Probability theory 1

10/12/21

Recall: $\{X_n\}_{n\geq 1}$ i.i.d. $X_n = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$

The random walk is $\{S_n\}_{n\geq 1}$ where $S_n = \sum_{i=1}^{n} X_i$.

Reflection principle: $A = (a, \alpha)$ and $B = (b, \beta) \in \mathbb{Z}^2$. ($b > a \geq 0$, $\alpha \geq 1$, $\beta \geq 1$)

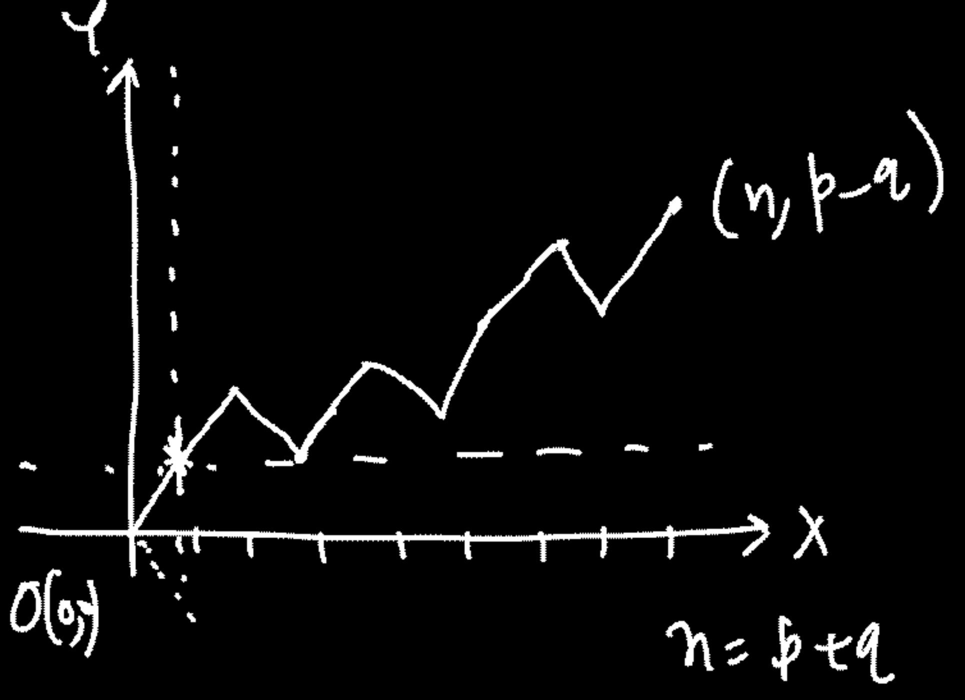
The # paths from A to B that touch the horizontal axis

equal to the # paths from $A' = (a, -\alpha)$ to B.

Theorem (Ballot theorem):

In an election where candidate A receives p votes and candidate B has received q votes with $p>q$, the probability that A will be ahed of B throughout the count is $\frac{p-q}{p+q}$.

Proof:



paths from $(0,0)$ to $(n, p-q)$

$$= \binom{p+q}{p} = \binom{n}{p}$$

paths from $(1,1)$ to $(n, p-q)$

$= \#$ paths from $(0,0)$ to $(n-1, p-q-1)$

$= \binom{n-1}{p-1}$ $\quad x \mapsto x-1,\ y \mapsto y-1$

Set $n = p+q$. The required probability is

$$= \frac{1}{\binom{n}{p}} \left\{ \text{the number of paths from } (0,0) \text{ to } (n, p-q) \text{ that do not touch the horizontal axis after } (0,0) \right\}$$

$$= \frac{1}{\binom{n}{p}} \left\{ \text{the number of paths from } (1,1) \text{ to } (n, p-q) \text{ that do not touch the horizontal axis} \right\}$$

$$= \frac{1}{\binom{n}{p}} \left[\text{the \# paths from } (1,1) \text{ to } (n, p-q) - \text{the number of paths from } (1,1) \text{ to } (n, p-q) \text{ that touch the horizontal axis} \right\}$$

$p+q+1$ — $p+1$ ups, q downs.

