# Probability theory 1

3/12/21

Recall: Joint conditional distribution:

$$p_A(x,y) = \mathbb{P}(X=x, Y=y \mid A) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(A)}.$$

Joint conditional expectation:

$$E(X \mid A) = \sum_n x_n \, \mathbb{P}(X=x_n \mid A)$$

Theorem: $E(X) = \sum_n x_n P(X = x_n) = \sum_m E(X | Y = y_m) P(Y = y_m)$.

· Tossed two coins untill the first coin gives a Tail. $X \sim$ # Heads for the second coin.

We observe $P(X = i | Y = n) = P(\text{Binomial}(n, p) = i)$

$E(X | Y = n) = E(\text{Binomial}(n, p)) = np$.

Applying the theorem above,

$$E(X) = \sum_{n \geq 1} E(X | Y = n) P(Y = n) = \sum_{n \geq 1} np\, p^{n-1}(1-p) = \cdots = \frac{p}{1-p} \quad \square$$

Example: Suppose that the number of people who enter a post office on a given day is a Poisson $(\lambda)$ random variable. Show that if each person who enters the post office is a male or female with probability $p$ and $1-p$ respectively, then the number of males and females entering the post office are independent Poisson $(\lambda p)$ and Poisson $(\lambda(1-p))$ respectively.

Ans: Let $X$ = # males entering the post office

$Y$ = # females entering the post office.

$X, Y \in \{0, 1, \cdots\}$

Need to show that $X \sim \mathrm{Poi}(\lambda p)$, $Y \sim \mathrm{Poi}(\lambda(1-p))$ and

$$\underline{P}(X=i, Y=j) = \underline{P}(X=i)\,\underline{P}(Y=j).$$

$\underline{P}(A) = \underline{P}(A \cap B) + \underline{P}(A \cap B^c)$

Now, fix $i, j \in \mathbb{N} \cup \{0\}$.

$$\underline{P}(X=i, Y=j) = \underline{P}(X=i, Y=j \mid X+Y = i+j)\,\underline{P}(X+Y = i+j) + \underline{P}(X=i, Y=j \mid X+Y \neq i+j)\,\underline{P}(X+Y \neq i+j)$$

$$= \underline{P}(X=i, Y=j \mid X+Y = i+j)\,\underline{P}(X+Y = i+j) \quad \left(\because \underline{P}(X=i, Y=j, X+Y \neq i+j) = 0\right)$$

$$= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} = \frac{p^i (1-p)^j e^{-\lambda} \lambda^{i+j}}{i!\,j!}$$

$$= e^{-\lambda} \frac{(\lambda p)^i ((1-p)\lambda)^j}{i!\, j!}$$

$$\cdot\ \mathbb{P}(X=i) = \sum_{j=0}^{\infty} e^{-\lambda} \frac{(\lambda p)^i ((1-p)\lambda)^j}{i!\, j!} = e^{-\lambda} \frac{(\lambda p)^i}{i!} \sum_{j=0}^{\infty} \frac{((1-p)\lambda)^j}{j!}$$

$$= e^{-\lambda} \frac{(\lambda p)^i}{i!} e^{(1-p)\lambda} = e^{-p\lambda} \frac{(\lambda p)^i}{i!}$$ Hence $X \sim \mathrm{Poi}(\lambda p)$.

$$\cdot\ \mathbb{P}(Y=j) = \cdots\cdots = e^{-(1-p)\lambda} \frac{(\lambda(1-p))^j}{j!}$$ Hence $Y \sim \mathrm{Poi}(\lambda(1-p))$

$$\mathbb{P}(X=i, Y=j) = \frac{e^{-\lambda} (\lambda p)^i ((1-p)\lambda)^j}{i!\, j!} = e^{-p\lambda-(1-p)\lambda} \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^j}{j!} = \mathbb{P}(X=i)\, \mathbb{P}(Y=j) \quad \Box$$

HW: Assume that the # road accidents in kolkata follows Poisson($\lambda$).
Each accident is reported with probability $p$, independently of each other. Let $X$ and $Y$ be the # accidents that are reported and not reported respectively.
Find the joint distribution of $X$ and $Y$.

