Probability theory 1

7/12/21

$k \in \{1, \dots, n\}$

$X \sim Bin(n, p) \Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\frac{X}{2} \sim \underbrace{Bin\left(\frac{n}{2}, p\right)} \Rightarrow P\left(\frac{X}{2} = \frac{k}{2}\right) = \binom{n}{k} p^k (1-p)^{n-k}$

ℓ

$\ell \in \{1, \frac{1}{2}, \dots, \frac{n}{2}\}$

$X+Y \sim Bin(m+n, p)$

$\frac{X}{n}$

$Bin\left(\frac{n}{2}, p\right)$

$P(X=k) = \underbrace{\binom{\frac{n}{2}}{k} p^k (1-p)^{\frac{n}{2}-k}}$

<u>Recall:</u> Random variables $\{X_n\}_{n\geq 1}$ is independent and identically distributed (i.i.d.) if for any $m\geq 2$, $X_1, X_2, \dots, X_m$ are independent and $X_i \overset{d}{=} X_j$ for all $i, j \in \mathbb{N}$.

In the above situation $E(e^{X_1+\dots+X_n}) = (E(e^{X_1}))^n$, whenever $E(e^{X_1}) < \infty$.

<u>Theorem:</u> $E(X) = \sum_{n=1}^{\infty} E(X \mid Y = y_n)\, P(Y = y_n)$.

Exercise: Suppose that $N, X_1, X_2, \cdots$ are i.i.d Poisson(λ). Find

$$E\left(X_1 + \cdots + X_N\right).$$

Ans:

$$E\left(X_1 + \cdots + X_N\right) = \sum_{n=1}^{\infty} E\left(X_1 + \cdots + X_N \mid N = n\right) P(N=n)$$

$$= \sum_{n=1}^{\infty} E\left(X_1 + \cdots + X_n \mid N = n\right) P(N=n)$$

$$= \sum_{n=1}^{\infty} \left(E(X_1) + \cdots + E(X_n)\right) P(N=n)$$

$$= \sum_{n=1}^{\infty} n P(N=n) \lambda = \lambda \sum_{n=1}^{\infty} n P(N=n)$$