Reflection principle

$$= \frac{1}{\binom{n}{p}} \left[\binom{n-1}{p-1} - \text{the number of paths from } (1,-1) \text{ to } (n, p-q) \right]$$

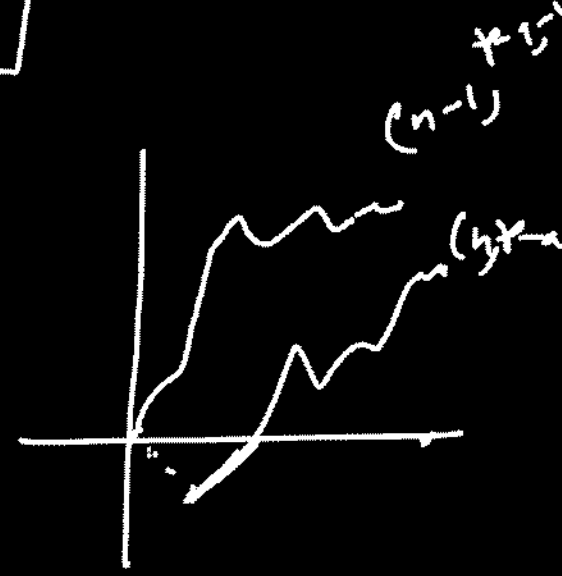
p ups, $q-1$ downs

$$= \frac{1}{\binom{n}{p}} \left[\binom{n-1}{p-1} - \text{the number of paths from } (0,0) \text{ to } (\underbrace{n-1}, p-q+1) \right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (n-1, p-(q-1))$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \quad \uparrow$$



$$= \frac{1}{\binom{n}{p}} \left[\binom{n-1}{p-1} - \binom{n-1}{p} \right] \qquad \underline{\underline{(\text{think!})}}$$

$$= \frac{p!\,(n-p)!}{n!} \left(\frac{(n-1)!}{(p-1)!\,(n-p)!} - \frac{(n-1)!}{p!\,(n-p-1)!} \right) = \frac{\cancel{(n-1)!}\ \cancel{p!}\ \cancel{(n-p)!}}{\cancel{n!}\ \cancel{(p-1)!}\ \cancel{(n-p-1)!}} \left(\frac{1}{n-p} - \frac{1}{p} \right)$$

$$= \frac{p(n-p)}{n} \; \frac{2p-n}{p(n-p)} = \frac{2p-n}{n} = \frac{2p-n}{p+q} = \frac{p-q}{p+q}. \quad \square$$

Example: A gambler enters a casino with C rupees, where $C \in \mathbb{N}$. At each game, the gambler bets ₹ 1 and gets back either ₹ 2 or nothing, with probability $\frac{1}{2}$. Given that after n games, he has won ₹ K rupees, find the conditional probability that he was out of money at some point in the middle.

Ans: The required probability equals $P\left(\min\limits_{0 \le i \le n} S_i \le 0 \;\middle|\; S_n = K+C\right)$ — (1)

$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases} \qquad S_m = C + \sum_{i=1}^{m} X_i \quad \text{for } 1 \le m \le n.$$

$$= \frac{\# A}{\# \Omega}$$ where

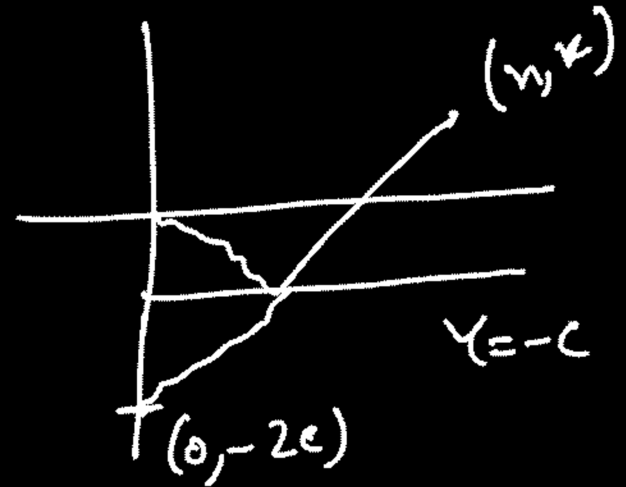
$A = \min S_i \le 0 \,\|\, S_n = K+C$

$\Omega =$ all paths from $(0,C)$ to $(n, K+C)$.

$$① = \frac{\#\text{ paths from } (0,c) \text{ to } (n,k+c) \text{ which touch horizontal axis } (Y=0)}{\#\text{ paths from } (0,c) \text{ to } (n,k+c)}$$

Reflection principle.

