7/1/22

HW 7

HW7/1

Distribution	PGF $\phi(s)$	
NegBin (k, p)	$\left(\dfrac{ps}{1-qs}\right)^k$	$(q = 1-p)$
Bin (n, p)	$(q+ps)^n$	$(q = 1-p)$
Poisson (λ)	$e^{\lambda(s-1)}$	
Geometric (p)	$ps/(1-qs)$	
Uniform on $\{1, \dots, n\}$	$\begin{cases} \dfrac{s-s^{n+1}}{n(1-s)}, & -1 \le s < 1. \\ 1, & s = 1 \end{cases}$	

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HW7/2 Follows from HW7/1 by noting the PGFs of NB(k,p) and Geo(p) with the help of the following result.

Thm. If X and Y are random variables taking values $0,1,2,\ldots$ with respective PGFs ϕ_X and ϕ_Y, then

$$X \stackrel{d}{=} Y$$

if and only if

$$\phi_X(s) = \phi_Y(s), \quad 0 \le s \le 1.$$

HW7/3 Similar to HW7/2.

HW7/4. The PGF of $\sum_{i=1}^{k} X_i$ is

$$\phi(s) = \prod_{i=1}^{k} \phi_i(s),$$

where ϕ_i is the PGF of X_i. Thus $\phi_i(s) = e^{\lambda_i(s-1)}$, $0 \le s \le 1$.

Hence $\phi(s) = e^{(s-1)\sum_{i=1}^{k}\lambda_i}$. Thus $X \sim \mathrm{Poi}\left(\sum_{i=1}^{k}\lambda_i\right)$.

HW7/5. Let ϕ be the PGF of X_1.

The assumption implies that ϕ^k is the PGF of Poisson (λ), that is,

$$\phi(s)^k = e^{\lambda(s-1)}, \quad 0 \le s \le 1.$$

Recalling that $\phi(s) \in [0,1]$ for $0 \le s \le 1$, the above implies that

$$\phi(s) = e^{\frac{\lambda}{k}(s-1)}, \quad 0 \le s \le 1.$$

Hence, $X_1 \sim \mathrm{Poi}\left(\frac{\lambda}{k}\right)$.

HW7/6. For $|s| \le 1$,

$$E(s^Z) = \sum_{n=0}^{\infty} E(s^Z \mid N=n) P(N=n)$$

$$= \sum_{n=0}^{\infty} E(s^{X_1+\cdots+X_n} \mid N=n) P(N=n)$$

$$= \sum_{n=0}^{\infty} E(s^{X_1+\cdots+X_n}) P(N=n)$$

$$= \sum_{n=0}^{\infty} \phi(s)^n P(N=n)$$

$$= E\left[\phi(s)^N\right]$$

$$= \psi \circ \phi(s).$$

HW 7/7 In this case, the PGF of Z is

$$\phi_Z(s) = \psi \circ \phi(s),$$

where ϕ and ψ are the PGFs of Ber(p) and Poi(λ), resp.

Recall that $\phi(s) = q + ps$ and $\psi(s) = e^{\lambda(s-1)}$ for $0 \le s \le 1$.

Thus, $\phi_Z(s) = e^{\lambda(q+ps-1)} = e^{\lambda p(s-1)}$, $0 \le s \le 1$.

Therefore, $Z \sim \text{Poi}(\lambda p)$.

In the problem where the # accidents in a city follows $\text{Poi}(\lambda)$ and each accident is reported with probability p independently, the # reported accidents follows $\text{Poi}(\lambda p)$. This is exactly what is done above.

