10/11

HW 2

Class Test 1: November 24th, Wed
during 11:30 – 12:30.

Syllabus: Whatever is covered till then.

HW2/1

Define the events as follows:

D = [The event that the subject has the <u>disease</u>],

PR = [The test yields a <u>positive result</u>],

NR = [The test yields a <u>negative result</u>],

C = [The test result is <u>correct</u>].

It is given that C and D are independent events.

We are asked to compute $P(D|PR)$.

Note that $C = (D \cap PR) \cup (D^c \cap NR)$. (*)

The defn implies that

$$P(D|PR) = \frac{P(D \cap PR)}{P(PR)}$$

$$= \frac{P(D \cap PR)}{P(D \cap PR) + P(D^c \cap PR)}$$

(by (*)) $$= \frac{P(C \cap D)}{P(C \cap D) + P(D^c \cap C^c)}$$

$$\boxed{\begin{aligned} C &= (D \cap PR) \cup (D^c \cap NR). \\ \text{Hence, } C^c &= (D^c \cap PR) \cup (D \cap NR). \\ C^c \cap D^c &= D^c \cap PR. \end{aligned}}$$

(because C & D are indep, and hence so are C^c & D^c)

$$= \frac{P(C)P(D)}{P(C)P(D) + P(C^c)P(D^c)}$$

$$= \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.1 \times 0.99}$$

$$= \frac{9}{9+99} = \frac{1}{12}.$$

Caution: While $A \cap B$ makes sense as an event, $A|B$ DOES NOT. What does make sense is $P(A|B)$.

HW2/2

Let D, NR, PR and C be as defined in HW2/1.

Now, $P(D) = p$,

$P(C) = 0.9$.

(a) The conditional probability of interest is

$$P(D \mid PR).$$

Proceeding as in HW2/1, we get

$$P(D \mid PR) = \frac{P(D)\,P(C)}{P(D)\,P(C) + P(D^c)\,P(C^c)} = \frac{0.9\,p}{0.9p + 0.1(1-p)}.$$

Thus,

$$\lim_{p \to 0} P(D \mid PR) = 0.$$

(b) The conditional probability in qn is

$$P(D \mid NR) = \frac{P(D \cap NR)}{P(D \cap NR) + P(D^c \cap NR)}$$

$$= \frac{P(D \cap C^c)}{P(D \cap C^c) + P(D^c \cap C)}$$

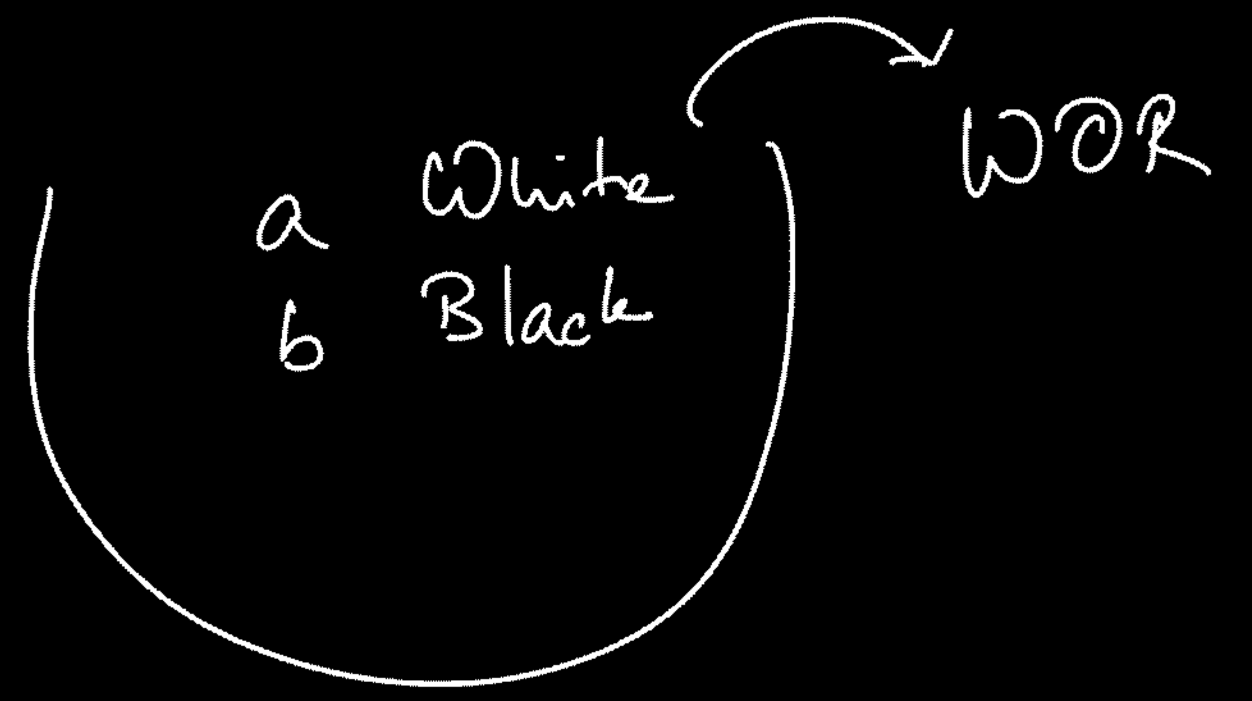
$$= \frac{P(D)\,P(C^c)}{P(D)\,P(C^c) + P(D^c)\,P(C)}$$