$$= \frac{\#\text{ paths from } (0,-c) \text{ to } (n,k+c)}{\#\text{ paths from } (0,0) \text{ to } (n,k)}$$

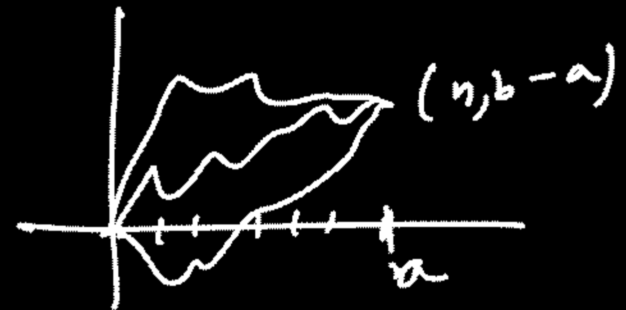


$$= \frac{\#\text{ paths from } (0,0) \text{ to } (n,k+2c)}{\binom{n}{\frac{n+k}{2}}}$$

(translate the point $(0,-c)$ to $(0,0)$).

\# paths from $(0,0)$ to (a,b)

$a>0, b>0$



$$p+q=a$$
$$p-q=b$$
$$2p=a+b$$
$$p=\frac{a+b}{2}$$

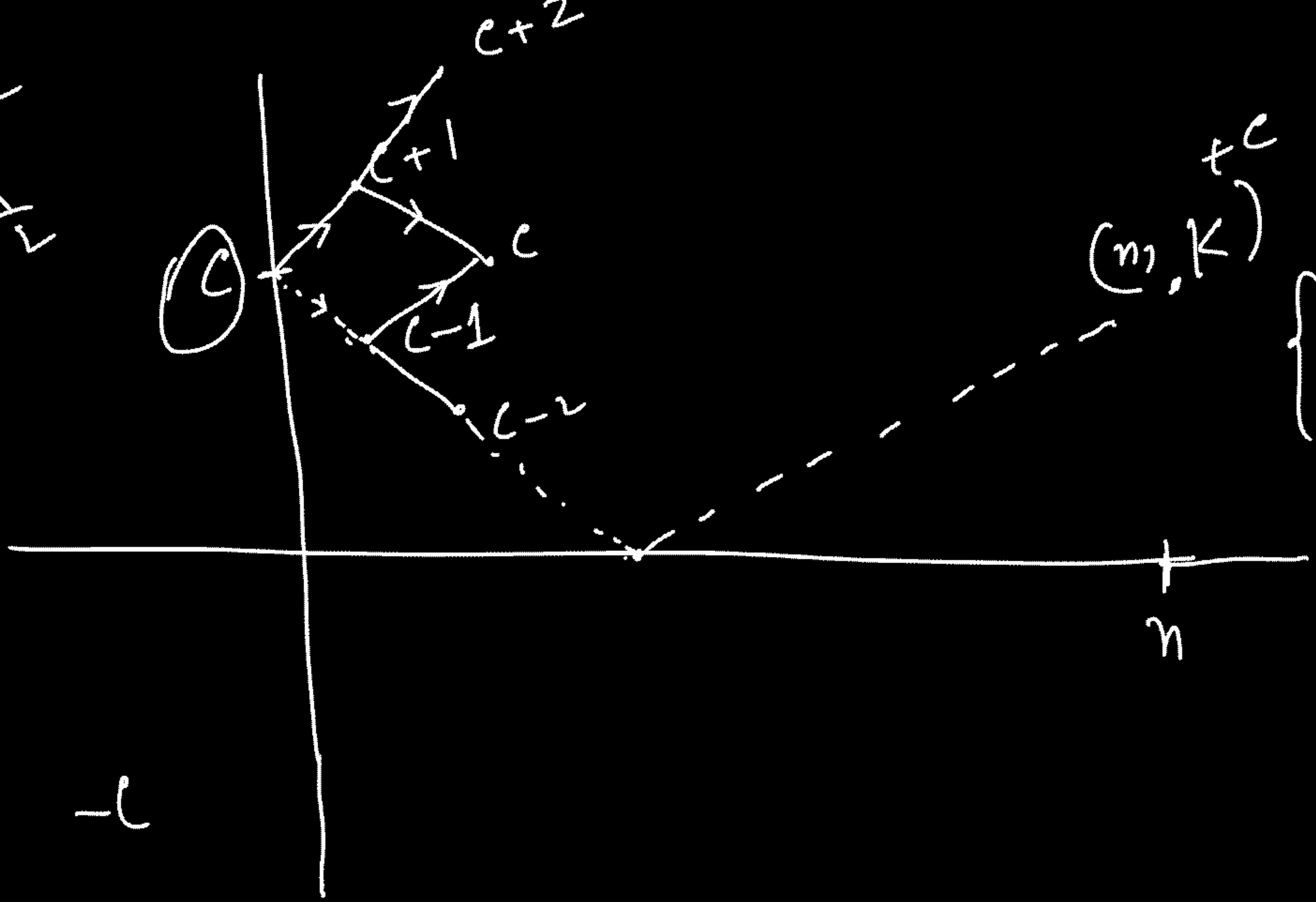
$$\binom{a}{\frac{a+b}{2}}$$

$$= \binom{n}{\frac{n+k+2c}{2}} \Big/ \binom{n}{\frac{n+k}{2}} = \frac{\binom{n}{c+\frac{n+k}{2}}}{\binom{n}{\frac{n+k}{2}}} \qquad \square$$

$X_i = 1$ w.p. $\frac{1}{2}$

$= -1$ w.p. $\frac{1}{2}$