$$Y = X_1 + \cdots + X_N$$

$$Y(\omega) = X_1(\omega) + \cdots + X_{N(\omega)}(\omega)$$

$$\sum_{k=0}^{\infty} k P(X_1 + \cdots + X_N = k)$$

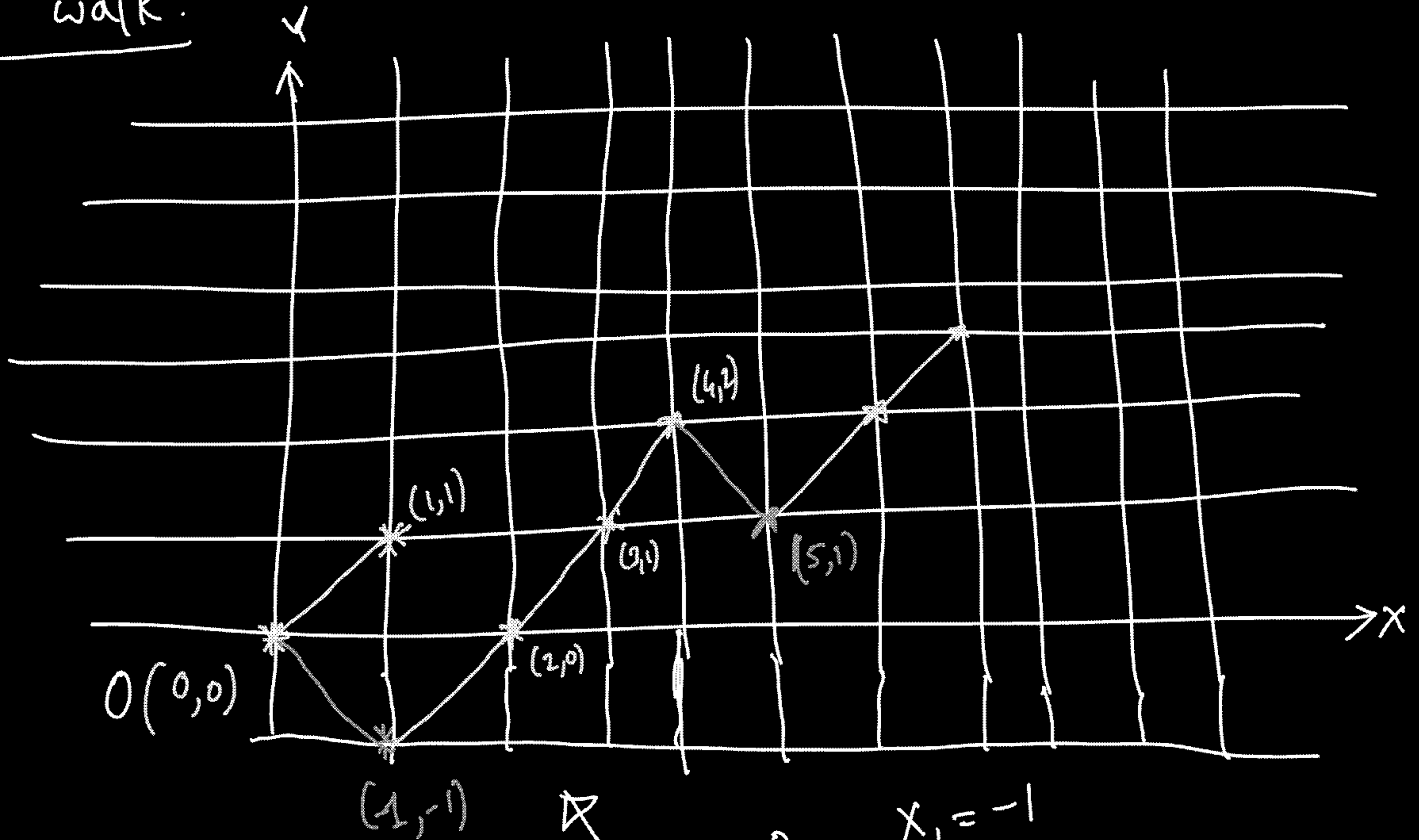
If $N(\omega) = k$

$$X_N(\omega) := X_{N(\omega)}(\omega) = X_k(\omega)$$

$$= \lambda E(N) = \lambda^2$$

Random walk:

$\mathbb{Z}^2$



$S_2 = 0$

$X_1 = -1$
$x_2 = 1$

$X_n = S_n - S_{n-1}$

$\{X_i\}_{i \geq 1}$ i.i.d.

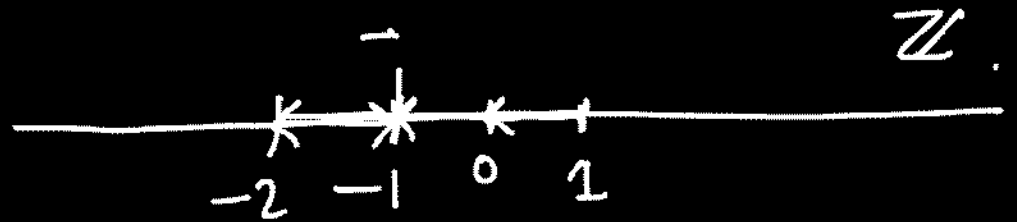
$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$\{S_n\}_{n \geq 1}$

$$S_{n+1} = S_n + X_{n+1}$$

$$S_n = \sum_{i=1}^{n} X_i$$

Simple symmetric random walk:



Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d Rademacher random variables, i.e.,

$$X_n = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases} \quad \text{for all } n \geq 1$$

$E(X_n) = 0$ $\forall n \geq 1$

$Var(X_n) = 1$

For each $n \in \mathbb{N}$ define the sums

$$S_n = X_1 + \cdots + X_n, \forall n \geq 1 \quad \text{and} \quad S_0 = 0.$$

Then $E(S_n) = 0$ and $Var(S_n) = n$.

The sequence of random variables $\{S_n\}_{n\geq 1}$ is called a simple symmetric random walk on the integers.

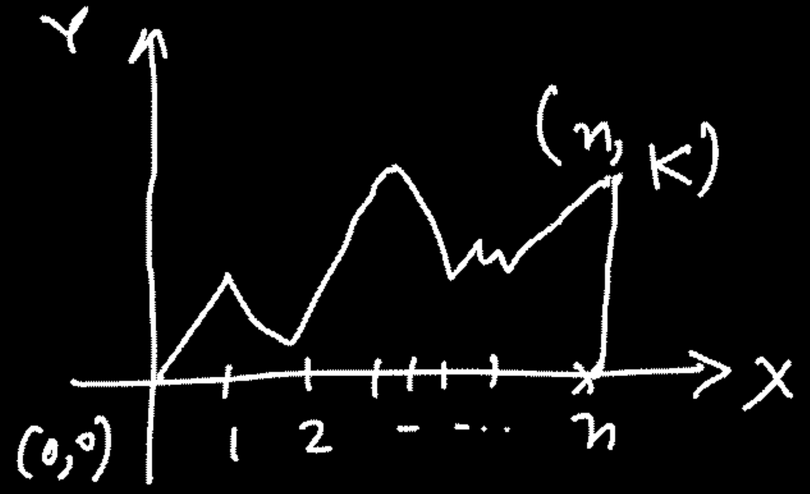
Observe that S_n is even/odd iff n is even/odd respectively.

