15/12

HW 4

HW4/2 The possible values for $X+Y$ are $0, 1, \dots, m+n$.

For a fixed $k \in \{0, 1, \dots, m+n\}$,

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i)$$

$$\text{(independence of } X, Y\text{)} = \sum_{i=0}^{k} P(X=i)\, P(Y=k-i)$$

$$\left(\text{where } q = 1-p\right) \quad = \sum_{i=0}^{k} \binom{m}{i} p^{i} q^{m-i} \binom{n}{k-i} p^{k-i} q^{n-(k-i)}$$

$$= \sum_{i=0}^{k} \frac{m!}{i!\,(m-i)!} \frac{n!}{(k-i)!\,(n-k+i)!} p^{k} q^{n+m-k}$$

$$= m!\,n!\,p^{k} q^{n+m-k} \sum_{i=0}^{k} \frac{1}{i!\,(k-i)!\,(m-i)!\,(n-k+i)!}$$

$$= \frac{m!\,n!}{k!\,(m+n-k)!} p^{k} q^{m+n-k} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \frac{(m+n-k)!}{(m-i)!\,(n-k+i)!}$$

$$= \frac{m!\; n!}{k!\,(m+n-k)!}\, p^{k} q^{m+n-k} \sum_{i=0}^{k} \binom{k}{i}\binom{m+n-k}{m-i}$$

Claim $\displaystyle\sum_{i=0}^{k} \binom{k}{i}\binom{m+n-k}{m-i} = \binom{m+n}{m}$

$\underbrace{\overbrace{\circ\ \circ\ \cdots\ \circ}^{k}}_{\text{Black}}\ \underbrace{\overbrace{\cdots\ \circ}^{m+n-k}}_{\text{White}}$ $\qquad (m+n) \longrightarrow$ Choose m

$$= \frac{m!\; n!}{k!\,(m+n-k)!} \binom{m+n}{m} p^{k} q^{m+n-k}$$

$$= \frac{(m+n)!}{k!\,(m+n-k)!} p^{k} q^{m+n-k}$$

$$= \binom{m+n}{k} p^{k} (1-p)^{m+n-k} .$$

That is, $X+Y \sim \mathrm{Bin}(m+n, p)$.

HW4/4

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= E(X^3) - E(X)E(X^2).$$

Since

$E(X) = 0$ and $E(X^3) = 0$,

we have $\mathrm{Cov}(X,Y) = 0$.

Since

$$P(X=1, Y=1) = P(X=1, X^2=1)$$

$$= P(X=1)$$

$$= \frac{1}{2N+1}.$$

$$P(X=1) = \frac{1}{2N+1}$$

$$P(Y=1) = P(X=1) + P(X=-1) = \frac{2}{2N+1}.$$

Clearly, $P(X=1, Y=1) \neq P(X=1)\,P(Y=1)$.

Hence, X and Y are dependent.

HW4/5(b).

$X \sim \mathrm{Geo}(p_1)$

$Y \sim \mathrm{Geo}(p_2)$

independent.

Intuitively $\min\{X, Y\} \sim \mathrm{Geo}\left(1 - (1-p_1)(1-p_2)\right)$.

First note that $\min\{X, Y\}$ takes values in $\{1, 2, \ldots\}$.

For $n \geq 1$,

$$P\left[\min\{X,Y\} \geq n\right] = P(X \geq n,\ Y \geq n)$$

(independence) $$= P(X \geq n)\, P(Y \geq n)$$

$$= (1-p_1)^{n-1} \ (1-p_2)^{n-1}$$

$$= \left[(1-p_1)(1-p_2) \right]^{n-1}.$$

Thus,

$$P\left[\min\{X,Y\} = n\right] = P\left[\min\{X,Y\} \geq n\right] - P\left[\min\{X,Y\} \geq n+1\right]$$

$$= \left[(1-p_1)(1-p_2)\right]^{n-1} - \left[(1-p_1)(1-p_2)\right]^{n}$$

$$= \left[(1-p_1)(1-p_2)\right]^{n-1} \left[1 - (1-p_1)(1-p_2)\right]$$

HW4/6. Fix $n \geq 1$ and write

$$X_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\text{Box } i \text{ is empty}). \quad (*)$$

Thus,

$$E(X_n) = \frac{1}{n} \sum_{i=1}^{n} P(\text{Box } i \text{ is empty})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n.$$

Therefore, $\lim_{n\to\infty} E(x_n) = e^{-1}$.

