Probability theory 1

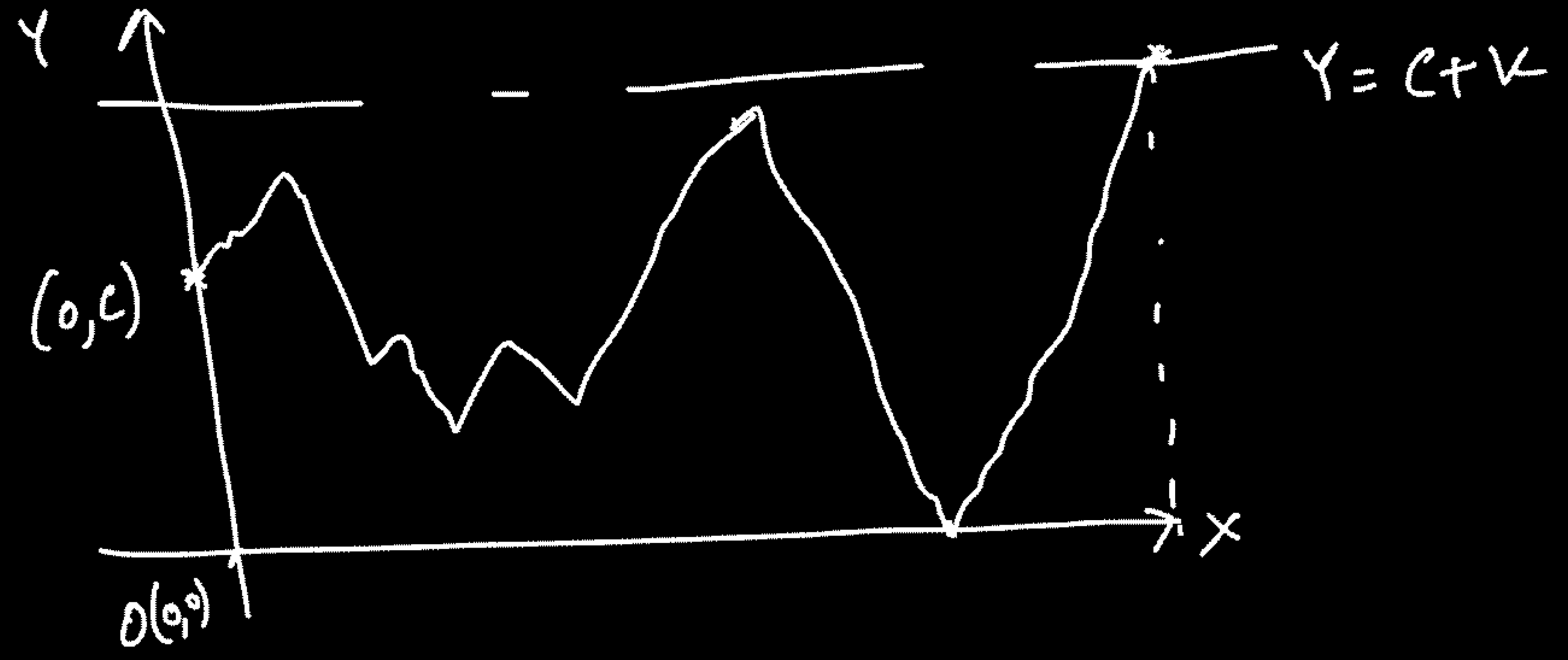
14/12/21

<u>Recall:</u> (We did the following problem last day but shall see again)

<u>Example:</u> A gambler enters a casino with ₹ C, where $C \in \mathbb{N}$. At each game, the gambler bets ₹ 1, and gets back either ₹ 2 or nothing, each with probability $\frac{1}{2}$. Given that after n games, he has won ₹ K, find the conditional probability that he was out of money at some point in the middle.

<u>Answer:</u> Let
$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$S_m = C + \sum_{i=1}^{m} X_i \quad \text{for } 1 \le m \le n.$$



$K \in \mathbb{Z}$

The sample space Ω consists of all paths from $(0,c)$ in steps. Hence,

$$\# \Omega = 2^n .$$

Let us define the events, $A = \{ \min_{0 \le i \le n} S_i \le 0 \}$

and $B = \{ S_n = C + K \}$

Then, the required probability in the question is,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{\#(A\cap B)}{\#\Omega}}{\frac{\#B}{\#\Omega}} = \frac{\#(A\cap B)}{\#B} \qquad (\#\Omega = 2^n < \infty)$$

$$= \frac{\#\text{ paths from } (0,c) \text{ to } (n, c+k) \text{ which touch the x-axis}}{\#\text{ paths from } (0,c) \text{ to } (n, k+c)}$$

(exactly as in the class

$= \ldots\ldots\ldots\ldots$

$$= \frac{\binom{n}{c+\frac{n+k}{2}}}{\binom{n}{\frac{n+k}{2}}}$$

paths from $(0,0)$ to (n, z)

$a+b=n$

$a-b=z$

$2a = n+z$

$a = \frac{n+z}{2}$

$\binom{n}{\frac{n+z}{2}}$

Recall: Find the probability that a random walk starting from 0 hits 0 for the first time after 2n steps.