If n is even then $S_n \in \{-n, -n+2, \ldots, -2, 0, 2, \ldots, n-2, n\} = [-n, n] \cap 2\mathbb{Z}$.

If n is odd then $S_n \in \{-n, \ldots, -3, -1, 1, 3, \ldots, n\} = [-n, n] \cap (2\mathbb{Z}+1)$

$\mathbb{P}(S_n = k)$? where k is such that k may be achived by S_n.

Let there be a many $+1$ values and b many -1 values in $\{S_n = k\}$.



Clearly,

$$a + b = n \quad \text{and}$$

$$a - b = k$$

$$\overline{2a = n + k}$$

$$a = \frac{n+k}{2}$$

$$+1 - 1 - 1 + 1$$

$$= \underbrace{1+1}\underbrace{-1-1}$$

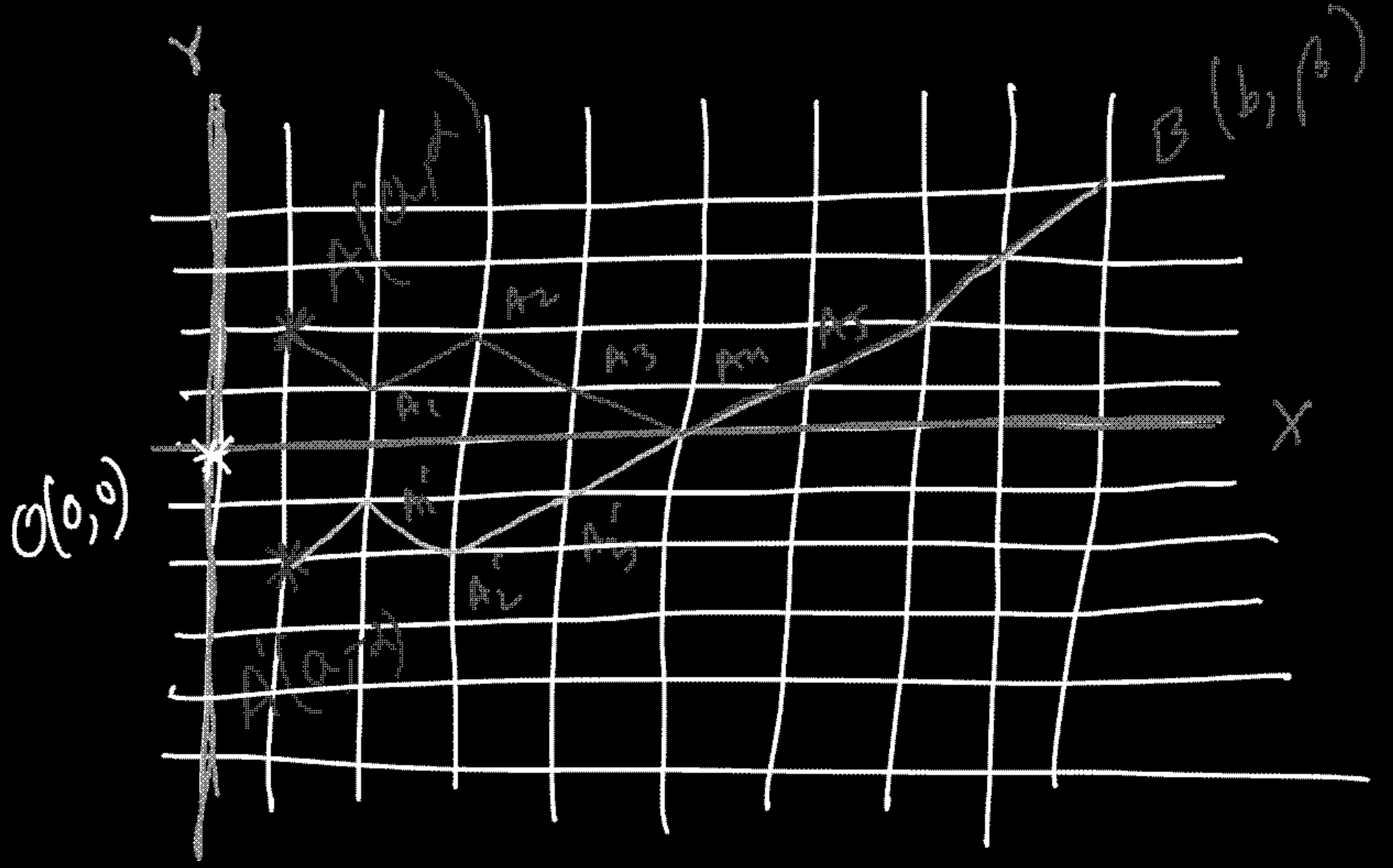
Therefore,

$$P(S_n = k) = \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^{\frac{n+k}{2}} \left(\frac{1}{2}\right)^{n - \frac{n+k}{2}} = \binom{n}{\frac{n+k}{2}} \left(\frac{1}{2}\right)^{n}$$

