My research interests lies in the field of Algebraic Topology and Deformation theory of algebraic structures and related fields.

**Thesis work**

In my thesis we developed an algebraic deformation theory, over commutative local algebra base, for Leibniz algebras and its homomorphisms and gave a concrete construction of a formal deformation which induces all other formal deformations of a given Leibniz algebra satisfying some cohomological condition, which is unique at the infinitesimal level — the so called “Versal deformation”.

**Motivation and Background**

Deformation theory dates back at least to Riemann’s 1857 memoir on abelian functions in which he studied manifolds of complex dimension one and calculated the number of parameters (called moduli) upon which a deformation depends. The modern theory of deformations of structures on manifolds was developed extensively by Frölicher-Nijenhuis [FN57], Kodaira-Spencer [KS58b], [KS58a], Kodaira-Nirenberg-Spencer [KNS58], and Spencer [Spe62a], [Spe62b], [Spe65].

The study of deformations of algebraic structures was initiated by M. Gerstenhaber through his monumental works [Ger63], [Ger64], [Ger66], [Ger68], [Ger74]. He introduced deformation theory for associative algebras and remarked that his methods would extend to any equationally defined algebraic structure. The basic theorems and features of algebraic deformation theory are all due to him. A comparative study of algebraic and analytic deformation theory is given in [Pip67].

The theory of Gerstenhaber was extended to Lie algebras by A. Nijenhuis and R. W. Richardson, Jr. [NR66], [NR67a], [NR67b]. The deformation theory of Hopf algebras, which relates to quantum groups, was studied by M. Gerstenhaber and S. D. Schack in [GS90]. An algebraic deformation theory for associative algebra homomorphisms was developed by Gerstenhaber and Schack [GS83, GS85].

More recent results on deformation theory following Gerstenhaber are obtained in [MM02], [Yau06], [Yau07], [Man07], [Yau08]. Gerstenhaber’s theory was generalized in [Bal97] by D. Balavoine to develop formal one parameter deformation theory for algebras over any quadratic operad, which includes all the classical cases. He also deduced formal one parameter deformation theory of Leibniz algebras from his theory.

Although formal deformation theory was developed in various categories following Gerstenhaber and computations were made, but the question of obtaining all non-equivalent deformations of a given object was not properly discussed for a long time. The right approach to this is the notion of versal deformation — a
deformation which includes all non-equivalent ones. The existence of such a versal deformation for algebraic categories follows from the work of M. Schlessinger [Sch68].

For Lie algebras it was worked out in detail, in [Fia88], [FF99], following the general construction in deformation theory as in [Ill71, Pal76, Lau79, GM88, Kon94]. It turns out that (under some minor cohomology restrictions) there exists a versal element, which is universal at the infinitesimal level. In the thesis we gave a concrete construction of versal deformation for Leibniz algebras [FMM08]. Leibniz algebras are some new type of algebras which are algebras over the quadratic operad Leib [Lod01], for which there is a Koszul dual. These algebras was introduced by J.-L. Loday [Lod93, Lod97] in connection with the study of periodicity phenomenon in algebraic $K$-theory, as a non-antisymmetric analogue of Lie algebras. A (co)homology theory associated to Leibniz algebras has been developed by J.-L. Loday and T. Pirashvili [LP93].

**Thesis Summary**

We introduce by using Leibniz cohomology certain cohomology modules associated to a Leibniz algebra homomorphism, which will be relevant in the discussion of deformation of Leibniz algebra homomorphisms. Next, we introduce the notion of deformations of Leibniz algebras and Leibniz algebra homomorphisms over a commutative local algebra base with multiplicative identity, and define infinitesimal deformation and other basic definitions related to deformations of a Leibniz algebra. We give a construction of an infinitesimal deformation $\eta_1$ of a Leibniz algebra $L$ for which $\dim(HL^2(L;L))$ is finite. Subsequently we show that this infinitesimal deformation is universal among the infinitesimal deformations of $L$ with finite dimensional local algebra base.

We also prove a necessary and sufficient criterion for equivalence of two infinitesimal deformations of a Leibniz algebra. To this end we introduce the notion of infinitesimal deformations of Leibniz algebra homomorphisms and obtain a necessary and sufficient condition for equivalence of two infinitesimal deformations in this case.