HW7/8. For $-1 \le s \le 1$,

$$E(s^{XY}) = E(s^{XY} \mid Y=0)\, P(Y=0) + E(s^{XY} \mid Y=1)\, P(Y=1)$$

$$= (1-p) + E(s^{X})\, p$$

$$= p f(s) + 1 - p .$$

Thus, the PGF of XY is

$$\phi(s) = p f(s) + 1 - p, \quad -1 \le s \le 1.$$

HW7/9 If ϕ_n denotes the PGF of X_n, then

$$\phi_n(s) = \left(1 - p_n + p_n s\right)^n, \quad -1 \leq s \leq 1.$$

Fact: If $\alpha_n \in \mathbb{R}$ is such that $\alpha_n \to \alpha \in \mathbb{R}$, then

$$\left(1 + \frac{\alpha_n}{n}\right)^n \to e^{\alpha}.$$

Pf. Since $\alpha_n \to \alpha$, $\dfrac{\alpha_n}{n} \to 0$.

Hence $1 + \dfrac{\alpha_n}{n} \to 1$. In particular, $1 + \dfrac{\alpha_n}{n} > 0$ for n large enough.

For such n, write

$$\left(1+\frac{\alpha_n}{n}\right)^n = e^{n\log\left(1+\frac{\alpha_n}{n}\right)}.$$

Since $\displaystyle\lim_{x\to 0}\frac{1}{x}\log(1+x) = 1$,

it follows that

$$\lim_{n\to\infty}\frac{n}{\alpha_n}\log\left(1+\frac{\alpha_n}{n}\right) = 1$$

and hence $\displaystyle\lim_{n\to\infty} n\log\left(1+\frac{\alpha_n}{n}\right) = \alpha$. This completes the pf. $\boxed{\,}$

Rewrite

$$\phi_n(s) = \left(1 + \frac{np_n(s-1)}{n}\right)^n .$$

Since $np_n(s-1) \to \lambda(s-1)$, the above fact implies that

$$\lim_{n\to\infty} \phi_n(s) = e^{\lambda(s-1)}, \quad -1 \leq s \leq 1 .$$

Since the RHS is the PGF of $\mathrm{Poi}(\lambda)$, it follows that

$$\lim_{n\to\infty} P(X_n = k) = P(Y = k), \quad k = 0, 1, 2, \ldots,$$

where $Y \sim \mathrm{Poi}(\lambda)$.

HW 7/10. Let X and Y be independent RVs with resp PGFs f and g.

(a) The RV $X+Y$ has PGF fg.

(b) Toss a coin with probability of Heads p, independently of (X,Y).

Define
$$Z = \begin{cases} X, & \text{if Heads is obtained,} \\ Y, & \text{if Tails is } " \end{cases}.$$

Then $E(s^Z) = E(s^Z \mid \text{Heads})\, p + E(s^Z \mid \text{Tails})\,(1-p)$

$$= pf(s) + (1-p)\,g(s), \quad -1 \le s \le 1.$$

First of all $pX + (1-p)Y$ is not even a non-negative integer valued R.V. Ignoring this for a moment,

$$E\left(s^{pX+(1-p)Y}\right) = E\left(s^{pX}\right)E\left(s^{(1-p)Y}\right)$$

$$= E\left(\left(s^{p}\right)^{X}\right)E\left(\left(s^{(1-p)}\right)^{Y}\right)$$

$$= f\left(s^{p}\right)g\left(s^{1-p}\right), \quad 0 \le s \le 1.$$

Thus, $pX+(1-p)Y$ does NOT have the desired PGF.

(c) In view of flw7/6, $f \circ g$ is the PGF of

$$Y_1 + \dots + Y_X$$

where $(Y_1, Y_2, \dots)$ is a collection of iid RVs with PGF g, independent of X.

HW7/11. Note that

$$\sup_{|s|\le 1} \left| f_n(s) - f_\infty(s) \right| = \sup_{|s|\le 1} \left| \sum_{k=0}^{\infty} \left[P(X_n=k) - P(X_\infty=k) \right] s^k \right|$$

$$\le \sup_{|s|\le 1} \sum_{k=0}^{\infty} \left| P(X_n=k) - P(X_\infty=k) \right| \, |s^k|$$

$$\le \sum_{k=0}^{\infty} \left| P(X_n=k) - P(X_\infty=k) \right| .$$

To complete the soln, suffices to show that $\lim_{n\to\infty} \sum_{k=0}^{\infty} \left| P(X_n=k) - P(X_\infty=k) \right| = 0.$ $(*)$