Answer: Let $A = \{ \overset{S_0=0}{,} S_1 \neq 0, S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0 \}$

$S_0 = 0$ by definition

The sample space Ω = the set of all paths from $(0,0)$ in $2n$ steps

$S_n = \sum_{i=1}^{n} X_i$

Hence, $\# \Omega = 2^{2n}$

The required probability,

$$P(A) = P\left((S_1 = 1) \cup (S_1 = -1), S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0 \right)$$

$$= P(S_1 = 1, S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0) + P(S_1 = -1, S_2 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0)$$

$$= 2\,\mathbb{P}\left(S_1 = 1, S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 0\right)$$

$$= 2\,\mathbb{P}\left(\min_{1 \le i \le 2n-1} S_i \geq 1, S_{2n} = 0\right)$$

$$= 2\,\frac{\#\{\min_{1 \le i \le 2n-1} S_i \geq 1, S_{2n} = 0\}}{\#\Omega}$$ (by defn of prob.)

(We made a mistake in the last class here in the denominator, we put $\binom{2n}{n}$).

$$= 2\,\frac{\#\text{ paths from } (1,1) \text{ to } (2n-1,1) \text{ which do not touch the x-axis}}{2^{2n}}$$

$$= 2\,\frac{\text{"the computations here in the last day was correct"}}{2^{2n}} = \cdots\cdots = \frac{(2n-2)!}{n!\,(n-1)!}\,\frac{1}{2^{2n-1}} \qquad \square$$

Remark: # paths from (a,b) to (c,d) (when possible)

$(c,d) \neq (n, 2n)$ with initial start is $(0,0)$

$\|$

paths from $(0,0)$ to $(c-a, d-b)$ $\left| \begin{array}{l} X = X - a \\ Y = Y - b \end{array} \right|$ $\square$

Recall: Let $X_i = 1$ w.p. $\frac{1}{2}$

$= -1$ w.p. $\frac{1}{2}$

$$S_n = \sum_{i=1}^{n} X_i$$

Theorem ⊛: Denote $M_n = \max(S_0, S_1, \cdots, S_n)$. Then, for $r \geq 0$,

$$\mathbb{P}(M_n = r) = \mathbb{P}(S_n = r) + \mathbb{P}(S_n = r+1).$$

In particular,

$$\mathbb{P}(M_n = 0) = \mathbb{P}(S_n = 0) + \mathbb{P}(S_n = 1)$$

<u>Theorem</u>: Let $(S_n : n \geq 0)$ be a random walk and

$$M_n = \max(S_0, S_1, \ldots, S_n), \quad n \geq 0,$$

Then, for $n, r \geq 0$,

$$\mathbb{P}(M_n \geq r) = \mathbb{P}(S_n = r) + 2\mathbb{P}(S_n > r)$$

$$F(x) = \mathbb{P}(X \leq x)$$

$$F_{M_n}(x) = \mathbb{P}(M_n \leq x)$$

$$= 1 - \mathbb{P}(M_n \geq x)$$

<u>Proof</u>: Using the just previous Theorem (Theorem ⊛),

$$\mathbb{P}(M_n \geq r) = \sum_{k=r}^{\infty} \mathbb{P}(M_n = k) = \sum_{k=r}^{\infty} \left(\mathbb{P}(S_n = k) + \mathbb{P}(S_n = k+1)\right)$$

$$= \mathbb{P}(S_n = r) + \mathbb{P}(S_n = r+1)$$
$$+ \mathbb{P}(S_n = r+1) + \mathbb{P}(S_n = r+2)$$
$$+ \mathbb{P}(S_n = r+2) + \mathbb{P}(S_n = r+3) + \cdots$$

$$= \mathbb{P}(S_n = r) + 2\,\mathbb{P}(S_n \geq r+1)$$

$$= \mathbb{P}(S_n = r) + 2\,\mathbb{P}(S_n > r).$$

* $\mathbb{P}(M_n = n+1) = 0$

$$\parallel$$

$$\mathbb{P}(S_n = n+1) + \mathbb{P}(S_n = n+2) = 0$$

<u>Another proof:</u> $\mathbb{P}(M_n \geq r) = \mathbb{P}(M_n \geq r, S_n \geq r) + \mathbb{P}(M_n \geq r, S_n < r)$ —— (1)

