Probability theory 1

29/12/21

- **Recall:** We were proving Stirling's approximation theorem.

- **Statement:** As $n \to \infty$, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

- **Lemma 1:** $$\lim_{n\to\infty} \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n} = C$$

  for some constant $C \in (0,\infty)$.

- We proved Lemma 1 in the previous class.

Lemma 2: (Wallis formula):

As $n \to \infty$, $$\prod_{k=1}^{n} \left(\frac{2k}{2k-1} \frac{2k}{2k+1} \right) \longrightarrow \frac{\pi}{2}.$$

Proof: Fix $n \geq 2$. Using integration by parts,

$$A_n := \int_0^{\pi/2} \sin^n x \, dx$$

$$= \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx = \left[-\sin^{n-1} x \cos x \right]_0^{\pi/2} + \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos^2 x \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \left(1 - \sin^2 x \right) dx = (n-1) \int_0^{\pi/2} \sin^{n-2} dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$$

So, $A_n = (n-1) A_{n-2} - (n-1) A_n$

$$\Rightarrow \quad A_n = \frac{n-1}{n} A_{n-2} \quad , \quad n \geq 2.$$

Hence, for every $n \geq 1$

$$A_{2n} = \frac{2n-1}{2n} A_{2n-2} = \frac{(2n-1)}{2n} \frac{(2n-2-1)}{2n-2} A_{2n-4} = \cdots$$

$$= \left(\prod_{k=1}^{n} \frac{2k-1}{2k} \right) \int_0^{\pi/2} dx = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}$$

and $$A_{2n+1} = \left(\prod_{k=1}^{n} \frac{2k}{2k+1} \right) \int_0^{\pi/2} \sin x \, dx = \prod_{k=1}^{n} \frac{2k}{2k+1}$$

Hence,
$$\frac{A_{2n+1}}{A_{2n}} = \frac{2}{\pi} \prod_{k=1}^{n} \left(\frac{2k}{2k+1} \quad \frac{2k}{2k-1} \right)$$

In order to complete the proof it suffices to show that

$$\lim_{n \to \infty} \frac{A_{2n+1}}{A_{2n}} = 1 .$$

To that end, note that $\sin^{2n-1} x \geq \sin^{2n} x \geq \sin^{2n+1} x$ for $0 \leq x \leq \frac{\pi}{2}$,

and hence
$$\int_0^{\pi/2} \sin^{2n-1} x\,dx \geq \int_0^{\pi/2} \sin^{2n} x\,dx \geq \int_0^{\pi/2} \sin^{2n+1} x\,dx$$

$$\therefore \quad \underline{A_{2n-1} \geq A_{2n}} \geq A_{2n+1}$$

$$A_n = \frac{n-1}{n} A_{n-2} \quad \text{———} \circledast$$

So,

$$1 \geq \frac{A_{2n+1}}{A_{2n}} \overset{\circledast}{=} \frac{\frac{2n}{2n+1} A_{2n-1}}{A_{2n}}$$

$$\geq \frac{2n}{2n+1} \qquad \left(\because \frac{A_{2n-1}}{A_{2n}} \geq 1 \right)$$

$$\therefore \quad \frac{2n}{2n+1} \leq \frac{2n}{2n+1} \frac{A_{2n-1}}{A_{2n}} \leq \frac{A_{2n+1}}{A_{2n}} \leq 1$$

Letting $n \to \infty$, we get that $\displaystyle\lim_{n\to\infty} \frac{A_{2n+1}}{A_{2n}} = 1$.

Proof of stirlings approximation theorem:

All that needs to be shown is $c = \sqrt{\pi}$. The LHS of Wallis formula, for a fixed n, equals.

$$\prod_{k=1}^{n} \left(\frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right) = \frac{4^n (n!)^2}{(1.3.5 \ldots 2n-1)(1.3.5 \ldots (2n-1))(2n+1)}$$

$$= \frac{4^n (n!)^2 (2.4. \ldots 2n)^2}{((2n)!)^2 (2n+1)}$$

$$= \frac{16^n (n!)^4}{((2n)!)^2 (2n+1)}$$

So, Wallis formula is equivalent to

$$\lim_{n\to\infty} \frac{16^n (n!)^4}{((2n)!)^2 (2n+1)} = \frac{\pi}{2} \quad \text{——— ⓐ}$$

From Lemma 1, $\dfrac{(n!)^4}{(2n!)^2} \sim \dfrac{\left(c\sqrt{2n}\left(\frac{n}{e}\right)^n\right)^4}{\left(c\sqrt{4n}\left(\frac{2n}{e}\right)^{2n}\right)^2} = c^2 \dfrac{(2n)^2}{4n} \dfrac{n^{4n}}{(2n)^{4n}}$

$$= c^2 n \frac{1}{16^n} \quad \text{——— ⓑ}$$

Combining ⓐ & ⓑ we get, $\dfrac{\pi}{2} = \lim\limits_{n\to\infty}\left(\dfrac{16^n}{(2n+1)} \times \dfrac{(n!)^4}{((2n)!)^2}\right)$

$$= \lim_{n\to\infty} \frac{16^n}{2n+1} c^2 n \frac{1}{16^n}$$

$$= \lim_{n\to\infty} c^2 \frac{n}{2n+1} = \frac{c^2}{2}$$

Hence, $c = \sqrt{\pi}$ as desired. This completes the proof.

* Steps of the proof of the 'if part' of previous day's theorem:

<u>Assumption:</u> $\lim_{n\to\infty} f_n(s) = f_\infty(s)$ for every $|s| < 1$.

Step 1: By Cantor's diagonalization argument there exists $1 \le n_1 < n_2 < \cdots$ such that

$$p_k := \lim_{r\to\infty} \mathbb{P}(X_{n_r}=k) \quad \text{exists for all } k\in\{0,1,2,\dots\}.$$

Step 2: Argue that $\lim_{r\to\infty} f_{n_r}(s) = \sum_{k=0}^{\infty} p_k s^k$ by the proof of 'only if' part of this theorem. for all $|s|<1$.

Step 3: Using assumption and step 2, conclude that

$$p_k = \mathbb{P}(X_\infty = k) \quad \text{for all } k\in\mathbb{N}\cup\{0\},$$

and hence $\lim_{r\to\infty} \mathbb{P}(X_{n_r}=k) = \mathbb{P}(X_\infty = k)$ for all $k \in \mathbb{N} \cup \{0\}$.

Step 4: Repeat step 1 to Step 3 with a subsequence i.e. Show that for every subsequence $\{X_{n_r}\}_{r\geq 1}$ of $\{X_n\}_{n\geq 1}$ has a further subsequence $\{X_{n_{k_r}}\}_{r\geq 1}$ such that

$$\lim_{r\to\infty} \mathbb{P}\left(X_{n_{k_r}} = j\right) = \mathbb{P}\left(X_\infty = j\right), \quad j \in \{0,1,2,\dots\}.$$

Step 5: Finally,

Step 4 and previous day's Theorem 1 implies,

$$\lim_{n\to\infty} \mathbb{P}(X_n = j) = \mathbb{P}(X_\infty = j), \quad \text{for all } j \in \{0, 1, 2, \cdots\}. \qquad \square$$

* Next class Friday: by Arijit Sir.

(Branching process).