Then we address the question of extending a given deformation $\mathcal{D} = (\lambda, \mu; f_{\lambda\mu})$ of a Leibniz algebra homomorphism $f : L \rightarrow M$ with a given base to a larger base. This extension problem can be described as follows. Suppose $\mathcal{D} = (\lambda, \mu; f_{\lambda\mu})$ is a given deformation of a Leibniz algebra homomorphism $f : L \rightarrow M$ with local base $A$. Let

\[ 0 \rightarrow M_0 \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0 \]

be a given finite dimensional extension of $A$ by $M_0$. The problem is to obtain
condition for existence of a deformation $\tilde{D}$ of $f$ with base $B$ which extends the given deformation, that is, $p_\ast \tilde{D} = D$. We shall measure the possible obstructions that one might encounter in the above extension process as certain 3-dimensional cohomology classes, vanishing of which is a necessary and sufficient condition for an extension to exist. The set of equivalence classes of possible extensions of a given deformation $\lambda$ of $L$ with base $A$, admits certain natural actions and we shall investigate their relationship. We first take up the case of extending deformations of Leibniz algebras and then consider the relative problem of extending deformations of Leibniz algebra homomorphisms. Then we study formal deformations and obtain a necessary condition for non-triviality of a formal deformation. The results discussed above will also enable us to obtain a sufficient criterion for existence of a formal deformation with a given differential and infinitesimal part. Lastly we give definition of a versal deformation.

Next we give a construction of versal deformation of a given Leibniz algebra $L$ with $\dim(HL^2(L; L)) < \infty$. We begin with the universal infinitesimal deformation $\eta_1$ of $L$ with base $C_1$, and consider the extension problem for the trivial extension of $\mathbb{K}$ by the dual of $HL^2(L; L)$, to get a finite dimensional extension $\eta_2$ with base $C_2$. We kill off the possible obstruction associated to the extension problem for a specific extension of $C_1$ to obtain $\eta_2$ with base $C_2$. We repeat this procedure successively to get a sequence of finite dimensional extensions $\eta_k$ with base $C_k$. The projective limit $C = \lim_{\leftarrow k} C_k$ is a complete local algebra and $\eta = \lim_{k \to \infty} \eta_k$ is a formal deformation of $L$ with base $C$. We show that the algebra base $C$ can be described as a quotient of the formal power series ring over $\mathbb{K}$ in finitely many variables. Finally we prove that the formal deformation $\eta$ is a versal deformation of $L$ with base $C$.

It is well known that the construction of one parameter deformations of various algebraic structures, like associative algebras or Lie algebras, involves certain conditions on cohomology classes arising as obstructions. These condition are expressed in terms of Massey brackets [Ret77, Ret93], which are, in turn, the Lie counterpart of classical Massey products [Mas54]. The connection between obstructions in extending a given deformation and Massey products was first noticed in [Dou61]. Next our aim was to study this relationship in our context. More precisely, we use Massey $n$-operations as defined in [Ret77] to establish this connection in the case of one parameter deformation of a Leibniz algebra. Next, we use a general treatment of Massey brackets as introduced in [FW01] to express the obstructions arising at different steps in the inductive construction of a versal deformation as described earlier, in terms of these general Massey brackets.

Finally, we discuss two examples to illustrate the theory developed in this
thesis. The first example is a three dimensional nilpotent Leibniz algebra for which we compute a versal deformation. The next example is the three dimensional Heisenberg Lie algebra. We deform this example viewing it as a Leibniz algebra to show that not only we recover all the usual Lie algebra deformations, but we get some new deformations which are Leibniz algebras and not Lie algebras. This example also illustrate the fact that versal deformation of a Lie algebra \( L \) and that of \( L \) when viewed as a Leibniz algebra may differ.

Current Work

1. **Deformation of Leibniz algebra morphisms:**
   In [Man07] we study formal deformation of Leibniz algebra morphisms \( f \). The associated deformation cohomology that controls deformations is constructed using the cochain complex \( CL^\bullet(f; f) \) defining Leibniz cohomology. We define the notion of equivalence of deformations, and study rigidity. The standard results we proved here are: (i) The infinitesimals are 2-cocycles in \( CL^2(f; f) \). (ii) The vanishing of the second cohomology module \( HL^2(f; f) \) implies that \( f \) is rigid. (iii) The obstructions to extending a partial deformation are 3-cocycles in \( CL^3(f; f) \).

