20/10

Tutorial

HW1/1. For this experiment, the sample space is

$$\Omega = \{T, HT, HHT, HHHT, \ldots\},$$

and the probability assignment is

$$p(\underbrace{H\ldots H}_{n}T) = p^n q \quad \text{where } q = 1-p,$$

$$\text{for } n \in \{0,1,2,\ldots\}.$$

The event in the question is

$$A = \{T, HHT, HHHHT, \ldots\}.$$

Thus, $$P(A) = q + p^2 q + p^4 q + \ldots = \frac{q}{1-p^2}$$

$$= (1+p)^{-1}.$$

HW1/2. Number of ways of obtaining 15 as the sum of the numbers obtained in 9 rolls of a die

$= $ Coefficient of x^{15} in $(x + x^2 + \cdots + x^6)^9$

Write for $|x| < 1$,

$$(x + x^2 + \cdots + x^6)^9 = x^9 (1 + x + \cdots + x^5)^9$$
$$= x^9 (1 + \cdots + x^5)^9 (1-x)^9 (1-x)^{-9}$$

$$= x^9 (1-x^6)^9 (1-x)^{-9}$$

$$= x^9 \left(1 - 9x^6 + \binom{9}{2} x^{12} - \cdots - x^{54}\right)\left(1 + 9x + \binom{10}{2} x^2 + \binom{11}{3} x^3 + \cdots\right).$$

Therefore, # ways of getting a sum of 15

$$= \text{Coeff of } x^6 \text{ in } \left(1 - 9x^6 \cdots - x^{54}\right)\left(1 + 9x + \binom{10}{2} x^2 + \cdots\right)$$

$$= \binom{14}{6} - 9.$$

Hence, the probability of the given event

$$= 6^{-9}\left[\binom{14}{6} - 9\right].$$

HW1/3. Let A be the event that the number of Tails is div by 4.

Then, $P(A) = \dfrac{\# A}{2^n}$.

Clearly,

$$\# A = \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots + \binom{n}{4\left[\frac{n}{4}\right]},$$

where $[x]$ denotes the largest integer less than or equal to x.

Recall that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2\left[\frac{n}{2}\right]} = 2^{n-1}.$$

Recall the binomial thm:

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j .$$

Putting $x = i = \sqrt{-1}$, we get

$$(1+i)^n = \sum_{j=0}^{n} \binom{n}{j} i^j . \qquad (*)$$

Recall: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Equating the real parts in (*), we get

$$\Re\left((1+i)^n\right) = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots .$$

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots$$

$$2^{n-1} + \Re\left((1+i)^n\right) = 2\left[\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots\right]$$

That is,

$$\# A = 2^{n-2} + \frac{1}{2} Q\left((1+i)^n\right).$$

Write $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right).$$

DeMoivre's thm implies

$$(1+i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}\right).$$

Hence, $\Re\left((1+i)^n\right) = 2^{n/2} \cos \frac{n\pi}{4}$.

Hence $\# A = 2^{n-2} + 2^{\frac{n}{2}-1} \cos \frac{n\pi}{4}$.

Hence $P(A) = 2^{-n} \# A$

$$= \frac{1}{4} + 2^{-\frac{n}{2}-1} \cos \frac{n\pi}{4}.$$

For $\theta \in \mathbb{R}$,

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n \in \mathbb{N}.}$$

HW1/4. Let A denote the event that out of the 4 chosen cards, at least 2 have the same rank.

~~Clearly,~~ $\cancel{\# A = \binom{4}{2} 13 \binom{4}{2} \binom{50}{2}}$

~~Has double counting~~

Notice A^c is the event that no two of the chosen cards have the same rank. Hence,

$$\# A^c = \binom{13}{4} 4^4 ,$$

and thus,

$$P(A^c) = \frac{\binom{13}{4} 4^4}{\binom{52}{4}}.$$

Therefore, $P(A) = 1 - P(A^c) = 1 - \dfrac{\binom{13}{4} 4^4}{\binom{52}{4}}.$

__Convention:__ In the absence of a qualifier, draws are made "without replacement".

__HW1/5__. Probability of the given event

$$= \frac{22!\ 5!}{26!}.$$

HW 1/b

Without Replacement

Define the following events:

B = [At least one black ball is drawn]

W = [" " " white " " "]

R = [" " " red " " "].

The event in the question is $B \cap W \cap R$.

Note that

$$P(B \cap W \cap R) = 1 - P\left([B \cap W \cap R]^c \right)$$

5 Black
5 White
10 Red

$$= 1 - P\left(B^c \cup W^c \cup R^c \right)$$

inclusion
exclusion

$$1 - \Big[P(B^c) + P(W^c) + P(R^c) - P(B^c \cap W^c) - P(B^c \cap R^c) - P(W^c \cap R^c) + P(B^c \cap W^c \cap R^c) \Big]$$

$$= 1 - \frac{1}{\binom{20}{7}} \left[\binom{15}{7} + \binom{15}{7} + \binom{10}{7} - \binom{10}{7} \right]$$

$$= 1 - \frac{2\binom{15}{7}}{\binom{20}{7}}.$$

With replacement

$$1-\left[\left(\frac{15}{20}\right)^7+\left(\frac{15}{20}\right)^7+\left(\frac{10}{20}\right)^7-\left(\frac{10}{20}\right)^7-\left(\frac{5}{20}\right)^7-\left(\frac{5}{20}\right)^7\right]$$