Tutorial

21/12

HW4/12(e)

$X =$ min of the n numbers drawn.

$Y =$ max " " " " ".

Fix $1 \leq i < j \leq N$.

$P(i \leq X, Y \leq j) = P(\text{All the chosen numbers are in } [i, j])$

$$= \left(\frac{j-i+1}{N} \right)^n .$$

$$\begin{aligned}
P(X=i, Y=j) &= P(X=i, Y\le j) - P(X=i, Y\le j-1) \\
&= P(i\le X, Y\le j) - P(i+1\le X, Y\le j) \\
&\quad - \left[P(i\le X, Y\le j-1) - P(i+1\le X, Y\le j-1) \right] \\
&= \left(\frac{j-i+1}{N} \right)^n - \left(\frac{j-i}{N} \right)^n - \left(\frac{j-i}{N} \right)^n + \left(\frac{j-i-1}{N} \right)^n .
\end{aligned}$$

Thus, the joint PMF of X and Y is given by

$$P(X=i, Y=j) = \begin{cases} \left(\frac{j-i+1}{N}\right)^n - 2\left(\frac{j-i}{N}\right)^n + \left(\frac{j-i-1}{N}\right)^n, & 1 \le i < j \le N, \\ \frac{1}{N^n}, & 1 \le i = j \le N, \\ 0, & \text{o.w.} \end{cases}$$

HW4|15. The PMF of X is

$$P(X=k) = \frac{\binom{W}{k}\binom{B}{n-k}}{\binom{W+B}{n}}, \quad k = 0, 1, \ldots, W,$$

with the convention that $\binom{i}{j} = 0$ if $j < 0$ or $j > i$.

Let us label the white balls as $1, \ldots, W$. Define for $1 \le i \le W$

$$X_i = \mathbb{1}_{(\text{The white ball labelled } i \text{ has been drawn})}.$$

Then, $$X = \sum_{i=1}^{W} X_i .$$

Thus, $$E(X) = \sum_{i=1}^{W} E(X_i)$$

$$= \sum_{i=1}^{W} P(\text{White ball labeled } i \text{ has been drawn})$$

$$= \sum_{i=1}^{W} \frac{\binom{W+B-1}{n-1}}{\binom{W+B}{n}}$$

$$= \sum_{i=1}^{W} \frac{n}{W+B} = \frac{nW}{W+B}.$$

$$\frac{(W+B-1)!}{(n-1)!\,\cancel{(W+B-n)!}} \quad \frac{n!\,\cancel{(W+B-n)!}}{(W+B)!}$$

HW4/11

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th sample unit} \le \frac{N}{2}, \\ 0 & \text{o.w.} \end{cases}$$

$$X_j = \begin{cases} 1 & \text{--- } j\text{-th --- ---}, \\ 0 & \text{o.w.} \end{cases}$$

HW5/3.

$$\mathrm{Cov}(U,V) = \mathrm{Cov}(X+Y, X-Y)$$

$$= \mathrm{Cov}(X,X) - \cancel{\mathrm{Cov}(X,Y)} + \cancel{\mathrm{Cov}(Y,X)} - \mathrm{Cov}(Y,Y)$$

$$= Var(X) - Var(Y)$$

$$= 9 \cdot \frac{1}{3} \cdot \frac{2}{3} - 2$$

$$= 0.$$

Hence $Corr(U,V) = 0$.

HW5|4. The possible values of Y are $0, 1, 2, \ldots$.

For $n \in \{0, 1, 2, \dots\}$,

$$P(Y=n) = \sum_{k=0}^{\infty} P(Y=n, X=k)$$

$$= \sum_{k=0}^{\infty} P(Y=n \mid X=k) P(X=k)$$

(Since Given $X=k$, $Y \sim Bin(k, p)$) $= \sum_{k=n}^{\infty} P(Y=n \mid X=k) P(X=k)$

$$= \sum_{k=n}^{\infty} \binom{k}{n} p^{n} (1-p)^{k-n} \frac{e^{-\lambda} \lambda^{k}}{k!}$$

$$= p^{n} e^{-\lambda} \sum_{k=n}^{\infty} \frac{\cancel{k!}}{n!\,(k-n)!} (1-p)^{k-n} \frac{\lambda^{k}}{\cancel{k!}}$$

$$(j = k-n) \qquad = \frac{p^{n} e^{-\lambda}}{n!} \sum_{j=0}^{\infty} \frac{(1-p)^{j} \lambda^{n+j}}{j!}$$

$$= \frac{(\lambda p)^n e^{-\lambda}}{n!} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!}$$

$$= \frac{(\lambda p)^n}{n!} e^{-\lambda} e^{\lambda(1-p)}$$

$$= e^{-\lambda p} \frac{(\lambda p)^n}{n!} .$$

Thus, $Y \sim Poi(\lambda p)$.

