Probability theory 1

28/12/21

- We were discussing about PGFs.

Theorem: Suppose that $X_1, X_2, \dots, X_\infty$ are random variables taking values $0, 1, 2, \dots$ and having PGFs $f_1, f_2, \dots, f_\infty$ respectively. Then,

$$\lim_{n\to\infty} P(X_n = k) = P(X_\infty = k) \quad \text{for every } k \in \mathbb{N} \cup \{0\}$$

if and only if

$$\lim_{n\to\infty} f_n(s) = f_\infty(s) \quad \text{for every } s \in (-1,1).$$

Proof: $(\rightarrow)$ Suppose that $\lim_{n\to\infty} P(X_n = k) = P(X_\infty = k)$ for every $k = 0, 1, 2, \cdots$.

Denote $p_{nk} := P(X_n = k)$, $k \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}\cup\{\infty\}$.

Fix $s \in (-1,1)$ and let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that

$$2\frac{|s|^{n+1}}{1-|s|} < \varepsilon \quad \forall\ n \geq N. \qquad \left[\ |s|^{n+1} < \frac{\varepsilon}{2}(1-|s|) \quad \forall\ n \geq N\ \right]$$

Now,
$$\limsup_{n\to\infty} |f_n(s) - f_\infty(s)| = \limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} p_{nk} s^k - \sum_{k=0}^{\infty} p_{\infty k} s^k \right|$$
$$= \limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} (p_{nk} - p_{\infty k}) s^k \right|$$

$$= \limsup_{n\to\infty} \left| \sum_{k=0}^{N} (p_{nk} - p_{\infty k}) s^k + \sum_{k=N+1}^{\infty} (p_{nk} - p_{\infty k}) s^k \right|$$

$$\le \limsup_{n\to\infty} \sum_{k=0}^{N} |p_{nk} - p_{\infty k}|\, |s|^k + \limsup_{n\to\infty} \sum_{k=N+1}^{\infty} |p_{nk} - p_{\infty k}|\, |s|^k$$

$$\le \sum_{k=0}^{N} |s|^k \limsup_{n\to\infty} |p_{nk} - p_{\infty k}| + 2 \limsup_{n\to\infty} \sum_{k=N+1}^{\infty} |s|^k$$

$$\le 0 + 2 \frac{|s|^{N+1}}{1 - |s|} < \varepsilon$$

Thus $0 \le \liminf_{n\to\infty} |f_n(s) - f_\infty(s)| \le \limsup_{n\to\infty} |f_n(s) - f_\infty(s)| < \varepsilon$ for any $\varepsilon > 0$.

Since ε is arbitrary, this proves the 'only if' part.

(⟵) For the if part we assume that,

$$\lim_{n\to\infty} f_n(s) = f_\infty(s) \quad \text{i.e.} \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} p_{nk} s^k = \sum_{k=0}^{\infty} p_{\infty k} s^k .$$

<u>Theorem (Cantor)</u>: Suppose that for each $k \in \mathbb{N}$, $\{a_{nk}, n \geq 1\}$ is bounded sequence. Then there exists $1 \leq n_1 < n_2 < \cdots$ such that

$$\lim_{r\to\infty} a_{n_r k} \text{ exists for all } k \geq 1 .$$

<u>Proof</u>: We know that every bounded sequence has a convergent subsequence. Therefore for the sequence $\{a_{n1}\}_{n=1}^{\infty}$ there exists $0 \leq n_1' \leq n_2' \leq n_3' < \cdots$ such that $\{a_{n_r' 1}\}_{r=1}^{\infty}$ is convergent.

$\{a_{n1}\}_{n=1}^{\infty} \to \{a_{n_\delta 1}\}_{n_\delta}^{\infty}$

$\{a_{n2}\}_{n=1}^{\infty} \to \{a_{n_2 2}\}_{n_2=1}^{\infty}$

a_{11}, a_{21}, a_{31} ↓

$1 \leq n_1 \leq n_2 < n_3 \cdots$

$\{a_{n_r k}\}_{r=1}^{\infty}$

Now for the sequence $\{a_{n^1_r 2}\}_{r=1}^{\infty}$ we can get a convergent subsequence, say $1 \le n^2_1 < n^2_2 < n^2_3 < \cdots$ from $\{n^1_r\}_{r=1}^{\infty}$ such that $\{a_{n^2_r 2}\}_{r=1}^{\infty}$ is convergent. Observe that also $\{a_{n^2_r 1}\}_{r=1}^{\infty}$ is convergent. We proceed by considering the sequence

$$\{a_{n^2_r 3}\}_{r=1}^{\infty}.$$

This has a convergent subsequence.