Observe that $\{M_n \geq r\} \supseteq \{S_n \geq r\}$

Hence, (1) $= \mathbb{P}(S_n \geq r) + \mathbb{P}(M_n \geq r, S_n < r)$

$$= P(S_n \geq r) + P(S_n > r)$$

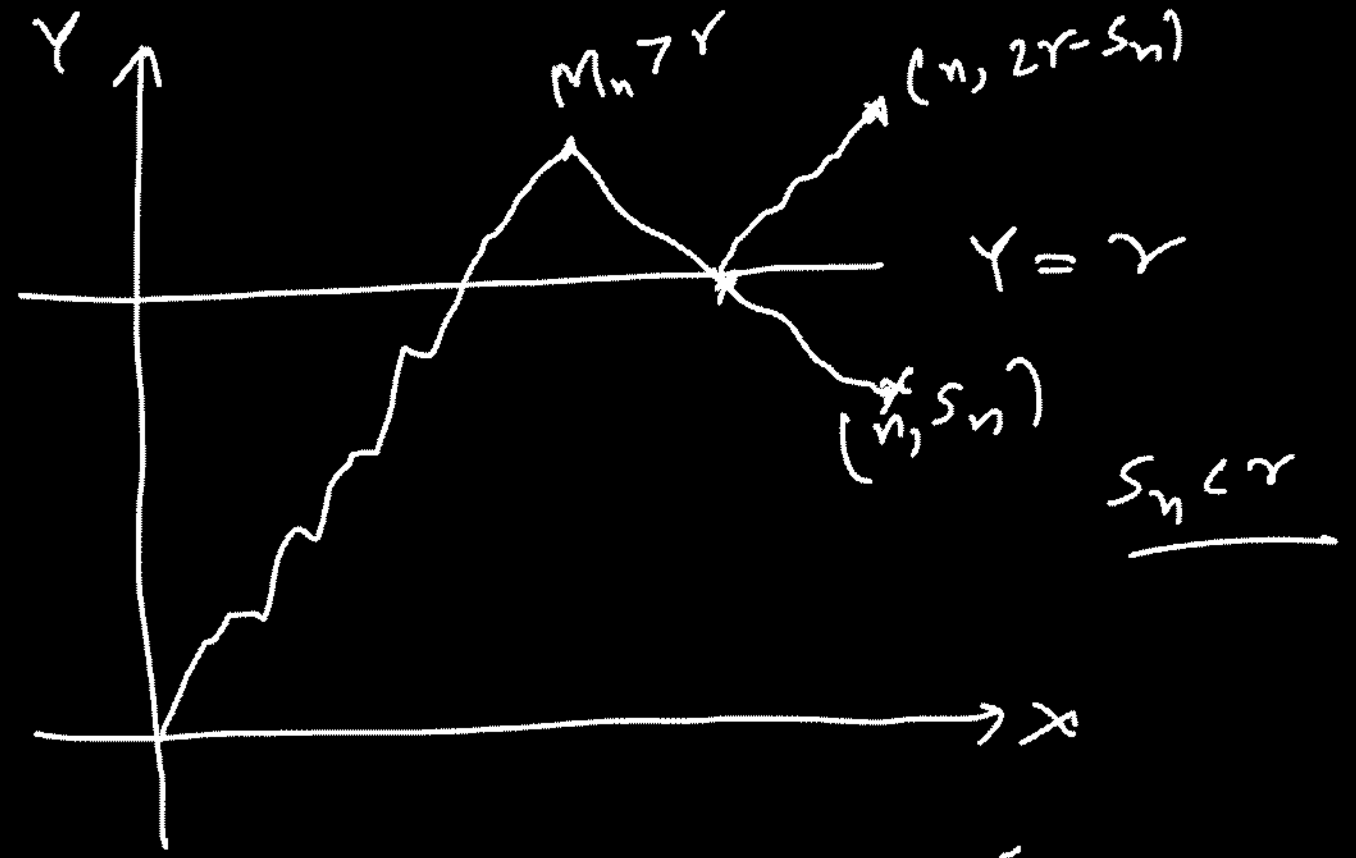
since