Example: Say X_n is degenerate at n. Then,

$$\lim_{n\to\infty} P(X_n = k) = 0 \text{ for all } k \in \{0, 1, \ldots\}.$$

However, $$\sum_{k=0}^{\infty} |P(X_n = k) - 0| = 1 \text{ for all } n.$$

Recall that $|a-b| = a + b - 2(a \wedge b)$, $a, b \in \mathbb{R}$,

which is same as $a \wedge b = \dfrac{a+b-|a-b|}{2}$.

Using this, write

$$\sum_{k=0}^{\infty} \left| P(X_n = k) - P(X_\infty = k) \right| = \sum_{k=0}^{\infty} P(X_n = k) + \sum_{k=0}^{\infty} P(X_\infty = k)$$

$$- 2 \sum_{k=0}^{\infty} P(X_n = k) \wedge P(X_\infty = k)$$

$$= 2 \left[1 - \sum_{k=0}^{\infty} P(X_n = k) \wedge P(X_\infty = k) \right].$$

Fix $\varepsilon > 0$. Since $\sum_{k=0}^{\infty} P(X_\infty = k) = 1$, there exists

K such that $\sum_{k=0}^{K} P(X_\infty = k) \geq 1 - \frac{\varepsilon}{2}$ $\left(\sum_{k=K+1}^{\infty} P(X_\infty = k) \leq \frac{\varepsilon}{2} \right)$

Therefore,

$$\limsup_{n \to \infty} 2 \left[1 - \sum_{k=0}^{\infty} P(X_n = k) \wedge P(X_\infty = k) \right]$$

$$= 2 \left[1 - \liminf_{n \to \infty} \sum_{k=0}^{\infty} P(X_n = k) \wedge P(X_\infty = k) \right]$$

$$\leq 2\left[1 - \liminf_{n\to\infty} \sum_{k=0}^{K} P(X_n=k)\wedge P(X_\infty=k)\right.$$

$$\left[\begin{array}{l}\liminf\,(a_n+b_n) \\ \geq \ \liminf a_n + \liminf b_n\end{array}\right]$$

$$\left. - \liminf_{n\to\infty} \sum_{k=K+1}^{\infty} P(X_n=k)\wedge P(X_\infty \wedge k)\right]$$

$$= 2\left[1 - \sum_{k=0}^{K} P(X_\infty=k) - \liminf_{n\to\infty} \sum_{k=K+1}^{\infty} P(X_n=k)\wedge P(X_\infty=k)\right]$$

$$= 2\left[\sum_{k=K+1}^{\infty} P(X_\infty=k) - \liminf_{n\to\infty} \sum_{k=K+1}^{\infty} P(X_n=k)\wedge P(X_\infty=k)\right]$$

$$\leq 2 \sum_{k=K+1}^{\infty} P(X_\infty = k)$$

$$\leq \varepsilon .$$

Since ε is arbitrary, $(*)$ follows and completes the soln.

HW7/12. The "only if" part follows from a result done in class.

That is,

$$\lim_{n\to\infty} P(X_n = k) = P(Z = k)$$

implies that the PGF of X_n converge pointwise to that of Z on $(-1,1)$, and hence ϕ is the PGF of Z. Therefore,

$$\lim_{s\uparrow 1} \phi(s) = 1.$$

For the "if part", proceed as in the proof of the said theorem to get $1 \le n_1 < n_2 < \dots$ s.t.

$$(*) \quad p_k = \lim_{r\to\infty} P(X_{n_r} = k) \text{ exists for all } k \in \{0,1,\dots\}.$$

Follow the said proof to argue

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k, \quad -1 < s < 1.$$

Since we are given $\lim_{s\uparrow 1} \phi(s) = 1$, this means $\sum_{k=0}^{\infty} p_k = 1$.

