Probability theory 1

1/12/21

$$\boxed{E(X) = \sum_{n \geq 1} P(X \geq n)} \qquad \times \qquad \left[E(X) = \sum_{n \geq 0} P(X > n) \right] \qquad \cancel{0\, P(X=0}$$

$X \in \{\cancel{0}, 1, 2, \cdots\}$

$$\underbrace{P(X \geq 0)}_{=1} + \sum_{i=1}^{\infty} P(X \geq i)$$

② $X \in \{3, 4, 5, \cdots\}$

$$P(X \geq i) = P(W_i \cap B_i^c \cap R_i^c) = 2\left[\left(\tfrac{3}{4}\right)^i - \left(\tfrac{1}{4}\right)^i\right] \quad \forall i \geq 3.$$

$$E(X) = \sum_{n=3}^{\infty} n\, P(X = n)$$

$$= 3P(X=3) + 4P(X=4) + \cdots$$

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$$+ P(X=4) + P(X=5) + \cdots$$

$$+ P(X=5) + \cdots$$

$$= 3P(X \geq 3) + P(X \geq 4) + P(X \geq 5) + \cdots$$

- Yesterday we saw the joint conditional distribution and joint conditional expectation.

- <u>Recall:</u> The joint conditional distribution of (X,Y) given A (such that $\mathbb{P}(A)>0$) is

$$p_A(x,y) := \mathbb{P}(X=x, Y=y \mid A) = \frac{\mathbb{P}(X=x, Y=y, A)}{\mathbb{P}(A)} \qquad x,y \in \mathbb{R}$$

and $$E(X|A) = \sum_n x_n \mathbb{P}(X=x_n \mid A)$$ where X takes the values $x_1, x_2, \ldots$

<u>Theorem</u>: Suppose that X and Y are jointly distributed random variables and X has an expectation. If $y_1, y_2, \ldots$ are the values that Y takes with positive probabilities, then

$$E(X) = \sum_{n} \left(E(X \mid Y = y_n)\right) P(Y = y_n)$$

$$\left[\, P(Y = y_n) > 0 \;\; \forall\, n \right)$$

<u>Pf</u>: $$\text{RHS} = \sum_{n \geq 1} \left(\sum_{m \geq 1} x_m \, P(X = x_m \mid Y = y_n) \right) P(Y = y_n)$$

$$= \sum_{n \geq 1} \sum_{m \geq 1} x_m \frac{P(X = x_m, Y = y_n)}{\cancel{P(Y = y_n)}} \cancel{P(X = y_n)}$$

$$\sum \sum |x_m| \, P(X = x_m, Y = y_n) \leq \sum |x_m| \, P(X = x_m)$$

$$= \sum_{m \geq 1} x_m \sum_{n \geq 1} P(X = x_m, Y = y_n) = \sum_{n \geq 1} x_m P(X = x_m) = E(X) = \text{LHS}.$$

Example: Two coins each with probability of Heads being p, are simulteneously tossed repeatedly, untill the first coin gives a tails. Let X be the number of Heads for the second coin. Calculate $E(X)$.

Ans: Let Y be the number of tosses needed to get the first tail for the first coin. Then $Y \sim \text{Geometric}(1-p)$, i.e.,

$$\mathbb{P}(Y=k) = p^{k-1}(1-p) \quad \text{for} \quad k \in \{1,2,3,\dots\}.$$

Now, $$\mathbb{P}(X=0) = \sum_{k\geq 1} \mathbb{P}(X=0, Y=k) = \sum_{k\geq 1} \underbrace{\mathbb{P}(X=0 \mid Y=k)}\, \mathbb{P}(Y=k)$$

$$\mathbb{P}(X=a) = \sum_{n} \mathbb{P}(X=a, Y=y_n)$$

$$=\sum_{k\geq 1}\binom{k}{0}p^0(1-p)^k p^{k-1}(1-p)=\sum_{k\geq 1}p^{k-1}(1-p)^{k+1}=\sum_{k\geq 1}\left(p(1-p)\right)^{k-1}(1-p)^2$$

$$=(1-p)^2\frac{1}{1-p+p^2}=\frac{1-2p+p^2}{1-p+p^2}.$$

Similarly, let $i\geq 1$, $P(X=i)=\sum_{k\geq i}P(X=i\mid Y=k)\,P(Y=k)=\sum_{k\geq i}\binom{k}{i}p^i(1-p)^{k-i}(1-p)p^{k-1}$

$$=\sum_{k\geq i}\binom{k}{i}p^{k+i-1}(1-p)^{k-i+1}$$

$$=\sum_{\underline{k-i}\geq 0}\binom{\overline{k-i}+i}{i}p^{\overline{k-i}+2i-1}(1-p)^{\overline{k-i}+1}$$

$$=\sum_{k\geq 0}\binom{k+i}{k}p^{k+2i-1}(1-p)^{k+1}$$

$Z\sim$Negative binomial(r,p)

$P(Z=j)$

$\binom{j+r-1}{j}(1-p)^r p^j$

$$= \frac{1}{(1-p(1-p))^{i+1}} \underbrace{\sum_{k \geq 0} \binom{k+i+1-1}{k} \left(p(1-p)\right)^k \left(1-p(1-p)\right)^{i+1}}_{=1} p^{2i-1} (1-p)$$

$$= \frac{p^{2i-1}(1-p)}{(1-p+p^2)^{i+1}}$$

$$\therefore \; P(X=i) = \frac{1-2p+p^2}{(1-p+p^2)} \qquad \text{if } i=0$$

$$= \frac{p^{2i-1}(1-p)}{(1-p+p^2)^{i+1}} \qquad \text{if } i \geq 1$$

Check! $\sum_{i \geq 0} P(X=i) = 1$. HW

$$E(X) = \sum_{j=0}^{\infty} j\, P(X=j) = \sum_{j=1}^{\infty} j \frac{p^{2j-1}(1-p)}{(1-p+p^2)^{j+1}} = \frac{(1-p)p^2}{p(1-p+p^2)^2} \sum_{j=1}^{\infty} j \left(\frac{p^2}{1-p+p^2}\right)^{j-1}$$

$$= \frac{(1-p)p^2}{p(1-p+p^2)^2} \left(\frac{1}{\left(1 - \frac{p^2}{1-p+p^2}\right)^2} \right)$$

$$= \frac{\cancel{(1-p)}\, p^{\cancel{2}}}{\cancel{p}(1-p+p^2)^2} \frac{(1-p+p^2)^2}{(1-p)^{\cancel{2}}}$$

$$= \frac{p}{1-p} \quad \square$$

$1 + x + x^2 + \cdots = \frac{1}{1-x}$

diff

$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$

$x = \frac{p^2}{1-p+p^2}$