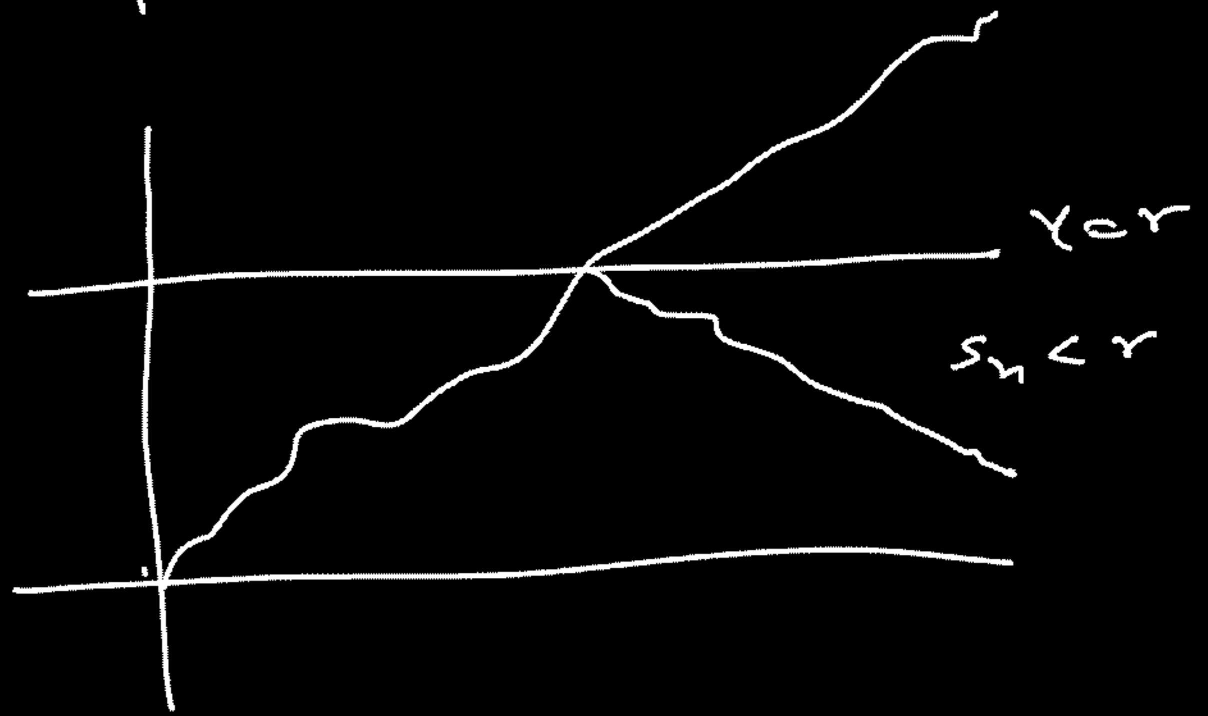
$$\#\{M_n \geq r, S_n < r\} = \#\{M_n \geq r, S_n > r\}$$

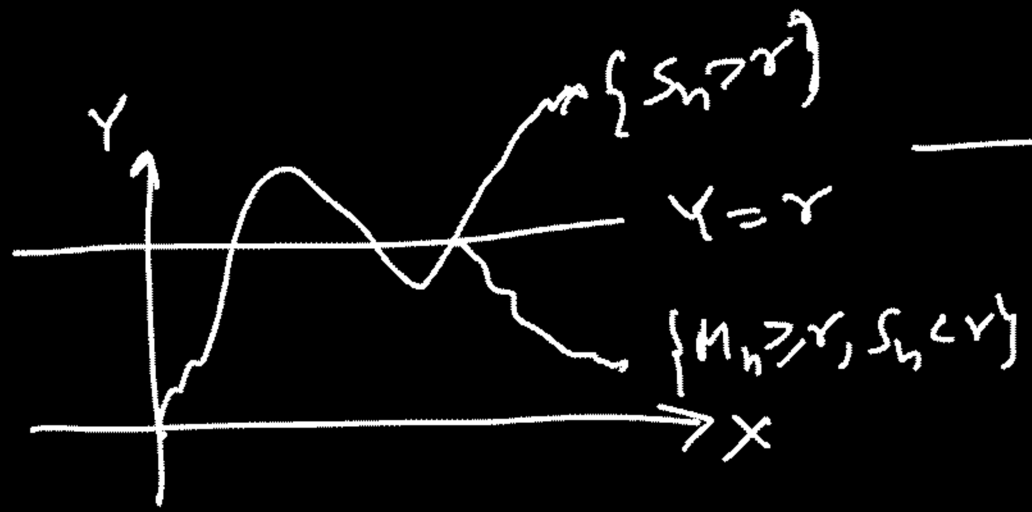
$$\| $$

$$\#\{S_n > r\}$$

$$= P(S_n = r) + 2\,P(S_n > r)$$







Theorem: (Arcsine law for the last visit)

