27/10

Tutorial 2 (HW 1 contd.)

Revised weightages for Probability I:

Class test 1 (sometime in the week of 22 Nov, online) : 15%

Class test 2 (" " " " " 20 Dec, mode to be decided later) : 15%

Final exam : 70%

HW 1

HW1/7 : Denote

A_i = [The ith long part is joined with the i-th short part], $i = 1, \dots, 100$.

The event in question is $A_1 \cup \ldots \cup A_{100}$ whose probability can be calculated as follows using IEP:

$$P\left(\bigcup_{i=1}^{100} A_i\right) = \sum_{i=1}^{100} P(A_i) - \sum_{1 \le i < j \le 100} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le 100} P(A_i \cap A_j \cap A_k) - \ldots - P(A_1 \cap \ldots \cap A_{100})$$

$$= 100 \cdot \frac{1}{100} - \binom{100}{2} \frac{98!}{100!} + \binom{100}{3} \frac{97!}{100!} - \cdots - \frac{1}{100!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{100!} = \sum_{i=1}^{100} (-1)^{i+1} \frac{1}{i!}.$$

HW 1 | 8

For this experiment, the sample space Ω has $\binom{16}{10}$ equally likely outcomes.

If A denotes the event that no box gets more than 2 balls, then

$$\# A = \text{Coefficient of } x^{10} \text{ in } (1+x+x^2)^7 .$$

This is because

$$\text{Coefficient of } x^{10} \text{ in } (1+x+x^2)^7 = \# \{ (i_1, \dots, i_7) : i_j \in \{0,1,2\} \text{ and } i_1 + \dots + i_7 = 10 \}$$

$$= \# A .$$

For $|x| < 1$,

$$(1+x+x^2)^7 = \underbrace{(1+x+x^2)^7(1-x)^7}(1-x)^{-7}$$

$$= (1-x^3)^7(1-x)^{-7}$$

$$= \left(1 - 7x^3 + \binom{7}{2}x^6 - \binom{7}{3}x^9 + \cdots - x^{21}\right)\left(1 + 7x + \binom{8}{2}x^2 + \cdots\right)$$

Hence,

$$\# A = \binom{16}{6} - 7\binom{13}{6} + \binom{7}{2}\binom{10}{6} - \binom{7}{3}7 .$$

Thus,

$$P(A) = \frac{\binom{16}{6} - 7\binom{13}{6} + \binom{7}{2}\binom{10}{6} - \binom{7}{3}7}{\binom{16}{6}}.$$

HW 1.19. Let the 3 boxes be labeled as Box 1, Box 2 and Box 3, and

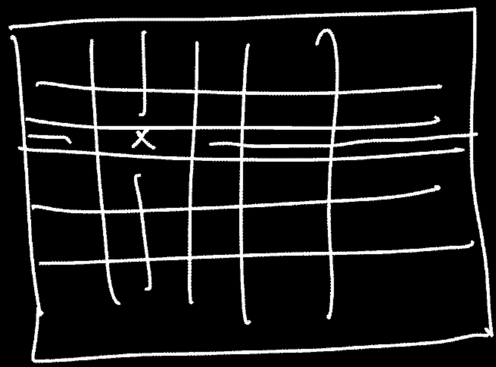
$A_i = [\text{Box } i \text{ is empty}]$, $i = 1, 2, 3$.

Using IEP,

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= \underbrace{3\left(\frac{2}{3}\right)^{10} - 3\left(\frac{1}{3}\right)^{10}}_{\text{Ans.}}$$

HW1 /10



Ans:

$$\frac{64 \times 14}{64 \times 63} = \frac{2}{9}.$$

HW1 /11

For this problem, the sample space Ω has n^n outcomes.

Ways in which exactly 1 box is empty

= # Ways in which $\underline{1}$ box is empty, $\underline{1}$ box gets 2 balls,

and the remaining $(n-2)$ boxes get 1 ball each

$= n(n-1)\binom{n}{2}(n-2)!$

Hence, Prob of exactly 1 box remaining empty

$$= n^{-n}\, n! \binom{n}{2}.$$

HW 1/12. Without replacement.

Probability that the largest number drawn is $M = \dfrac{\binom{M-1}{n-1}}{\binom{N}{n}}$, with the convention that $\binom{k}{r} = 0$ if $r > k$.

With replacement

$$P(\text{The largest number drawn is at most } M) = \frac{M^n}{N^n}.$$

Since

P(The largest number drawn is at most M)

= P(The .... at most M−1) + P(The .... exactly M),

therefore,

P(The largest number drawn is exactly M)

= P(the ... at most M) − P(The ... at most M−1)

$$= \frac{M^n - (M-1)^n}{N^n} \quad (\text{Ans}).$$

HW1/13

(a) $$\frac{\binom{n}{m} m!}{n^m}$$

$$\boxed{\# \text{ Functions from } \{1, \ldots, m\} \text{ to } \{1, \ldots, n\} \text{ is } n^m.}$$

(b) $n^{-n} n!$ (Here $m = n$)

HW1/14.

Ans: $p_n = \dfrac{[n/k]}{n}$,

where $[x]$ denotes the largest integer $\le x$.

$\lim_{n\to\infty} p_n = \dfrac{1}{k}$.

HW1/15.

(a) P(The same number appears on at least 2 dice)

$= 1 - P(\text{The numbers on the 3 dice are distinct})$

$$= 1 - \frac{\binom{6}{3} 3!}{6^3}.$$

(b) P(The same number appears on exactly 2 dice)

$$= \frac{6\binom{3}{2}5}{6^3}$$