Theorem: Let $X$ and $Y$ be random variables having expectation, and $A$ an event with positive probability. Then,

$$E(X+Y \mid A) = E(X \mid A) + E(Y \mid A)$$

Proof: $X \in \{x_1, x_2, \ldots\}$, $Y \in \{y_1, y_2, \ldots\}$. Thus $X+Y \in \{x_i + z_j \mid i, j \in \mathbb{N}\}$.

$$E(X+Y|A) = \sum_{i,j} z_{ij}\, P(X=x_i, Y=y_j | A) \quad \left( \sum_{\substack{i,j: \\ x_i+y_j=z_{ij}}} \underbrace{z_{ij}\, P(Z=z_{ij}|A)} \right) \qquad x_i + y_j =: z_{ij}$$

$$= \sum_i \sum_j (x_i + y_j)\, P(X=x_i, Y=y_j | A)$$

$$x_0 + y_0 = 2$$
$$x_1 + y_2 = 2$$

$$= \sum_i \sum_j x_i\, P(X=x_i, Y=y_j | A) + \sum_i \sum_j y_j\, P(X=x_i, Y=y_j | A)$$

$$= \sum_i x_i \sum_j \frac{P(X=x_i, Y=y_j, A)}{P(A)} + \sum_j y_j \sum_i \frac{P(X=x_i, Y=y_j, A)}{P(A)}$$

$$= \sum_i \frac{x_i}{P(A)}\, P(X=x_i, A) + \sum_j \frac{y_j}{P(A)}\, P(Y=y_j, A)$$

$$= \sum_i x_i\, P(X=x_i|A) + \sum_j y_j\, P(Y=y_j|A) = E(X|A) + E(Y|A).$$

Theorem: If $X$ is a random variable with expectation and $A$ an event with $\mathbb{P}(A) > 0$, then

$$E(X \mid A) = \frac{1}{\mathbb{P}(A)} E(X \mathbb{1}_A)$$

Proof:

$$X\mathbb{1}_A(\omega) = x_n \quad \text{if } \left(X^{(\omega)} = x_n,\ \mathbb{1}_A^{(\omega)} = 1 \Rightarrow X^{(\omega)} = x_n,\ \omega \in A \Rightarrow \{\omega \mid X = x_n\} \cap A\right)$$

$$= 0 \quad \text{if } \left(X = x_n,\ \mathbb{1}_A = 0 \Rightarrow X(\omega) = x_n,\ \omega \notin A \Rightarrow \{\omega \mid X = x_n\} \cap A^c\right)$$

$X\mathbb{1}_A : \Omega \to \mathbb{R}$

$$E(X\mathbb{1}_A) = \sum_{n=1}^{\infty} x_n \mathbb{P}(X\mathbb{1}_A = x_n) + 0 \times \mathbb{P}(X\mathbb{1}_A = 0)$$

$$= \sum_{n=1}^{\infty} x_n \mathbb{P}(\{\omega : X(\omega) = x_n,\ \omega \in A\})$$

$$\mathbb{P}(\{\omega_1, \omega_2, \cdots\})$$

$$E(Y) = \sum_m y_m \mathbb{P}(Y = y_m)$$

$$= \sum_{n=1}^{\infty} x_n \, P(X = x_n, A) = \sum_{n=1}^{\infty} x_n \, P(X = x_n \mid A) \, P(A)$$

$$= P(A) \; E(X \mid A) \qquad \square$$

<u>Exercise:</u> Suppose that for random variables $X$ and $Y$, there exists functions $f$ and $g$ from $\mathbb{R}$ to itself and $c \in \mathbb{R}$ such that

$$P(X = x, Y = y) = c\, f(x)\, g(y) \qquad \text{for every } x, y \in \mathbb{R}.$$

Then $X$ and $Y$ are independent.

