22/11

HW 2 (contd)

HW2/7. Let X denote the number of rooms with full attendance.

Qn.: What is the distribution of X?

Ans: Binomial $(50, 0.8^{10})$

HW2|8

(a) Yes

(b) Probability that the product of the n digits is even

$$= P(A_1^c \cup \cdots \cup A_n^c) = 1 - P(A_1 \cap \cdots \cap A_n)$$

(due to independence) $$= 1 - P(A_1) \cdots P(A_n) = 1 - 2^{-n}.$$

HW 2|9 : Proof by mathematical induction, as done in class for $c = 0$.

HW 2|10

(a) $RHS = \underbrace{P(A_1)\,P(A_2 \mid A_1)}\,P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap \dots \cap A_{n-1})$

$= \underbrace{P(A_1 \cap A_2)\,P(A_3 \mid A_1 \cap A_2)} \cdots P(A_n \mid A_1 \cap \dots \cap A_{n-1})$

$$\begin{aligned}
&= P(A_1 \cap A_2 \cap A_3)\, P(A_4 \mid A_1 \cap A_2 \cap A_3) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1}) \\
&\;\;\vdots \\
&= P(A_1 \cap \cdots \cap A_{n-1})\, P(A_n \mid A_1 \cap \cdots \cap A_{n-1}) \\
&= P(A_1 \cap \cdots \cap A_n) \\
&= \text{LHS}.
\end{aligned}$$

(b) $\text{RHS} = \dfrac{P(A_1 \cap A_n \mid A_2 \cap \cdots \cap A_{n-1})}{P(A_n \mid A_2 \cap \cdots \cap A_{n-1})}$

(defn of cond prob) $$= \frac{P(A_1 \cap A_n \cap A_2 \cap \dots \cap A_{n-1})}{\cancel{P(A_2 \cap \dots \cap A_{n-1})}} \; \frac{\cancel{P(A_2 \cap \dots \cap A_{n-1})}}{P(A_n \cap A_2 \cap \dots \cap A_{n-1})}$$

$$= \frac{P(A_1 \cap A_2 \cap \dots \cap A_n)}{P(A_2 \cap \dots \cap A_n)}$$

(defn) $$= P(A_1 \mid A_2 \cap \dots \cap A_n) = \text{LHS}.$$

<u>HW2/11</u>. Done in class.

HW2/12. Given : A and B are independent events.

$$RHS = P(C|A\cap B)\,P(B) + P(C|A\cap B^c)\,P(B^c)$$

$$= \frac{P(C|A\cap B)\,P(A)\,P(B) + P(C|A\cap B^c)\,P(A)\,P(B^c)}{P(A)}$$

(Since A and B are indep, so are A and B^c)

$$= \frac{P(C|A\cap B)\,P(A\cap B) + P(C|A\cap B^c)\,P(A\cap B^c)}{P(A)}$$

$$\text{(defn)} = \frac{P(A \cap B \cap C) + P(A \cap B^c \cap C)}{P(A)}$$

because $(A \cap B \cap C)$ and $(A \cap B^c \cap C)$ are disjt whose union is $A \cap C$

$$= \frac{P(A \cap C)}{P(A)} = P(C \mid A)$$

$$= \text{LHS}$$

HW 2/13. Need to show that $P(B \mid A_i) = P(B)$ for all i. Denote

$$p = P(B \mid A_1).$$

Then $P(B \mid A_i) = p$ for all i.

Since $A_1, A_2, \ldots$ are exhaustive, that is, $\bigcup_{i \geq 1} A_i = \Omega$,

$$P(B) = P\left(B \cap \left(\bigcup_{i \geq 1} A_i\right)\right)$$

$$= P\left(\bigcup_{i \geq 1} (A_i \cap B)\right)$$

$$\left(\begin{array}{l}\text{Because for } i \neq j, \\ (A_i \cap B) \cap (A_j \cap B) = \phi\end{array}\right) = \sum_{i \geq 1} P(A_i \cap B)$$

$$= \sum_{i \geq 1} P(B \mid A_i)\, P(A_i)$$

$$= \sum_{i \geq 1} p\, P(A_i)$$

$$= p \sum_{i \geq 1} P(A_i)$$

$$= p\, P\Big(\bigcup_{i \geq 1} A_i\Big)$$

$$= p\,P(\Omega)$$

$$= p$$

$$= P(B|A_j) \quad \text{for all } j.$$

Since $P(B) = P(B|A_j)$ for all j, it follows that B is independent of A_j for all j.

HW2|14

(a) $P(A^c|B) = 1 - P(A|B)$. True.

From the defn,

$$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)} = 1 - P(A|B).$$

(Since $(A \cap B)$ and $(A^c \cap B)$ are disjoint events whose union is B, $P(B) = P(A \cap B) + P(A^c \cap B)$)

(b) $P(A|B^c) = 1 - P(A|B)$. False.

Counter-example: Toss a coin with probability of heads $1/3$ twice and let

A = [The first toss yields a head]

B = [The second toss ---- head].

Then $P(A|B) = P(A) = P(A|B^c) = \frac{1}{3}$. The identity fails.

(c) $P(A|B^c) = P(A) - P(A|B)$. False.

In the previous counter-example, $RHS = 0$,
$LHS = \frac{1}{3}$.

Hence the claim again fails.

HW2|15. The events $A_1, \ldots, A_n$ are pairwise indep,
$P(A_i) = p$ for $i = 1, \ldots, n$,
and no three events can occur together.

Probability that none of $A_1, \dots, A_n$ occurs

$$= P(A_1^c \cap \dots \cap A_n^c)$$

$$= 1 - P(A_1 \cup \dots \cup A_n)$$

(inclusion exclusion) $$= 1 - \Big[\sum_{i=1}^{n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) - \dots \Big]$$

$$\begin{pmatrix}\text{because no} \\ \text{three can occur} \\ \text{together}\end{pmatrix} = 1 - \left[\sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)\right]$$

$$\begin{pmatrix}\text{pairwise} \\ \text{independence}\end{pmatrix} = 1 - np + \sum_{1 \le i < j \le n} P(A_i)\, P(A_j)$$

$$= 1 - np + \binom{n}{2} p^2 .$$