Say $1 \le n^3_1 < n^3_2 < n^3_3 < \cdots$ such that

$\{a_{n^3_r 1}\}_{r=1}^{\infty}$, $\{a_{n^3_r 2}\}_{r=1}^{\infty}$, $\{a_{n^3_r 3}\}_{r=1}^{\infty}$ all are convergent.

$\{a_{n1}\}_{n=1}^{\infty}$ $a_{11}, a_{131}, a_{1001}, \ldots$

$\{a_{n2}\}_{n=1}^{\infty}$ $a_{12}, a_{132}, a_{1002}, \ldots$

$1 < \{a_{n1}\}_{n=1}^{\infty} < 2$

$2 < \{a_{n2}\}_{n=1}^{\infty} < 3$

$3 < \{a_{n3}\}_{n=1}^{\infty} < 4$

$\{a_{n1}\} = 1$

$\{a_{n2}\} = 2$

finall consider the indexing sequence.

$$1 \le n_1^1 < n_2^2 < n_3^3 < \cdots$$

$$n_1^1 < n_2^1 < n_3^1 < \cdots$$

$$n_1^2 < n_2^2 < n_3^2 < \cdots$$

$$n_1^3 < n_2^3 < n_3^3 < \cdots$$

$$a_{n_r^r k}$$

and check that $\lim_{r \to \infty} a_{n_r^r k}$ exists for any $k \in \mathbb{N}$.

$$a_{n_r^r 3}$$

Coming back to the proof of if part:

Consider the sequence for each $k \in \mathbb{N}$, $\{p_{nk}\}_{n=1}^{\infty} =: \{\mathbb{P}(X_n = k)\}_{n=1}^{\infty}$. Applying Cantor's theorem we get $1 \le n_1 < n_2 < n_3 < \cdots$ such that

$$\lim_{r\to\infty} p_{n_r k} \text{ exists for any } k\in\mathbb{N}.$$

$$\lim_{r\to\infty} P(X_{n_r}=k) = p_k \quad \forall\, k\in\mathbb{N}$$

$P(X_n=k) \xrightarrow{n\to\infty} \underbrace{P(X_\infty=k)}_{p_k}$ $\forall k$.

$\Downarrow$

$\underbrace{\sum P(X_n=k)s^k}$ $\xrightarrow[n\to\infty]{}$ $\underbrace{\sum P(X_\infty=k)s^k}_{p_k s^k}$

Thus by the proof of the 'only if' part we conclude that

$$\underbrace{\sum_{k=0}^{\infty} P(X_{n_r}=k)s^k}\ \xrightarrow[r\to\infty]{}\ \sum_{k=0}^{\infty} p_k s^k$$

$$\left\{\sum_{k=0}^{\infty} P(X_{n_r}=k)s^k\right\}_{r=1}^{\infty} \text{ is a subsequence of } \left\{\sum P(X_n=k)s^k\right\}_{n=1}^{\infty}$$

and $\sum_{k=0}^{\infty} \mathbb{P}(X_n=k)s^k \left(= f_n(s)\right)$ converges. and converges to $f_\infty(s)$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X_\infty=k)s^k$$

in $s \in (-1,1)$. Thus $p_k = \mathbb{P}(X_\infty=k)$.

$$p_k = \lim_{r\to\infty} \mathbb{P}(X_{n_r}=k)$$

So we have proved that $\lim_{r\to\infty} \mathbb{P}(X_{n_r}=k) = \mathbb{P}(X_\infty=k)$. — ⊛

but we need $\lim_{n\to\infty} \mathbb{P}(X_n=k) = \mathbb{P}(X_\infty=k)$.

Theorem 1 Consider $\{a_n\}_{n=1}^{\infty}$. If the sequence has the property that for any subsequence of $\{a_n\}_{n=1}^{\infty}$ has a further subsequence which is convergent to a (say). Then $\lim_{n\to\infty} a_n = a$.