Theorem (Reflection principle):

Let $A=(a,\alpha)$ and $B=(b,\beta)$ be points in $\mathbb{Z}^2$ satisfying $b>a\geq 0$, $\boxed{\alpha\geq 1 \text{ and } \beta\geq 1.}$ The number of paths from A to B that touch the horizontal axis equals the number of all paths from $A'=(a,-\alpha)$ to B.

Proof:



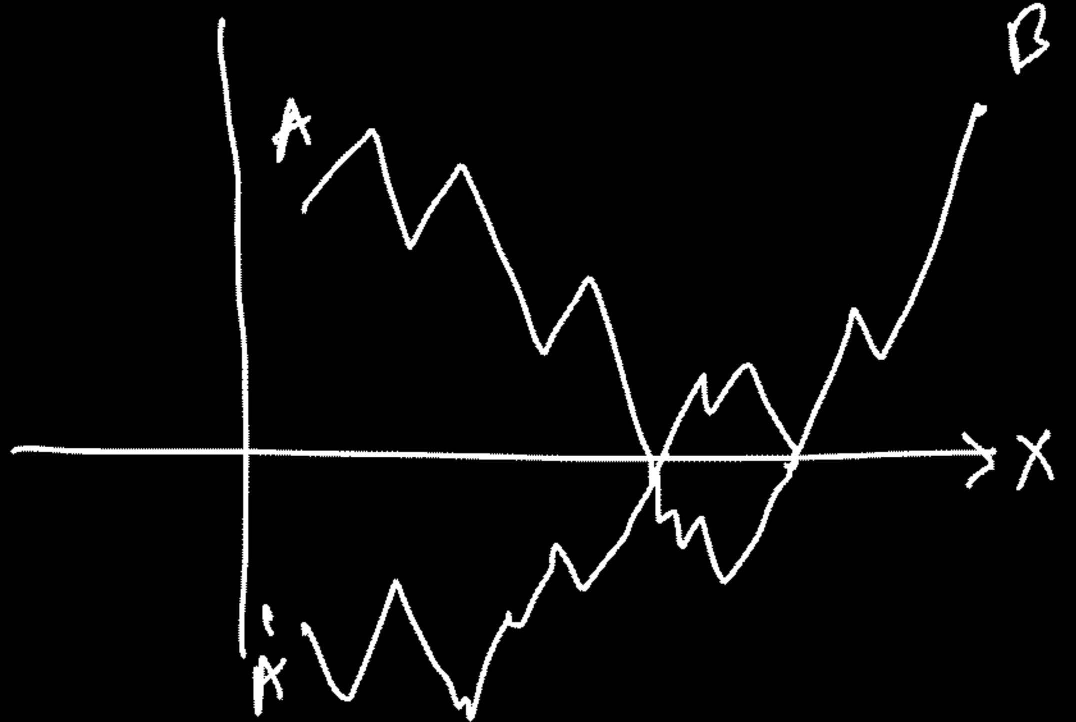
$A_i'=(x,-y)$ iff $A_i=(x,y)$

$(A, A_1, A_2, \dots, B)$

$\Updownarrow$

$(A', A_1', A_2', A_3', A_4, A_5, \dots, B)$

Consider a path from A to B that touches the horizontal axis. Let A_4 be the first point at which it touches. Reflect the portion of the path from A to A_4 along the x-axis. That gives a path from A' to B. It is elementary to check that the function, thus defined is a bijection.