$S_m = \sum_{i=1}^{m} X_i + c$



$$\left\{ \min_{0 \le i \le n} S_i \le 0 \;\middle|\; S_n = k \right\}$$

<u>Example:</u> Find the probability that a random walk starting from 0 hits 0 for the first time after $2n$ steps.

<u>Ans:</u> The required probability equals

$$\mathbb{P}\left(S_1 \neq 0, S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0\right)$$

$$= \mathbb{P}\left(S_i \geq 1 \ \forall\, 1 \leq i \leq 2n-1, S_{2n} = 0\right)$$

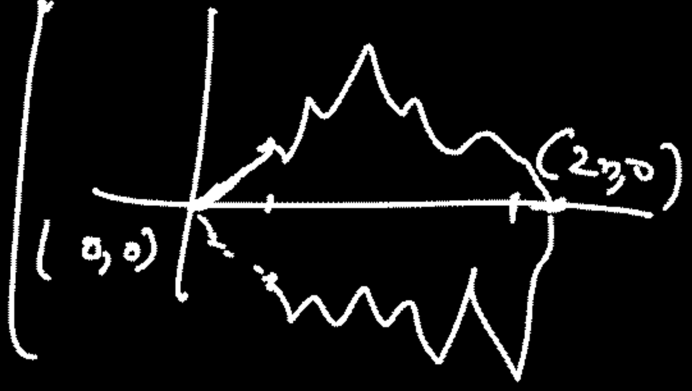
$$+ \ \mathbb{P}\left(S_i \leq -1, \forall\, 1 \leq i \leq 2n-1, S_{2n} = 0\right)$$

$$= 2\, \mathbb{P}\left(S_i \geq 1 \ \forall\, 1 \leq i \leq 2n-1, S_{2n} = 0\right)$$