When $X \sim Bin(n, r)$, we expect the dist of Y to be $Bin(n, pr)$ because of the following intuition. Toss a coin with probability of Heads r n times. Count each Heads obtained with probability p, independently of the other Heads. Let X denote the number of H actually obtained, and Y be those counted.

Exc. Calculate the distn to verify the above intuition.

HW5|5. Let

$A = $ [All the rolls fill the first 6 given even numbers].

Clearly X takes values in $\{1, 2, 3, \ldots\}$. For $n \geq 1$,

$$P(X=n \mid A) = \frac{P([X=n] \cap A)}{P(A)}$$

$$= \frac{P(\text{the first } (n-1) \text{ rolls yield 2 or 4, } n\text{th roll gives } 6)}{P(A)}$$

$$= \frac{\left(\frac{1}{3}\right)^{n-1} \frac{1}{6}}{P(A)}$$

The conditional distribution of X given A is Geo$(2/3)$.

Exc. Calculate $P(A)$ to verify the above.

Thus,

$$E(X \mid A) = \frac{3}{2}.$$

Wrong intuition: Since each of 2, 4, 6 are equally likely, X

the conditional dist of X given A is Geo$\left(\frac{1}{3}\right)$.

HW5/6.

Let X_i be the number obtained on the i-th roll, and N be the roll on which the first 6 is obtained. Let Z be the random variable in the question, that is, the sum of all the numbers obtained.

Then,

$$Z = X_1 + \cdots + X_N.$$

Conditioning on N,

$$E(Z) = \sum_{n=1}^{\infty} E(Z \mid N=n)\, P(N=n).$$

Conditionally on $[N=n]$, $X_1, X_2, \ldots X_{n-1}$ are uniformly distributed on $\{1, \ldots, 5\}$ and $X_n = 6$.

Thus, $$E(Z \mid N=n) = \sum_{i=1}^{n} E(X_i \mid N=n) = 3(n-1) + 6.$$

$$E(Z) = \sum_{n=1}^{\infty} \left[3(n-1) + 6\right] P(N=n)$$

$$= E\left[3(N-1)+6\right]$$

Recall

$$\sum_{n \in \{\text{Possible values}\}} f(n)\, P(N=n) = E\left[f(N)\right]$$

$$= 3E(N) + 3$$

$$= 3 \times 6 + 3$$

$$= 21.$$

HW5/8(b).

Let X be the number of tosses needed to get 2 consecutive H.

Write

$$E(X) = E\left(X\,\mathbb{1}(\text{First toss is T})\right) + E\left(X\,\mathbb{1}\left(\begin{array}{l}\text{First toss gives H,}\\ \text{second T}\end{array}\right)\right)$$
$$+ E\left(X\,\mathbb{1}(\text{First 2 tosses give H})\right)$$

$$= E\left(X \mid \text{first toss is } T\right) P\left(\text{first toss is } T\right)$$

$$+ E\left(X \mid \text{first toss is } H,\ \text{second toss is } T\right) P\left(\qquad \right)$$

$$+ E\left(X \mid \text{first 2 tosses } H\right) P\left(\qquad \right)$$

$$= \left(1 + E(X)\right)(1-p) + \left(2 + E(X)\right) p(1-p) + 2p^2$$

In other words, we get the equation

$$E(X) = (1-p)\big(1 + E(X)\big) + p(1-p)\big(2 + E(X)\big) + 2p^2 .$$

$$\underbrace{\big(1 - (1-p) - p(1-p)\big)}_{\substack{= \\ p - p(1-p) \\ = p^2}} E(X) = 1 - p + 2p(1-p) + 2p^2$$

$$E(X) = \frac{1 - p + 2p(1-p) + 2p^2}{p^2} .$$

<u>Exc.</u> Try HW 5/8 with $p = 1/2$.

<u>Exc.</u> Toss a coin with probability of H p till the first H is obtained. Denote by X the number of tosses needed. Calculate $E(X)$. [<u>Ans</u>: $\frac{1}{p}$].

Write $E(X) = E(X \mid \text{First toss gives } H)\, P(\text{First toss is } H)$

$$+ E(X \mid \text{First toss gives } T)\, P(\text{First toss gives } T)$$

$$= 1\, p + \left(1 + E(X)\right)(1-p).$$

$p\,E(X) = 1$. Hence $E(X) = \dfrac{1}{p}$.