$$= \frac{0.1\,p}{0.1\,p + 0.9\,(1-p)}.$$

Thus,

$$\lim_{p \to 0} P(D \mid NR) = 0.$$

HW2/3



Let $n = a+b$.

Suppose all the n balls are drawn from the box, one by one, in a random order. The probability of getting a white ball in the last draw is the same as the probability of arriving at a situation where only white balls remain. This is because if and only if a white ball is drawn in the n-th draw, after ~~when~~ the last time a black ball was drawn, then all the remaining balls in the urn were white. Thus, the probability in question equals the following:

$$P(\text{Getting a white ball in the nth draw}) = \frac{a}{a+b}.$$

Polya: An urn has a white & b black balls. A typical outcome is a permutation of a many W's & b many B's. Therefore, for any $k \in \{1, \dots, n = a+b\}$,

$$P(\text{Getting white in the } k\text{-th draw})$$

$$= \frac{\#\text{ Permutations of } a \text{ W's } \& \ b \text{ B's with W in the } k\text{-th place}}{\#\text{ Permutations of } a \text{ W's } \& \ b \text{ B's}}$$

$$= \frac{\frac{(a+b-1)!}{(a-1)!\, b!}}{\frac{(a+b)!}{a!\, b!}} = \frac{a}{a+b}.$$

HW2/4. Easy application of Bayes' thm.

HW2/5. Ans: $\frac{1}{13}$.

HW2/6 (a) If $A_1 \cap A_2 = \emptyset$, then $P(A_1 \cap A_2) = 0$ and hence $P(A_1 | A_2) = 0$.

(b) If $A_1 \subset A_2$, then

$$P(A_1|A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = \frac{P(A_1)}{P(A_2)} = \frac{p_1}{p_2}.$$

(c) If $A_1 \supset A_2$, then

$$P(A_1|A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = \frac{P(A_2)}{P(A_2)} = 1.$$

HW2 15. Let $D = \{(i,j) : i \in \{\text{Spades, Hearts, Diamonds, Clubs}\}, j \in \{\text{Ace}, 2, 3, \ldots, 10, \text{Jack, Queen, King}\}\}$.

The sample space is

$$\Omega = \{\{u, v\} : u, v \in D \text{ and } u \neq v\}.$$

Then $\# \Omega = \binom{52}{2}$. Define the events

$A = [\text{The suits are different}]$, $B = [\text{The ranks are same}]$.

Then,

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{13\binom{4}{2} \Big/ \binom{52}{2}}{\binom{4}{2}13^2 \Big/ \binom{52}{2}} = \frac{1}{13}.$$

If $\Omega = \left\{(u,v) : u, v \in D \text{ and } u \neq v\right\}$, then $\# \Omega = 52 \times 51$. Convince yourself that $P(B|A)$ is still $1/13$.

HW2/7. Label the rooms as 1, ..., 50 and let

$$A_i = [\text{Room \# } i \text{ has full attendance}], \quad i = 1, \dots, 50.$$

If E is the event that at least one room has full attendance, then

$$E = \bigcup_{i=1}^{50} A_i.$$

Thus $P(E) = 1 - P(E^c)$

$$= 1 - P\left(\bigcap_{i=1}^{50} A_i^c\right).$$

It has been given that $A_1, \ldots, A_{50}$ are indep, and hence so are $A_1^c, \ldots, A_{50}^c$. Thus,

$$P(E) = 1 - \prod_{i=1}^{50} P(A_i^c) = 1 - \prod_{i=1}^{50} (1 - 0.8^{10})$$

$$= 1 - (1 - 0.8^{10})^{50}.$$

$X = \#$ Rooms with full atendance.

Write $X = \sum_{i=1}^{50} \mathbb{1}_{A_i}$.

Then $E(X) = \sum_{i=1}^{50} E(\mathbb{1}_{A_i}) = 50\, P(A_1)$

$= 50 \times 0.8^{10}$.

<u>Exc.</u> Find the distribution of X.