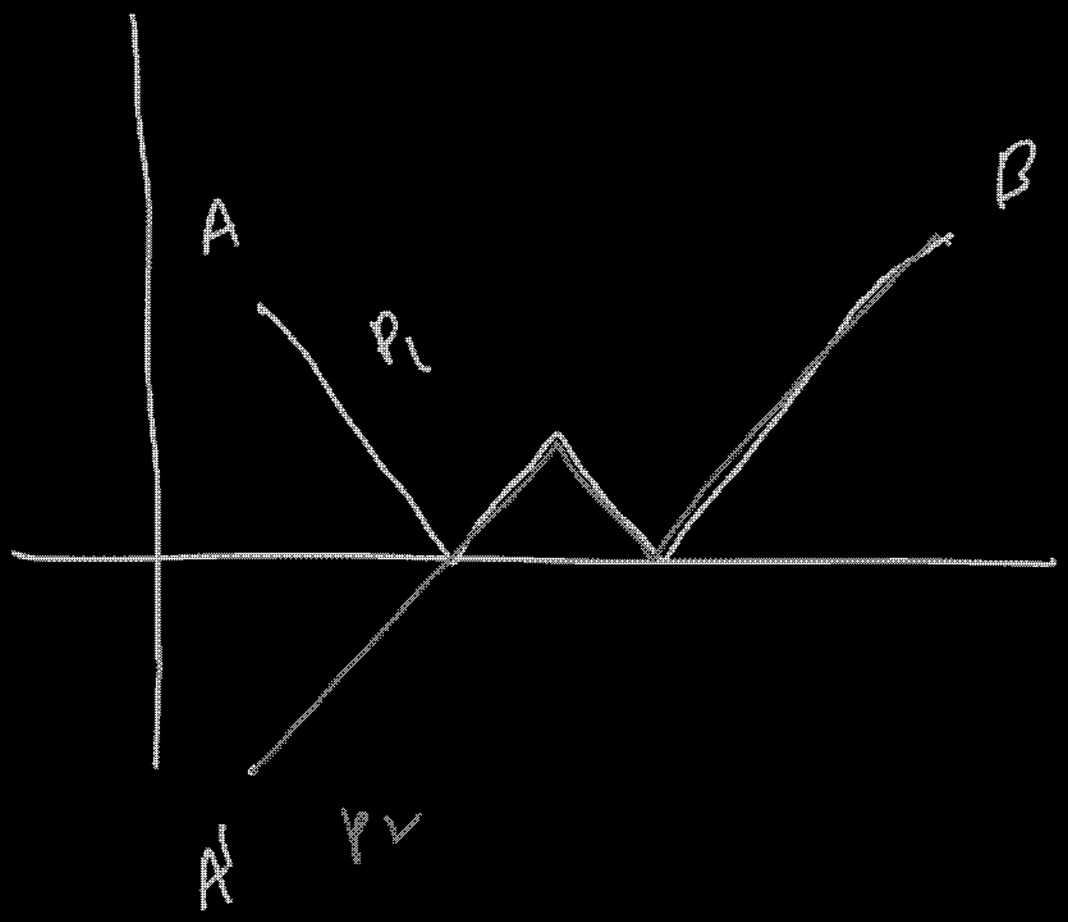
2

A

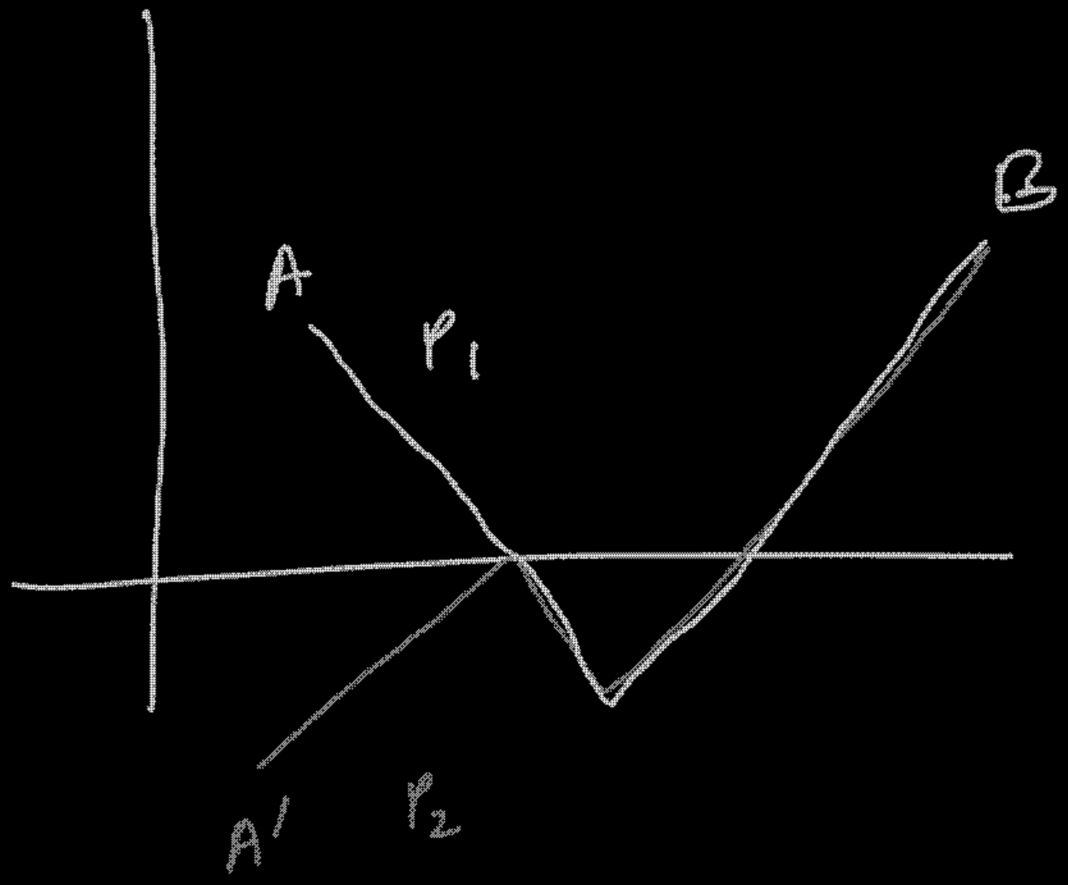
$f(p_1) = p_1'$

$f(p_2) = p_2'$