HW5/14

(a) $P(\min(X_1, \dots, X_n) \geq k)$ $\quad (k \in \{1, 2, \dots\})$

$= P(X_1 \geq k, X_2 \geq k, \dots\dots, X_n \geq k)$

$= P(X_1 \geq k) \dots P(X_n \geq k) = (1-p)^{k-1} \cdots (1-p)^{k-1} = \left((1-p)^n\right)^{k-1}$

Thus, $\min\{X_1, \dots, X_n\} \sim \mathrm{Geo}\left(1 - (1-p)^n\right)$.

(b) Ans: Negative Bin (n, p).

Exc. Check.

HW 5/15.

Ans. Poisson $(\lambda_1 + \cdots + \lambda_n)$.

Induction on n.

HW 6

HW 6/2.

(a) For $n \geq 1$,

$$P(T > 2n) = P(\text{Till time } 2n,\ 0 \text{ has not been visited})$$

$$= P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$$

$$\left(\text{Done in the class}\right) = P(S_{2n} = 0)$$

$$= \binom{2n}{n} 2^{-2n}.$$

(b) Let A be the event that the random walk never visits the origin (after time 0). We have to show

$$P(A) = 0.$$

For all n,

$$A \subseteq [T > 2n].$$

Therefore,

$$P(A) \leq P(T > 2n).$$

Thus, it suffices to show

$$\lim_{n \to \infty} P(T > 2n) = 0. \qquad (*)$$

In (a), we showed

$$P(T > 2n) = \binom{2n}{n} 4^{-n}$$

$$= \frac{(2n)!}{(n!)^2} 4^{-n} .$$

Recall: Stirling: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n , n \to \infty .$

As $n \to \infty$,

$$P(T > 2n) \sim \frac{\sqrt{2\pi\, 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^{2}}\, 4^{-n}$$

$$= \frac{\cancel{2}\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{\cancel{2}\pi n\left(\frac{n}{e}\right)^{2n}}\, 4^{-n}$$

$$= \frac{\cancel{2^{2n}}}{\sqrt{n\pi}}\, \cancel{4^{-n}} = \frac{1}{\sqrt{n\pi}}.$$

That is,

$$\lim_{n\to\infty} \sqrt{n\pi}\, P(T > 2n) = 1 .$$

Therefore

$$\lim_{n\to\infty} P(T>2n) = \lim_{n\to\infty} \left(\sqrt{n\pi}\, P(T>2n)\right) \frac{1}{\sqrt{n\pi}}$$

$$= 0 .$$

This proves (*) which shows $P(A) = 0$, as desired.

(c) If T has an expectation,

$$E(T) = \sum_{n=1}^{\infty} P(T \geq n)$$

$$\geq \sum_{n=1}^{\infty} P(T > 2n).$$

Since $\lim_{n \to \infty} \sqrt{n}\, P(T > 2n) = \frac{1}{\sqrt{\pi}}$,

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$,

Ratio test tells us $\sum_{n=1}^{\infty} P(T > 2n) = \infty$.

Thus, T does not have an expectation.

HW6 1)

Write

$$T_n = \sum_{j=0}^{n} \mathbb{1}(S_j = 0).$$

Thus,

$$E(T_n) = \sum_{j=0}^{n} P(S_j = 0) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} P(S_{2i} = 0)$$

$$= \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{2i}{i} 4^{-i} \underset{\substack{\uparrow \\ \text{Prove.}}}{\sim} \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{1}{\sqrt{\pi i}}$$

<u>Hint</u>. Use the estimate for $\binom{2i}{i} 4^{-i}$ obtained in HW6/2, and then use the integral approximation.

HW 6/3.

Hint: Recall that

$$P(M \geq r) = P(S_n = r) + 2P(S_n > r).$$

HW 6/7. Define a random walk as follows. Given a string of left and right braces, take a positive step for each left brace and a negative step for each right brace. Thus,

P(Getting a legitimate arrangement of n left braces & n rt braces)

$$= P(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0 \mid S_{2n} = 0).$$

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ways of arranging n left braces & n right braces to get a legitimate expression

= # Paths from $(0,0)$ to $(2n, 0)$ that never touch -1.