$\max\{0 \le k \le n : S_{2k} = 0\}$ $= 0$

$\downarrow$

$S_0 = 0$

$S_{2k} = 0$

For $n \geq 1$, denote $L_n = \frac{1}{n} \max\{0 \le k \le n : S_{2k} = 0\}$

Then, $\lim_{n \to \infty} P(x < L_n \leq y) = \frac{2}{\pi}\left(\sin^{-1}(\sqrt{y}) - \sin^{-1}(\sqrt{x})\right)$ for $0 < x < y < 1$

$P(x < L_n \leq y) = F_{L_n}(y) - F_{L_n}(x)$

$\frac{2}{\pi}\left(\sin^{-1}(\sqrt{y}) - \sin^{-1}(\sqrt{x})\right) = F_X(y) - F_X(x)$

where $F_X(z) = P(X \le z) = \frac{2}{\pi} \arcsin(\sqrt{z})$ $\forall\ z \in (0,1)$.

The setup is
$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$S_m = \sum_{i=1}^{m} X_i$ for $0 < m \le 2n$; $S_0 = 0$.

To prove the above theorem we shall prove the following two lemmas.

<u>Lemma 1:</u> For $n \geq 1$ and $0 \leq k \leq n$,

$$\mathbb{P}\left(L_n = \frac{k}{n}\right) = \mathbb{P}(S_{2k} = 0)\,\mathbb{P}(S_{2n-2k} = 0)$$

<u>Pf of the lemma:</u> We start with showing that the claim is true for $k=0$,

i.e.,

$$\mathbb{P}(L_n = 0) = \mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = \mathbb{P}(S_0 = 0)\,\mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n} = 0)$$

i.e.,

$$\boxed{\underbrace{\mathbb{P}(S_1 \neq 0, \ldots, S_{2n} \neq 0) = \mathbb{P}(S_{2n} = 0)}_{\text{We shall show this.}}}$$

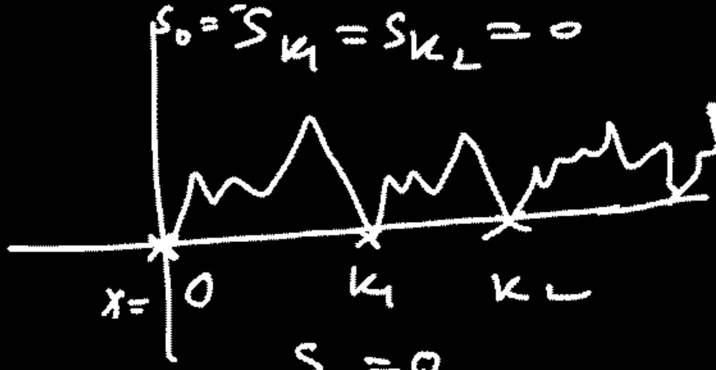
(since $S_0 = 0$ by definition, $\therefore \mathbb{P}(S_0 = 0) = 1$).

$$\text{LHS} = P\big((S_1=1)\cup(S_1=-1),\ S_2\neq 0,\ \dots,\ S_{2n}\neq 0\big)$$

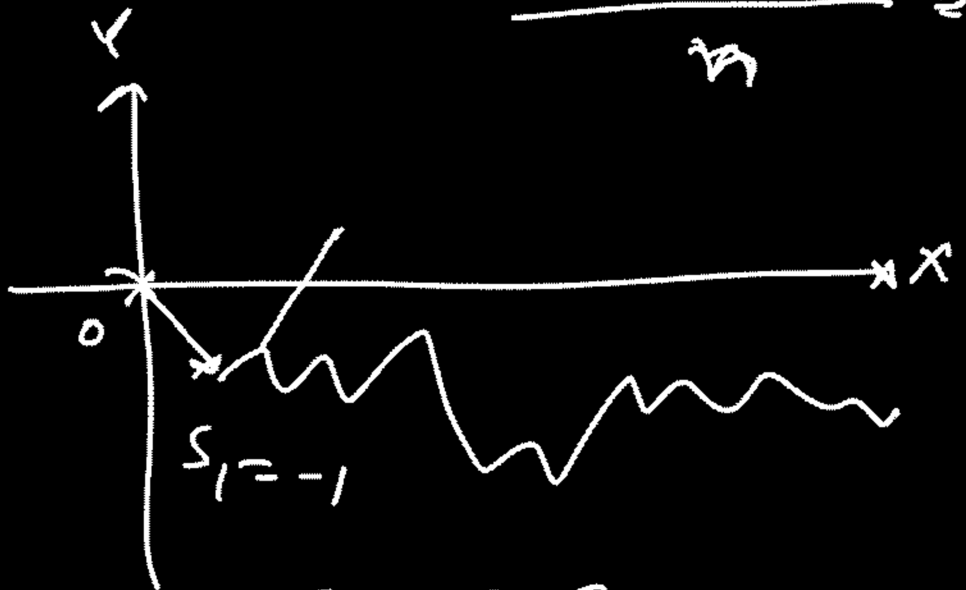
$$= P(S_1=1,\ S_2\neq 0,\ \dots,\ S_{2n}\neq 0) + P(S_1=-1,\ S_2\neq 0,\ \dots,\ S_{2n}\neq 0)$$