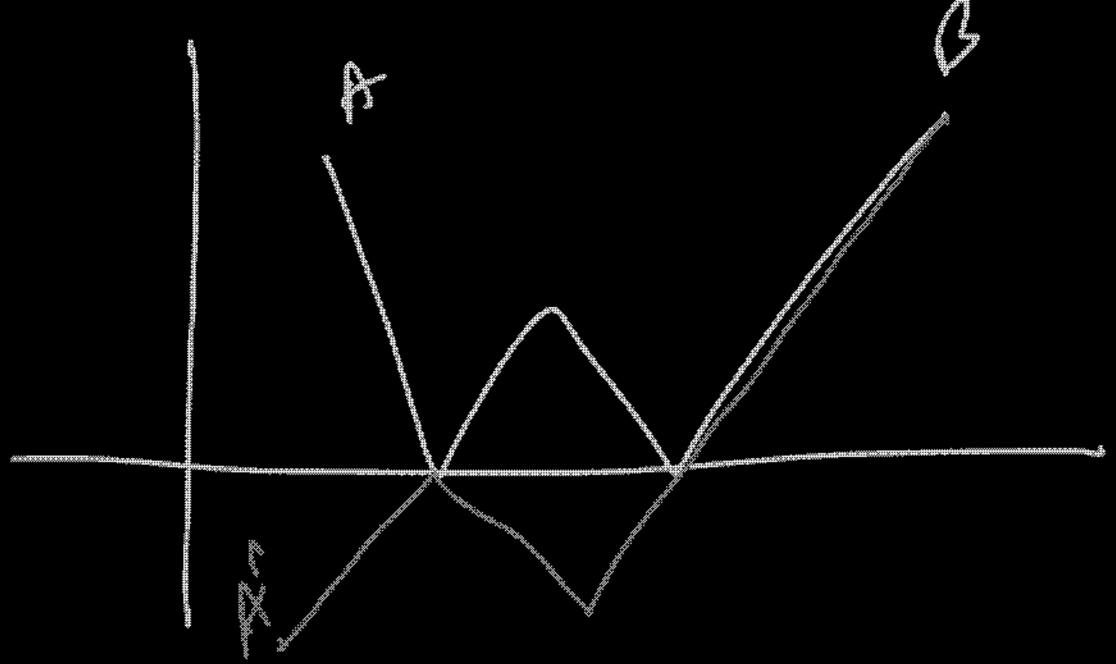
1st point reflection

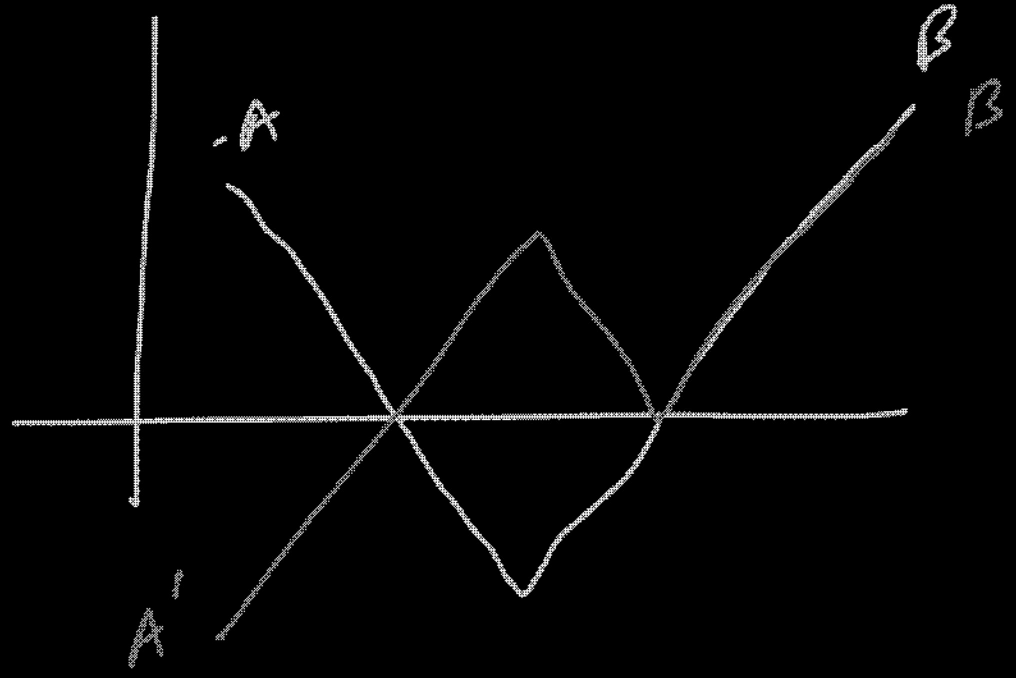