Now fix $k_0 \in \mathbb{N}$. Consider the sequence $\{P(X_n = k_0)\}_{n=1}^{\infty}$. The redo the whole thing done in the proof of if part and conclude from the theorem 1 & ⊛ that $\lim_{n\to\infty} P(X_n = k_0) = P(X_\infty = k_0)$.

Since k_0 was arbitrary, we have completed the proof.

Proof of Stirling's formula:

Theorem: (Stirlings approximation):

As $n \to \infty$, $\quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Lemma 1: Let $a_n = \dfrac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n}$, $\quad n \geq 1$.

Then there exists a constant $C \in (0,\infty)$ such that $\lim_{n\to\infty} a_n = C$.

Proof: Let $b_n := \log a_n$, $\quad n \geq 1$.

Note that $$b_n - b_{n+1} = \log \frac{a_n}{a_{n+1}} = \log \frac{n!\, \sqrt{2n+2}\, \left(\frac{n+1}{e}\right)^{n+1}}{\sqrt{2n}\, \left(\frac{n}{e}\right)^n \, (n+1)!}$$

$$= \log \frac{\sqrt{n+1}\, \left(\frac{n+1}{e}\right)^n \left(\frac{n+1}{e}\right)}{\sqrt{n}\, \left(\frac{n}{e}\right)^n (n+1)}$$

$$= \log \frac{\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}}}{e} = \left(n + \frac{1}{2}\right) \log\left(1 + \frac{1}{n}\right) - 1 .$$

Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \left(\frac{1}{x} + \frac{1}{2}\right) \log\left(1 + x\right) - 1$

Then, clearly, $f\left(\frac{1}{n}\right) = b_n - b_{n+1}$, $\forall\, n \geq 1$.

Recall for $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Hence,

$$\frac{1}{x}\log(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$$

$$\frac{1}{2}\log(1+x) = \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{6} - \cdots$$

$$\left(\frac{1}{x}+\frac{1}{2}\right)\log(1+x) = 1 + \frac{x^2}{12} + \sum_{n=3}^{\infty} r_n x^n, \quad 0 < x < 1 \text{ (say)}.$$

for some $r_3, r_4, \cdots \in \mathbb{R}$. Therefore,

$$\lim_{x \downarrow 0} \frac{f(x)}{x^2} = \frac{1}{12}$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(\frac{1}{n})}{\frac{1}{n^2}} = \frac{1}{12} \qquad \text{i.e.} \quad f\left(\frac{1}{n}\right) \sim \frac{1}{12n^2} \quad \text{as } n\to\infty.$$

$$\lim_{n\to\infty} \frac{f(\frac{1}{n})}{\frac{1}{12n^2}} = 1$$

$$\left| \frac{f(\frac{1}{n})}{(12n^2)^{-1}} - 1 \right| < \varepsilon$$

$\forall\ n \geq n_0.$

Observe that for any $\varepsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ s.t.

$$0 \leq \left| f\left(\tfrac{1}{n}\right) \right| \leq \frac{1+\varepsilon}{12n^2}$$

Summing we get, $\displaystyle\sum_{n=1}^{\infty} \left| f\left(\tfrac{1}{n}\right)\right|$ is convergent by comparison test.

$$\| \qquad\qquad$$

$$\sum_{n=1}^{\infty} |b_n - b_{n+1}|$$

$\sum a_n$ is convt

$\{a_n\}_{n=1}^{\infty} \to 0.$

Hence $\{b_n - b_{n+1}\}_{n=1}^{\infty}$ is convergent & converges to 0.

This shows that

$$K := \lim_{n\to\infty} b_n \quad \text{exists and finite.}$$

Therefore $K = \lim_{n\to\infty} \log a_n$

$$\Rightarrow \lim_{n\to\infty} a_n = \underset{\displaystyle c}{\overset{\|}{e^{K}}} \in (0,\infty).$$

$\square$

$$\lim_{n\to\infty} (b_n - b_{n+1}) = 0$$

$$S_N = \sum_{n=1}^{N} (b_n - b_{n+1})$$

$$= b_1 - b_2 + b_2 - b_3 + \cdots + b_N - b_{N+1}$$

$$= b_1 - b_{N+1}$$

$\lim_{N\to\infty} S_N$ exists.

$$\|$$

$$b_1 - b_{N+1}$$

$$b_1 - \lim_{N\to\infty} b_{N+1}$$

$k \geq 2,\quad f(k+2) - (1-p) f(k+1) = p(1-p) f(k).$