$$= 2\,P(S_1=-1,\ S_2\neq 0,\ \dots,\ S_{2n}\neq 0)$$

$$= 2\,P\big(S_1=-1,\ \max(\underbrace{S_1-S_1}_{=S_0'},\ \underbrace{S_2-S_1}_{=S_1'},\ \dots,\ \underbrace{S_{2n}-S_1}_{=S_{2n-1}'})=0\big) \quad \text{—} ②$$



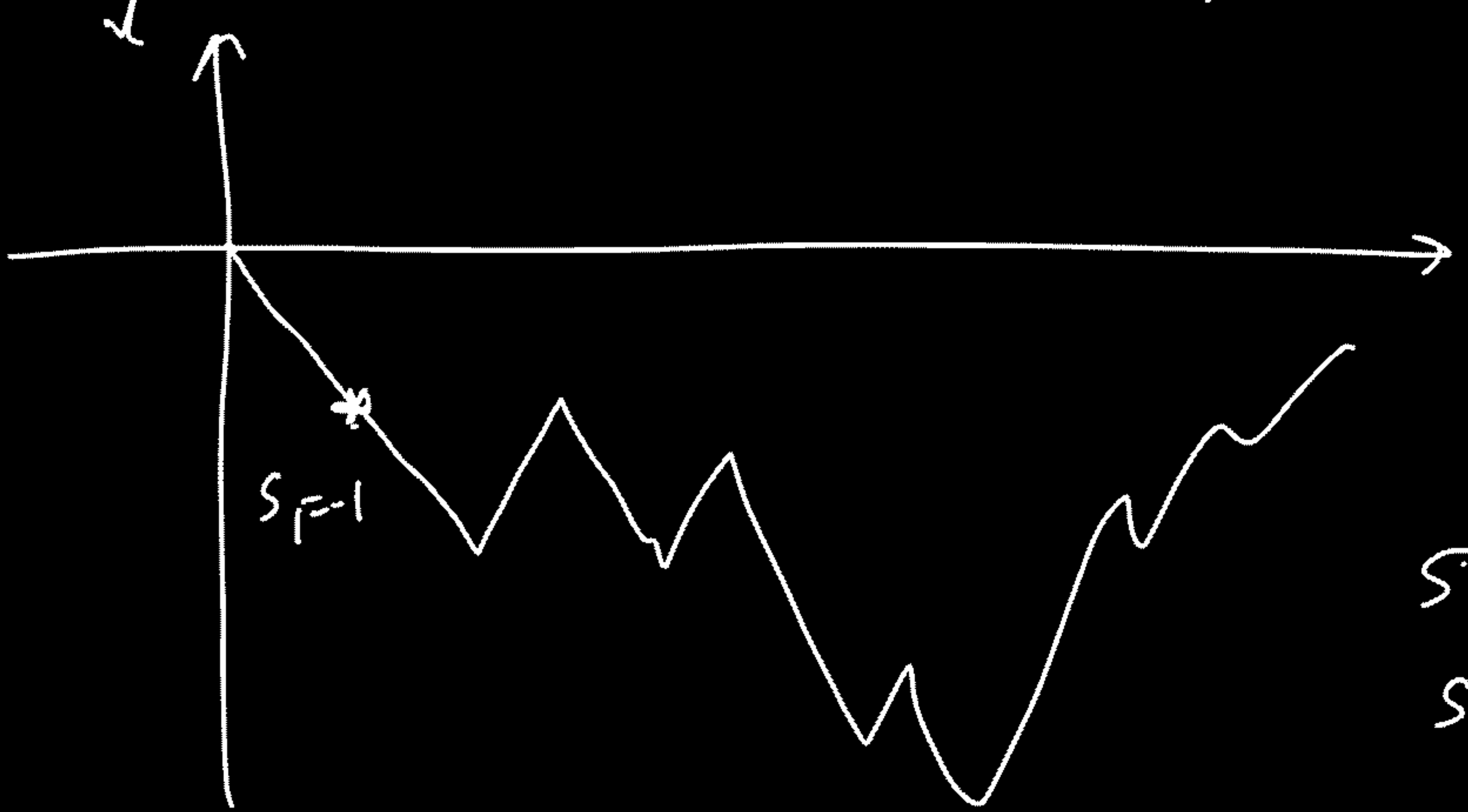
$$\frac{\max\{0, k_1, k_2\}}{n} = L_n$$

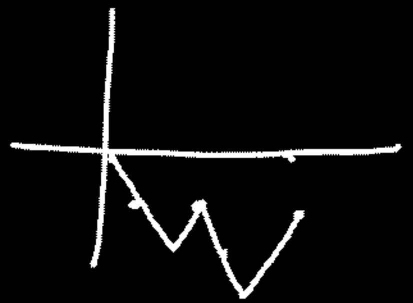


$S_2 - S_1 > 0$

$S_2 > S_1 > -1$

$S_2 > -1$





if $S_n - S_1 = 1$

$S_n = S_1 + 1$

$S_{2n} - S_1 = X_2 + X_3 + \dots + X_{2n}$

$S_2 - S_1 = X_2$

$S_3 - S_1 = X_2 + X_3$

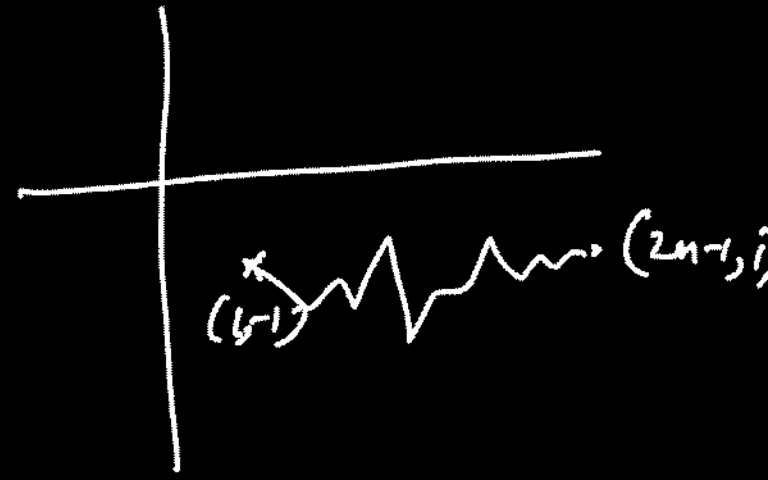
- - - -

$\{S_m'\}_{m=0}^{2n-1}$ is another random walk (with $S_m \neq 0$) starting from $(1,-1)$.

$$② = 2\,P(S_1=-1)\,P(\max(S_1-S_1, S_2-S_1, \dots, S_{2n}-S_1)=0)$$

$$= P(\max(S_0', S_1', \dots, S_{2n-1}')=0) \qquad \left(\because P(S_1=-1)=\tfrac{1}{2}\right)$$

$$= P(M_{2n-1}'=0) \qquad (\text{In the notation of Theorem ⊛})$$



$$= \cancelto{0}{P(S_{2n-1}'=0)} + P(S_{2n-1}'=1) \qquad (\text{By Theorem ⊛})$$

$$= P(S_{2n-1}'=1)$$

$$\overset{\text{(by defn of their prime random walks)}}{=} P(S_{2n}-S_1=1) = P(S_{2n}=0) \quad \text{ok}.$$

$$S_0' = S_1 - S_1$$
$$S_1' = S_2 - S_1$$
$$S_2' = S_3 - S_1$$
$$\vdots$$
$$S_{2n-1}' = S_{2n} - S_1$$