Using $(*)$ once again,

$$\operatorname{Var}(X_n) = \frac{1}{n^2} \operatorname{Var}\left(\sum_{i=1}^{n} \mathbb{1}(\text{Box } i \text{ is empty}) \right)$$

$$= \frac{1}{n^2} \left[\sum_{i=1}^{n} \operatorname{Var}\left(\mathbb{1}(\text{Box } i \text{ is empty}) \right) + 2 \sum_{1 \le i < j \le n} \operatorname{Cov}\left(\mathbb{1}(\text{Box } i \text{ is empty}), \mathbb{1}(\text{Box } j \text{ is empty}) \right) \right]$$

$$\mathrm{Var}\left(\mathbb{1}(\text{Box } i \text{ is empty})\right) = \left(\frac{n-1}{n}\right)^n - \left(\frac{n-1}{n}\right)^{2n}$$

$$\left[\text{Recall } \mathbb{1}(\qquad) \sim \text{Bernoulli}\left(\left(\frac{n-1}{n}\right)^n\right)\right]$$

For $1 \le i < j \le n$,

$$\mathrm{Cov}\left(\mathbb{1}(\text{Box } i \text{ is empty}), \mathbb{1}(\text{Box } j \text{ is empty})\right)$$

$$= E\left[\mathbb{1}(\ i \ldots)\,\mathbb{1}(\ j \ldots)\right] - \left(\frac{n-1}{n}\right)^{2n}$$

$$= \mathbb{E}\left[\mathbb{1}(\text{Boxes } i \text{ and } j \text{ are empty})\right] - \left(\frac{n-1}{n}\right)^{2n}$$

$$= \left(\frac{n-2}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}.$$

Thus,

$$\mathrm{Var}(X_n) = \frac{1}{n^2}\left[n\left(\left(\frac{n-1}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}\right) + n(n-1)\left(\left(\frac{n-2}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}\right)\right]$$

$$= \frac{1}{n}\left(\left(\frac{n-1}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}\right) + \frac{n-1}{n}\left(\left(\frac{n-2}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}\right)$$

$$\frac{1}{n} \to 0, \qquad \left(\frac{n-1}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n} \to e^{-1} - e^{-2}$$

$$\frac{1}{n}\left(\left(\frac{n-1}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n}\right) \to 0$$

$$\frac{n-1}{n} \to 1, \qquad \left(\frac{n-2}{n}\right)^{n} = \left(1-\frac{2}{n}\right)^{n} \to e^{-2}, \qquad \left(\frac{n-1}{n}\right)^{2n} \to e^{-2}$$

$$\left(\frac{n-2}{n}\right)^{n} - \left(\frac{n-1}{n}\right)^{2n} \to 0$$

Thus, $\lim\limits_{n\to\infty} \mathrm{Var}(X_n) = 0$.

HW4/9. Define

$Y_1 = 1$.

$Y_2 =$ Number of throws needed after the 1 st throw for the second box to have at least one ball.

$Y_3 =$ Number of throws needed after the $(Y_1 + Y_2)$-th throw for the third box ..... ball.

⋮

$Y_k =$ Number .... $(Y_1 + \dots + Y_{k-1})$-th throw for the k-th box... ball.

Thus, $X = Y_1 + \cdots + Y_k$.

Note that $Y_2 \sim \text{Geometric}\left(\frac{N-1}{N}\right)$;

$Y_3 \sim \text{Geometric}\left(\frac{N-2}{N}\right)$,

$\vdots$

$Y_k \sim \text{Geometric}\left(\frac{N-(k-1)}{N}\right)$.

$$\underbrace{\sqcup \; \odot \; \cdots \; \sqcup}_{N}$$

1st toss $\to p$ H

$\frac{1}{p}$ tosses ! H

Thus, $E(X) = \sum_{i=1}^{k} E(Y_i) = 1 + \sum_{i=2}^{k} \frac{N}{N-(i-1)}$.

Check that $Y_2, \ldots, Y_n$ are independent.

Hence

$$\mathrm{Var}(X) = \sum_{i=1}^{k} \mathrm{Var}(Y_i) = \sum_{i=2}^{k} \mathrm{Var}(Y_i).$$

Exc: Complete this.

Class Test 2: Dec 22nd 11:30. Tutorial on Dec 21st.