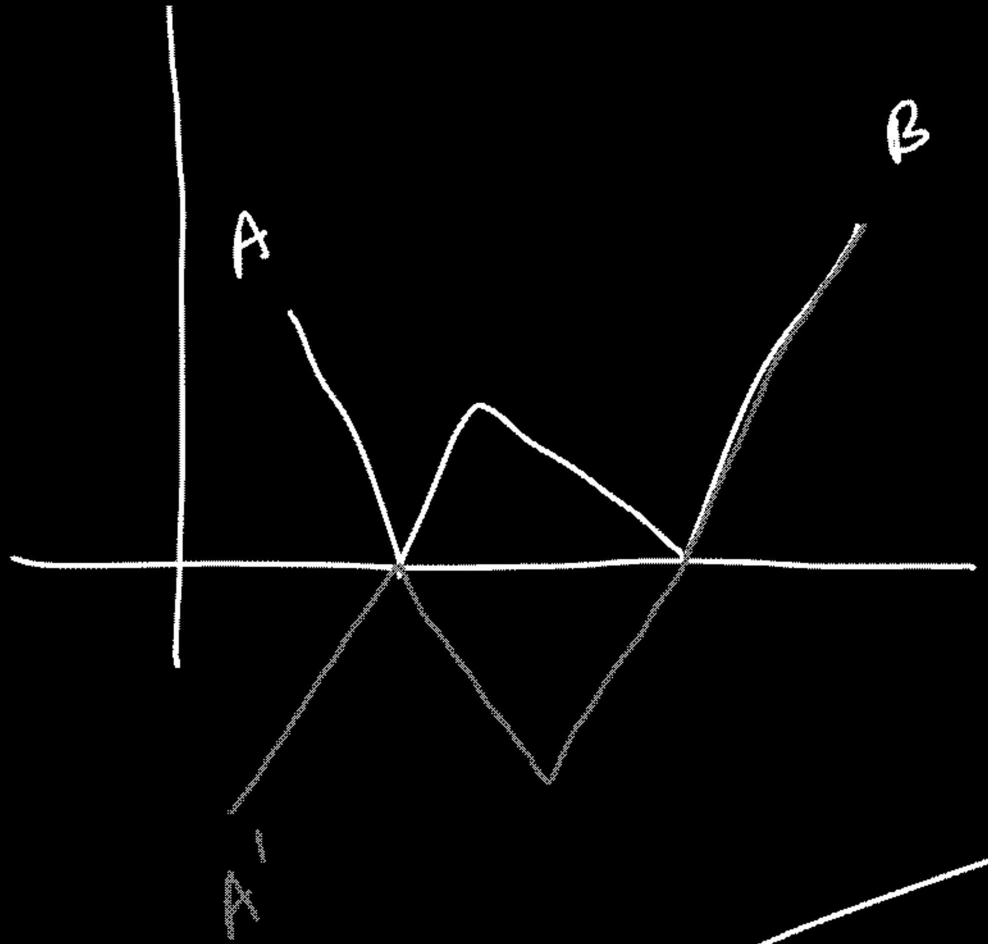
$f(p_1) = p_2$

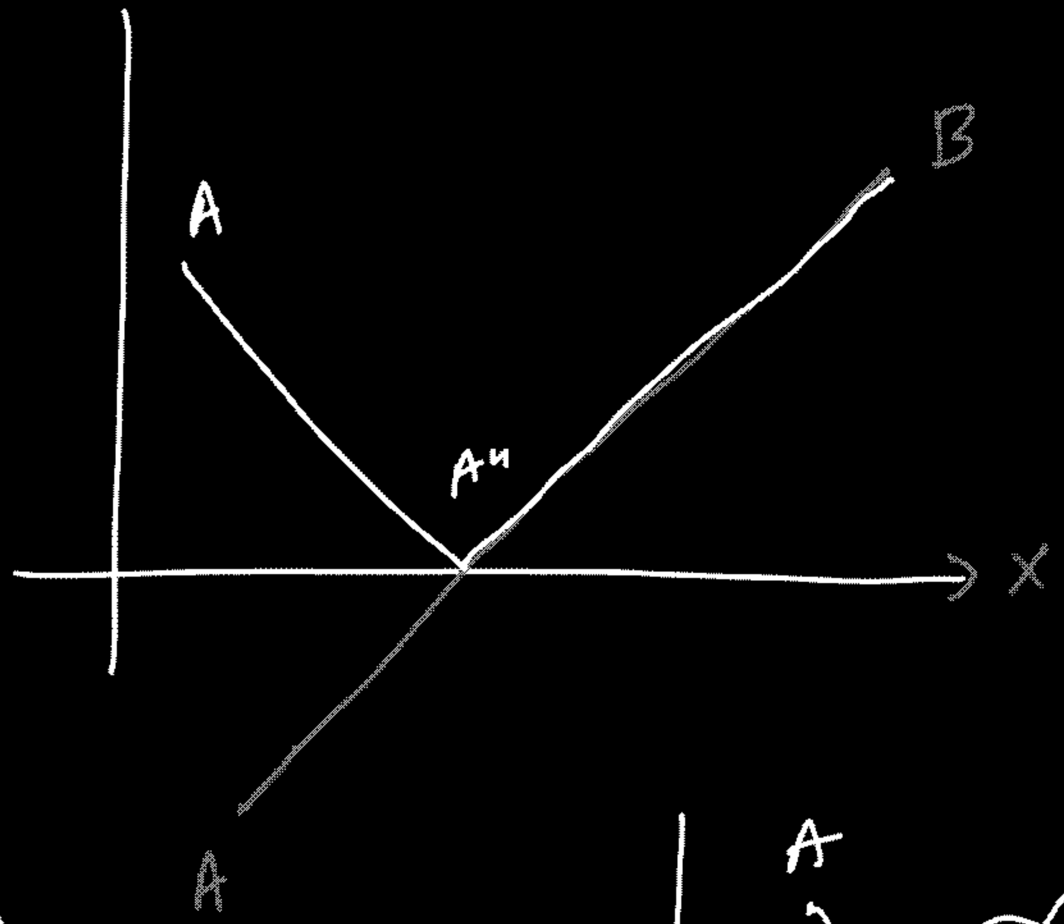


last point reflection









$\#\{\text{all paths } A \text{ to } B\}$?

$\geq \#\{\text{all paths from } A' \text{ to } B\}$