Therefore there exists a rv Z with

$$P(Z=k) = p_k, \quad k \in \{0, 1, \dots\}.$$

Use $(*)$ and argue using a subsequence to complete the soln.

HW7/13.

(a) The PGF of Z_n is

$$\phi_n(s) = \left(E\left(s^{X_1 Y_1}\right)\right)^n.$$

Write

$$E\left(s^{X_1 Y_1}\right) = E\left(s^{X_1 Y_1} \mid Y_1 = 0\right) P(Y_1 = 0) + E\left(s^{X_1 Y_1} \mid Y_1 = 1\right) P(Y_1 = 1)$$

$$= 1 - \frac{1}{n} + \frac{1}{n} f(s),$$

where f is the PGF of X_1.

Thus, $$\phi_n(s) = \left(1 + \frac{f(s) - 1}{n}\right)^n \longrightarrow e^{f(s) - 1}, \quad n \to \infty.$$

Let $\phi(s) = e^{f(s)-1}$, $-1 < s < 1$.

Clearly $\lim_{s \uparrow 1} \phi(s) = e^{f(1)-1} = \underline{1}$.

HW7/12 implies that there exists a RV Z taking values $0, 1, \ldots$ s.t.

$$\lim_{n \to \infty} P(X_n = k) = P(Z = k).$$

Besides, Z has PGF ϕ.

(b) $P(Z=0) = \phi(0) = e^{f(0)-1} = e^{p_0 - 1}$.

$P(Z=1) = \phi'(0) = e^{f(0)-1} f'(0) = e^{p_0-1} p_1$.

$$\boxed{\text{Recall that if } \psi \text{ is the PGF of } Y, \text{ then } P(Y=0) = \psi(0), \quad P(Y=1) = \psi'(0) \text{ etc.}}$$

HW7/14. (a) If ϕ and ψ denote the resp PGFs of X, and Y_1, then the given hypotheses imply

$$\phi(s)^k = \psi(s)^k \quad , \quad 0 \le s \le 1 .$$

Since for $0 \le s \le 1$, $\phi(s), \psi(s) \in [0,1]$, it follows that

$$\phi(s) = \psi(s), \quad 0 \le s \le 1 .$$

Thus, $X_1 \overset{d}{=} Y_1$.

(b) Let f be the PGF of N. The given hypotheses imply

$$(*) \qquad f(\phi(s)) = f(\psi(s)) \; , \; 0 \le s \le 1 .$$

Since $P(N=0) < 1$, f is a strictly increasing fn on $[0,1]$, and hence one-one. Thus, $(*)$ implies that

$$\phi(s) = \psi(s), \quad 0 \le s \le 1.$$

Therefore, $X_1 \stackrel{d}{=} Y_1$.

HW7/15. Let X_n be the number obtained on the n-th roll. Then

$$Z = X_1 + \dots + X_N,$$

where N is the first roll yielding 6. For $-1 \le s \le 1$,

$$E(s^Z) = \sum_{n=1}^{\infty} E(s^Z \mid N=n)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} E(s^{X_1+\cdots+X_n} \mid N=n)\, P(N=n)$$

$$= \sum_{n=1}^{\infty} s^6 \underbrace{E(s^{X_1+\cdots+X_{n-1}} \mid N=n)}_{\left(\frac{1}{5}(s+\cdots+s^5)\right)^{n-1}} P(N=n)$$

$$= \sum_{n=1}^{\infty} s^6 \left(\frac{s + \cdots + s^5}{5} \right)^{n-1} P(N=n)$$

$$= s^6 \, E\left[\left(\frac{s + \cdots + s^5}{5} \right)^{N-1} \right].$$

Recall $E\left[t^{N-1}\right] = \dfrac{1/6}{1 - \frac{5}{6}t} = \dfrac{1}{6 - 5t}.$

Thus, the PGF of Z is

$$\phi(s) = s^6 \frac{1}{6-(s+\cdots+s^5)}\,; \quad -1 \le s \le 1.$$

Final Exam: Date TBA. Open Notes.

Full Marks: 70

Time: 3 hours.