$$X_i = 1 \text{ w.p. } \tfrac{1}{2}$$
$$= -1 \text{ w.p. } \tfrac{1}{2}$$

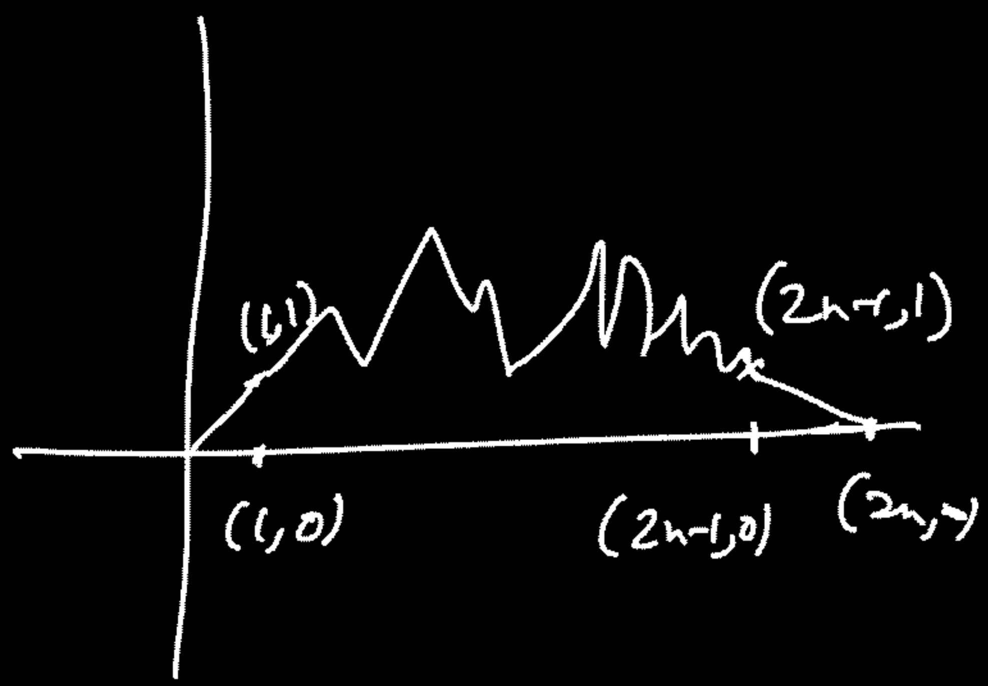
$$S_n = \sum_{i=1}^{n} X_i$$

$$S_0 = 0$$



$$= 2\, P\left(\min_{1 \le i \le 2n-1} S_i \geq 1,\ S_{2n} = 0 \right)$$

$$= 2 \, \frac{\#\text{ paths from } (1,1) \text{ to } (2n-1,1) \text{ which do not touch the } x\text{-axis. } (Y=0)}{\#\text{ paths from } (0,0) \text{ to } (2n,0)}$$



$$= 2 \, \frac{\#\text{ paths from } (1,1) \text{ to } (2n-1,1) - \#\text{ paths from } (1,1) \text{ to } (2n-1,1) \text{ which touch the } x\text{-axis}}{\binom{2n}{n}}$$

(Reflection principle).

$$= \frac{2}{\binom{2n}{n}} \left[\#\text{ paths from } (0,0) \text{ to } (2n-2,0) - \#\text{ paths from } (1,-1) \text{ to } (2n-1,1) \right]$$

$a+b=2n$

$a-b=0$

$2a = 2n \Rightarrow a = n$

$\binom{2n}{n}$

$$= \frac{2}{\binom{2n}{n}} \left[\binom{2n-2}{n-1} - \#\text{ paths from } (0,0) \text{ to } (2n-2, 2) \right]$$

