^{9} \frac{9!}{x!\,(9-x)!} \sum_{y=0}^{10-x-1} \frac{(9-x)\,!}{y!\,(9-x-y)!}$$

$$= \left(\frac{1}{3}\right)^{9} \frac{9!}{(9-x)!\,x!} \quad 2^{(9-x)}$$

Hence, $E(X|A) = \displaystyle\sum_{x=0}^{9} x\, p_A(x) = \sum_{x=0}^{9} x \frac{9!}{(9-x)!\,(x)!} \, 2^{(9-x)} \left(\frac{1}{3}\right)^{9}$

$$= \left(\frac{1}{3}\right)^9 \sum_{x=1}^{9} x \binom{9}{x} 2^{9-x} = \cdots\cdots = \square$$

HW.

Next, find $E(Y \mid X+Y=7)$

$E(X \mid X>0)$

$E(Y \mid X=i)$ for every i such that $P(X=i)>0$.

HW: $E(X \mid Y=i)$ for every i s.t. $P(Y=i)>0$.