2. **Versal deformations of Leibniz algebras (with A. Fialowski and G. Mukherjee):**
   In this work [FMM08], we develop an algebraic deformation theory over commutative local algebra base for Leibniz algebras over a field of characteristic zero. The main problem in deformation theory is to describe all non-equivalent deformations of a given object. We give a method to solve this problem completely, namely work out a construction of a versal deformation for a given Leibniz algebra, which induces all non-equivalent deformations and is unique on the infinitesimal level.

3. **An example of constructing Versal deformation for Leibniz algebras:**
   In [Man08] we compute a versal deformation of the three dimensional nilpotent Leibniz algebra over \( \mathbb{C} \), defined by the nontrivial brackets \( [e_1, e_3] = e_2 \) and \( [e_3, e_3] = e_1 \). This is denoted by \( \lambda_6 \) in the classification of three dimensional nilpotent Leibniz algebras, see [Lod93, AO01]. We compute cohomologies necessary for our purpose, Massey brackets and construct a versal deformation of our example.

4. **Deformation of Leibniz algebra morphisms over commutative local algebra base (with G. Mukherjee):**
In [MM] we study deformations of Leibniz algebra morphisms over a commutative local algebra base with multiplicative identity. The aim of this work is to show that when viewed from a general perspective, namely, deformation over a commutative local algebra base (rather than a one parameter formal deformation) the picture becomes more transparent. This approach also gives a nice relationship between the Harrison cohomology of the base of the deformation and the deformation cohomology. Obstructions in this case, arise as a map from the Harrison cohomology of the base to the deformation cohomology in question. Although we explain this by considering deformation of Leibniz algebra morphisms, deformation theory of morphisms of other algebras mentioned above can be treated similarly.

5. Leibniz algebra deformations of a Lie algebra (with A. Fialowski):
In this note [FM08] we compute Leibniz deformations of the three dimensional Heisenberg Lie algebra \( \mathfrak{n}_3 \) and compare it with its Lie deformations. It turns out that there are three extra Leibniz deformations. We also describe the versal Leibniz deformation of \( \mathfrak{n}_3 \) with the versal base. We compute infinitesimal Lie and Leibniz deformations and show that there are three additional Leibniz cocycles beside the five Lie cocycles. We will show that all these infinitesimal deformations are extendable without any obstructions. We also describe the versal Leibniz deformation.

Future Work

(1) Study of Leibniz algebra deformation for some important examples:
Since a Lie algebra is also a Leibniz algebra, the following natural question arises. If a Lie algebra is deformed as a Leibniz algebra does one get new deformations other than the Lie deformations of the original Lie algebra?

This question is answered for some specific examples in the thesis [FM08]. However, there are many more examples that we proposed to study, particularly, for those Lie algebras which are rigid in the Lie deformation sense. It may be noted that an object is called rigid if it does not admit any non trivial deformation.

(2) Comparative study of Leibniz algebras with other algebraic structures:
We would like to study algebraic relationship between Poisson and Leibniz algebras and also the possible relationship that might exist in context of algebraic deformations. Recently, the notion of Nambu- Poisson algebraic structures and related cohomology have been introduced in [IdLMP99]. We would like to study
them in the context of algebraic deformation theory.

(3) Cohomology comparison theorem:
In any deformation theory, a cohomology theory of the objects to be deformed plays an important role in encoding information about deformations. For instance, the infinitesimal of a deformation is captured by the second cohomology. Moreover, whether an arbitrary infinitesimal deformation can be extended to a full-blown deformation or not, is determined by obstructions which are elements of the third cohomology. In this process, it is an important step to prove that an obstruction cochain is a cocycle. It is usually a tedious job to prove this. To avoid this difficulty, in [GS83], M. Gerstenhaber and S. D. Schack, proposed a method of using an auxiliary cochain. This is known as ‘Cohomology comparison theorem’ (CCT in short) in the literature.

We propose to prove this theorem in the context of deformations of Leibniz algebras using Leibniz cohomology.

(4) I would also like to study and learn some new areas during my postdoc tenure.

References