$$\begin{aligned} a+b &= 2n-2 \\ a-b &= 0 \\ \hline a &= n-1 \end{aligned}$$

$$= \frac{2}{\binom{2n}{n}} \left[\binom{2n-2}{n-1} - \binom{2n-2}{n} \right]$$

$$\begin{aligned} a+b &= 2n-2 \\ a-b &= 2 \\ \hline a &= n \end{aligned}$$

<u>Recall:</u>

$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i .$$

Theorem: Denote $M_n = \max(S_0, S_1, \dots, S_n)$. Then, for $r \geq 0$,

$$\mathbb{P}(M_n = r) = \mathbb{P}(S_n = r) + \mathbb{P}(S_n = r+1).$$

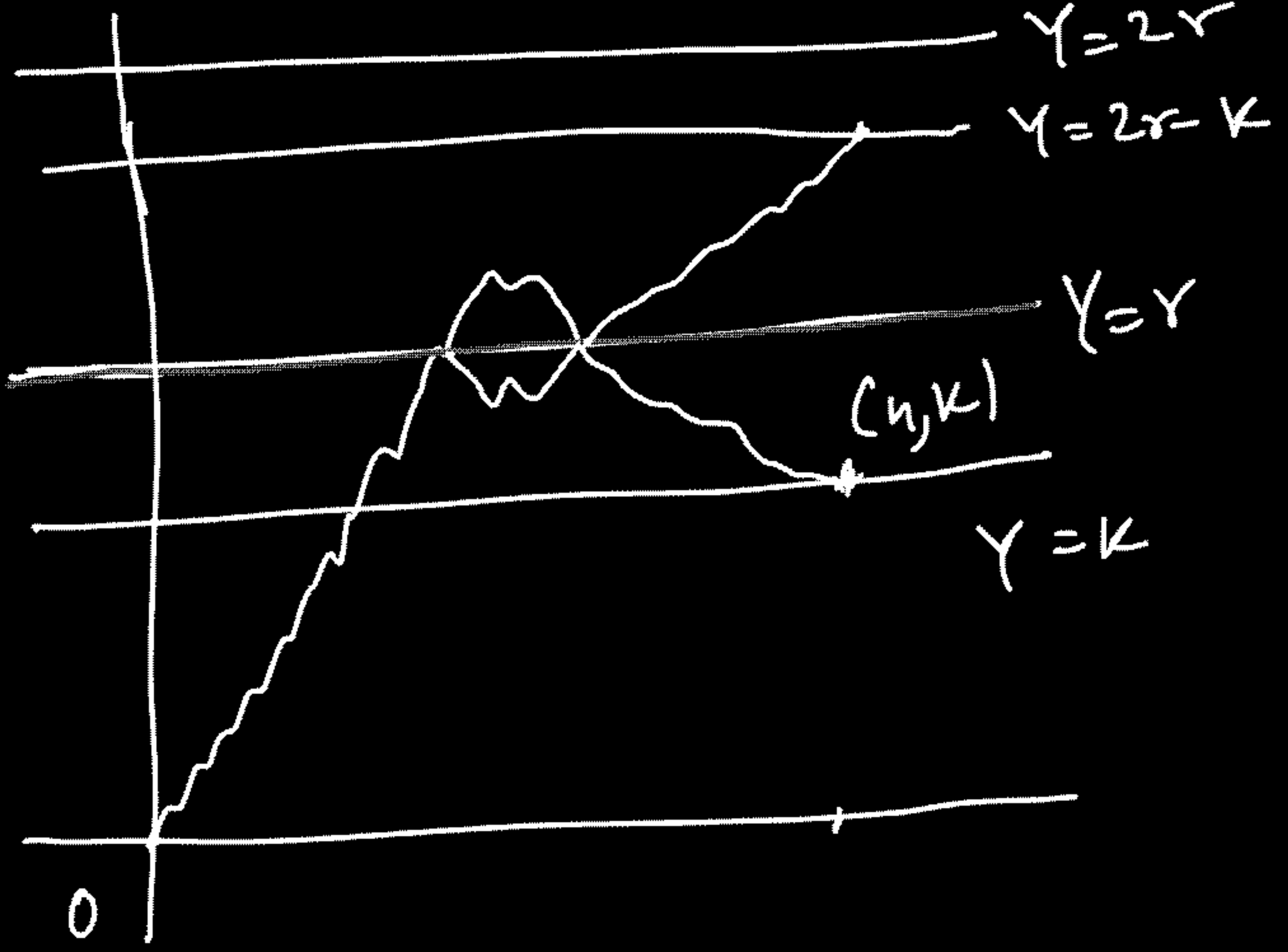
Proof: Notice that

$$\mathbb{P}(M_n < r) = \sum_{k=-n}^{r-1} \mathbb{P}(M_n < r, S_n = k) \quad \left(= \mathbb{P}\left(\Omega \setminus \{M_n \geq r\}, \{S_n = k\}\right)\right)$$

$$= \sum_{k=-n}^{r-1} \left(\mathbb{P}(S_n = k) - \mathbb{P}(M_n \geq r, S_n = k) \right)$$

Reflect around $Y = r$

$$= \sum_{k=-n}^{r-1} \left(\mathbb{P}(S_n = k) - \mathbb{P}(S_n = 2r - k) \right)$$



$$= \sum_{k=-n}^{r-1} P(S_n = k) - \sum_{k=-n}^{r-1} P(S_n = 2r-k)$$

$2r-k = l$

if $k = -n$ then $l = 2r+n$

if $k = r-1$ then $l = r+1$

$$= P(S_n < r) - \underbrace{\sum_{l=r+1}^{2r+n} P(S_n = l)}_{= \sum_{l=r+1}^{n} P(S_n = l)}$$

$$= P(S_n < r) - P(S_n > r)$$

Hence, $P(M_n < r) = P(S_n < r) - P(S_n > r)$ —— $\otimes$

Thus, $P(M_n = r) = P(M_n < r+1) - P(M_n < r)$

$$\overset{\text{by } \textcircled{*}}{=} \underline{P}(S_n < r+1) - \underline{P}(S_n > r+1) - \underline{P}(S_n < r) + \underline{P}(S_n > r)$$

$$= \underline{P}(S_n = r) + \underline{P}(S_n = r+1)$$