$$\underline{S_m' = S_{m+1} - S_1}$$

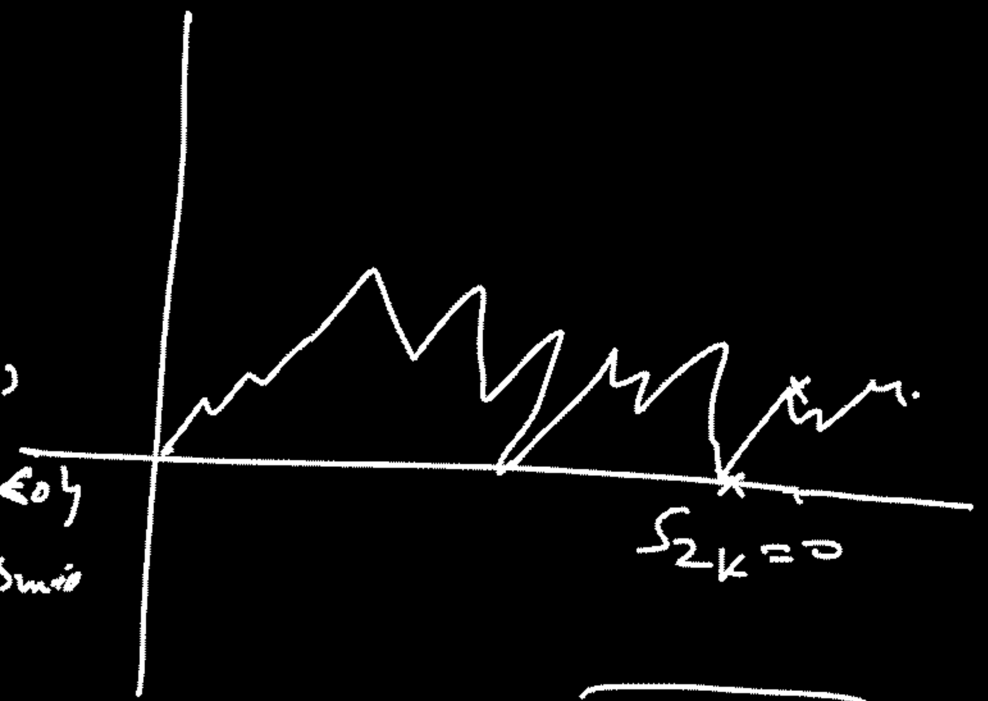
Now let $0 \le k \le n$,

$$P\left(L_n = \frac{k}{n}\right) = P\left(\max\{0 \le j \le n : S_{2j} = 0\} = k\right)$$

$$= P(S_{2k}=0, S_{2k+1} \neq 0, S_{2k+2} \neq 0, \dots, S_{2n} \neq 0) = P(S_{2n}=0, \max(\dots) \le 0\} \quad S_{2k+1}, S_{2n\dots}$$

$$= P(S_{2k}=0)\, P(\underset{\underset{S''_1}{\parallel}}{S_{2k+1}} \neq 0, \underset{\underset{S''_2}{\parallel}}{S_{2k+2}} \neq 0, \dots, \underset{\underset{S''_{2n-2k}}{\parallel}}{S_{2n}} \neq 0)$$

not clear !!



$2k + \boxed{2n-2k}$

$S''_m = S_{2k+m}$

$S''_0 = 0 = S_{2k}$

$S'' = S_{2n-2k}$

$\parallel$

$2k+m$

$m = 2n-4k$

$$= P(S_{2k}=0)\, \underbrace{P(S''_1 \neq 0, S''_2 \neq 0, \dots, S''_{2n-2k} \neq 0)}$$

$$= P(S_{2k}=0)\, P(S''_{2n-2k}=0)$$

$$= P(S_{2k}=0)\, P(S_{2n-2k}=0)$$

[the random walks $\{S''_m\}$ & $\{S_m\}$ have same properties].

<u>Lemma 2:</u> As $n \to \infty$ $\lim_{n\to\infty} \sqrt{\pi n}\; P(S_{2n}=0) = 1$.

<u>Pf of lemma 2:</u> By Stirling's approximation, it follows that

$$P(S_{2n}=0) = \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n)!}{(n!)^2}\,\frac{1}{2^{2n}}$$

$(n!)^2 \sim \left(\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}\right)^2$

$$\sim \frac{\sqrt{2\pi}\,(2n)^{2n+\frac{1}{2}}\, e^{-2n}}{\left(\sqrt{2\pi}\,(n)^{n+\frac{1}{2}}\, e^{-n}\right)^2}\,\frac{1}{2^{2n}} = \frac{1}{\sqrt{2\pi}}\,\frac{2^{2n+\frac{1}{2}}\, n^{2n+\frac{1}{2}}}{n^{2n+1}\, 2^{2n}}$$

$$= \frac{1}{\sqrt{2\pi}}\,\sqrt{2}\,\frac{1}{\sqrt{n}}$$

$$\Rightarrow \sqrt{\pi n}\ \mathbb{P}(S_{2n}=0) \sim 1$$

Therefore,

$$\lim_{n\to\infty} \sqrt{\pi n}\ \mathbb{P}(S_{2n}=0) = 1 \qquad \square$$

$$n! \sim \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}$$

$$\Downarrow$$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}} = 1$$

$$a_n \sim b_n \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$