<u>Ans:</u> <u>Hint:</u> Show that $P(X = x, Y = y) = P(X = x)\, P(Y = y)$ using

$$1 = \sum_x \sum_y P(X=x, Y=y) = c \sum_x \sum_y f(x) g(y) \Rightarrow c = \frac{1}{\sum_x \sum_y f(x) g(y)} \qquad \square$$

Exercise: (HW): If $X$ is a random variable and $\alpha < \beta$, such that

$P(\alpha \leq X \leq \beta) > 0$, then

$$\alpha \leq E(X \mid \alpha \leq X \leq \beta) \leq \beta.$$

Theorem: Suppose $X$ and $Y$ are random variables, taking values $x_1, x_2, \dots$ and $y_1, y_2, \dots$ respectively with positive probability. Then $\forall\, i, j$

$$P(X = x_i \mid Y = y_j) = \frac{P(Y = y_j \mid X = x_i)\, P(X = x_i)}{\sum_{k \geq 1} P(Y = y_j \mid X = x_k)\, P(X = x_k)} \qquad \square$$

Pf: HW

Exercise: Suppose $X$ forllows Geometric$(p)$. For a fixed $k \in \mathbb{N} \cup \{0\}$, find the conditional distribution of $X-k$ given $X>k$. (HW).

$\mathbb{P}(A \cap B \cap C \mid D)$
$= \mathbb{P}(A \mid B \cap C \cap D)$

## Several random variables:

For random variables $X_1, X_2, \ldots, X_n$ defined on the same sample space, their joint probability mass function is a function

$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$

$p : \mathbb{R}^n \to [0,1]$, defined by

$$p(x_1, x_2, \cdots, x_n) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

The marginal PMF of $X_i$ is

$$\mathbb{P}(X_i = x) = \sum_{\substack{(x_1, \dots, x_{i-1}, \\ x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}}} p(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is such that $f(X_1, \dots, X_n)$ has an expectation then

$$E\left(f(X_1, \dots, X_n)\right) = \sum_{\substack{x \in \mathbb{R}^n \mid p(x) > 0 \\ = (x_1, \dots, x_n)}} f(x)\, p(x)$$

<u>Definition:</u> For $n \geq 2$, $X_1, \dots, X_n$ are independent if

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n)$$

Random variable $X_1, \dots, X_n$ are pairwise independent if $\forall\ 1 \le i < j \le n$,

$X_i$ and $X_j$ are independent.

· **Definion:** ( $X = Y$ a.s. $\quad \mathbb{P}(X=Y) = 1 \Rightarrow \mathbb{P}(\omega : X(\omega) = Y(\omega)) = 1$ )

· **Definion:** Two random variables $X$ and $Y$ taking the same values $\{x_1, x_2, \dots\}$ are equal in distribution if $\mathbb{P}(X = x_n) = \mathbb{P}(Y = x_n)$ $\forall\ n \in \mathbb{N}$. (write $X \overset{d}{=} Y$)

**Definition:** $\{X_n\}_{n \ge 1}$ is called i.i.d. sequence of random variables if

$X_m \overset{d}{=} X_n$ $\forall\ m, n \in \mathbb{N}$ and $X_1, X_2, \dots, X_m$ are independent for any $m \ge 2$.

Theorem: Suppose $\{X_n\}_{n\geq 1}$ is an i.i.d sequence of random variables defined on $\Omega$ such that $E(e^{X_1+\cdots+X_n})$, $E(e^{X_1})$ exists. Then

$$E\left(e^{X_1+\cdots+X_n}\right) = \left(E\left(e^{X_1}\right)\right)^n$$

Pf: $$E\left(e^{X_1+X_2}\right) = E\left(e^{X_1} e^{X_2}\right) = E\left(e^{X_1}\right) E\left(e^{X_2}\right) \quad (\because X_1 \,\&\, X_2 \text{ are independent})$$

$$E\left(e^{X_1}\right) = \sum_n e^{x_n} P(X_1 = x_n) = \sum_n e^{x_n} P(X_2 = x_n) \quad (\because X_1 \stackrel{d}{=} X_2)$$

$$= E\left(e^{X_2}\right)$$

Combining ⊛ & ✡ we get $E\left(e^{X_1+X_2}\right) = \left(E\left(e^{X_1}\right)\right)^2.$

Q:

$$E\left(e^{X_1+X_2+X_3}\right) = E\left(e^{X_1+X_2} e^{X_3}\right) = E\left(e^{X_1+X_2}\right) E\left(e^{X_3}\right)$$