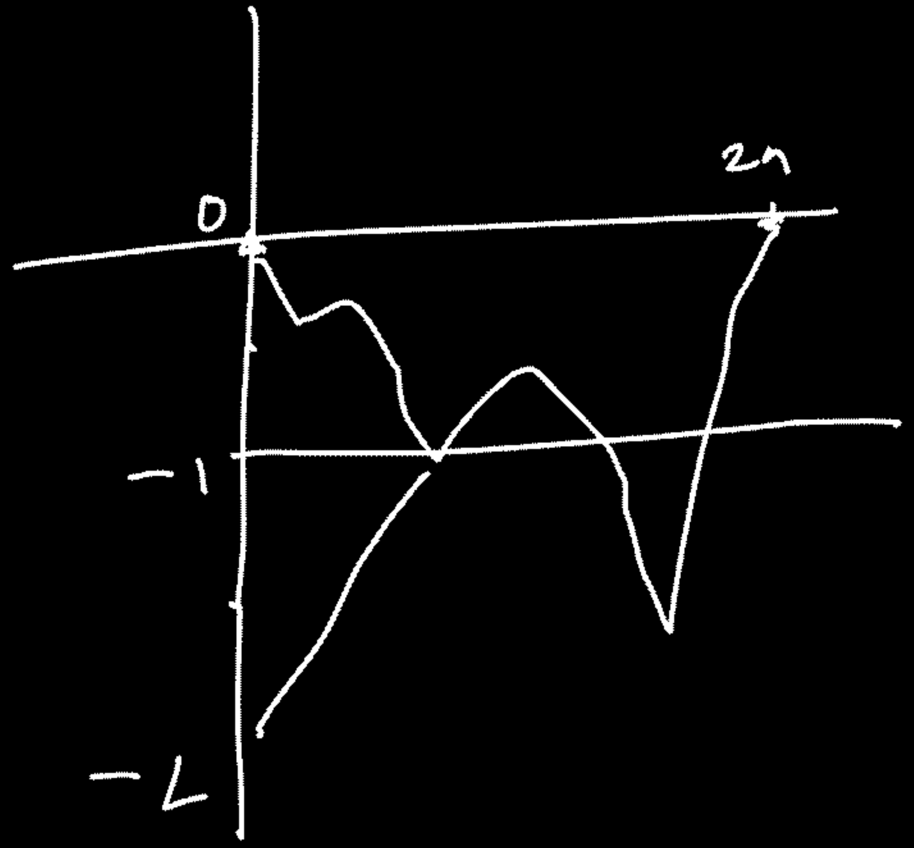
$= \#$ Paths from $(0,0)$ to $(2n,0)$

$- \#$ Paths from $(0,0)$ to $(2n,0)$ that touch -1

(Reflection principle) $\#$ Paths from $(0,0)$ to $(2n,0)$

$- \#$ Paths from $(0,-2)$ to $(2n,0)$

$$= \binom{2n}{n} - \binom{2n}{n+1}$$



$$= \frac{(2n)!}{(n!)^2} - \frac{(2n)!}{(n+1)!\,(n-1)!}$$

$$= \frac{(2n)!}{n!\,(n+1)!} \left[(n+1) - n \right]$$

$$= \frac{(2n)!}{n!\,(n+1)!} .$$

Since # Ways of arranging n left braces & n right braces $= \binom{2n}{n}$,

$$P(\text{Getting a legitimate expression}) = \frac{1}{n+1}.$$

$$P(S_1 \geq 0, \ldots, S_{2n} \geq 0 \mid S_{2n} = 0) = \frac{1}{P(S_{2n}=0)} P(S_1 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} = 0)$$

$$= \frac{\#\text{ Paths from } (0,0) \text{ to } (2n,0) \text{ that don't touch } -1}{\#\text{ Paths from } (0,0) \text{ to } (2n,0).}$$

HW6/9

For $n \geq 1$, S_n takes values in $-n, -n+2, \ldots, n-2, n$.

That is, S_n takes those values in $\{-n, \ldots, n\}$ that have the same parity as n. Fix $k \in \{-n, \ldots, n\}$ which has the same parity as n. Then

$$P(S_n = k) = P\left(\text{Of the first } n \text{ steps, } \frac{n+k}{2} \text{ are +ve and } \frac{n-k}{2} \text{ are -ve}\right)$$

$$= \binom{n}{\frac{n+k}{2}} p^{(n+k)/2} (1-p)^{(n-k)/2}.$$

If X is the number of +ve steps in the first n, then $S_n = X - (n - X) = 2X - n$.

Thus,

$$E(S_n) = 2E(X) - n = 2np - n$$
$$= 2n\left(p - \tfrac{1}{2}\right).$$

HW6/10.

$$P(M \geq k, S_n = j)$$

$$= p^{(n+j)/2}(1-p)^{(n-j)/2} \text{ \# Paths from } (0,0) \text{ to } (n,j) \text{ that touch the horizontal line } k.$$

Use Reflection principle. Exc: Complete!