Nonparametric Calibration of Differently Measured Components of a Sum

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Abstract

We consider a calibration problem, where two additive components of an aggregate quantity, the aggregate being independent of the proportion between the components, are measured with different degrees of fixed and multiplicative distortion. The goal is to estimate the ratio of the distortion factors, so that the measurements of the two components can be brought to a common scale. Such a problem may arise, for example, when the flows through different channels of a river controlling structure are measured by empirical formulae based on channel dimensions. We provide a class of consistent estimators of the desired ratio, by exploiting the independence of the aggregate and the proportion of the two measured components. We perform Monte Carlo simulations of two specific members of this class and compare their mean squared errors to the Cramer-Rao lower bound for the variance in the case of some choices of the underlying distributions. Illustrative analysis with a real data set is also presented.

Keywords: Bifurcation, Dependence measure, Independence, Kendall's tau, Multiplicative distortion, Spearman's rho.

1 Introduction

Consider an unobservable physical quantity Z that is split into two additive parts, X and Y, which are then measured by two different methods. Each method introduces a fixed amount of multiplicative distortion, producing observations X_M and Y_M. Thus, for two unknown but fixed parameters $\theta_1(>0)$ and $\theta_2(>0)$, we have

$$\begin{aligned} Z &= X + Y, \\ X_M &= \theta_1 X, \\ Y_M &= \theta_2 Y. \end{aligned} \tag{1}$$

Consider the problem of calibrating the measurements for comparability, by estimating the ratio of the distortion factors, $\theta_0 = \theta_1/\theta_2$. This has to be done on the basis of a set of paired measurements of the form (X_M, Y_M), subject to the knowledge that Z and X/Z are random and independent.

An inferential problem of this kind is encountered in a river barrage, where the incoming flow (Z) is bifurcated into two parts: flow through a diversion canal (X) and flow through the main stream (Y). Flow through each of the two channels is regulated by lock gates, which can be partially opened to control the flow. There is practically no storage facility at a barrage, which means that the entire incoming (upstream) flow must be immediately released through one of the two channels. Further, the flow through these two channels is generally measured indirectly by an empirically determined formula based on the dimensions of the lock gates (Subramanya, 2008). The empirical formula involves a constant multiplier. The gates of the river barrage have different dimensions than those on the canal side, and different multipliers may be used for the two sets of gates. The specified multipliers are typically determined from controlled experiments prior to installation, and their values may need adjustments in the installed set-up. Thus, the measured values of flow through the canal (X_M) and the main stream (Y_M) would have different factors of error, θ_1 and θ_2. If one only has access to the measurements X_M and Y_M, then the parameters θ_1 and θ_2 may not be identifiable. However, an estimate of the ratio $\theta_0 = \theta_1/\theta_2$ can be used for calibrating the measurements through the two channels with one another for appropriate decision making. In this problem, fluctuations in the upstream flow (Z) is a natural phenomenon, while the fraction of flow diverted through the canal (X/Z) is mostly governed by human decision. Dependence between these two quantities are generally attributable to seasonal variation in the flow, and can be removed by stratifying the data by seasons.

In diagnostic and therapeutic medicine, a ray of light is used to determine the optical properties of living tissue. The tissue surface receives short pulses of light (Z) emitted from a small source, through a laser beam or optical fiber. Amounts of absorption (X) and scattering (Y) are observed over a period of time for studying the relationship between the absorption and the scattering coefficients of the tissue (Patterson, Chance, & Wilson, 1989). The amounts of absorption and scattering are estimated by different approximation formulae, so that their estimated values, X_M and Y_M, may differ from X and Y, respectively. Calibration may be needed here (Section 4.2, (Wilson & Patterson, 2008)). The proportion of light absorbed (X/Z) is a property of the tissue, while the amount of incident light (Z) depends on external factors such as the intensity and distance of the light source. These two quantities should be independent, as long as the wavelength of the light does not change.

Another instance of bifurcation arises while determining light absorption in a system of particles for characterizing different transmission media such as highly scattering particles, colloids and composite materials, by using measurements of transmittance and reflectance (see (Duncle & Bevans, 1956); (Tassan & Ferrari, 2002)). In these examples, the aggregate amount Z and the fraction of bifurcation X/Z can have an unspecified distribution. Assumption of any specific distribution of these two quantities would induce an implied distribution of the measurements X_M and Y_M, which can be used for likelihood based inference. However, the estimate of θ_0 would depend crucially on the appropriateness of the distributional assumption. Instead, one can seek a distribution-free estimator, by making use of the basic fact that Z and X/Z should be independent. It may be recalled that the principle of independence between two random variables has been used in the past as the basis of inference in other situations also. A prominent example is independent component analysis, where one seeks to separate the contributions of different sources in multiple linear mixtures (e.g., identifying different audio sources from signals recorded through one or more microphones), by making use of the reasonable assumption that the sources are independent ((Comon, 1994); (Hyvrinen, Karhunen, & Oja, 2001); (Stone, 2004)).

In this article, we propose a class of non-parametric methods of estimating the calibration parameter θ_0, by exploiting the independence of the aggregate amount and the fraction of bifurcation. The parameter is estimated by minimizing an empirical measure of dependence. Different measures of dependence give rise to different estimators. After introducing the estimators in Section 2, we establish their consistency in Section 3 through a series of theorems. In Section 4, we study the small sample performance of two specific estimators belonging to the proposed class, through a simulation study. In Section 5, we illustrate the usefulness of the methods through the analysis of flow of the Teesta river at the Gajaldoba Barrage of West Bengal. We provide some concluding remarks in Section 6. The proofs of all the theorems and lemmas are given in the Appendix.

2 Derivation of the estimator

Consider paired samples from m independent strata, $(X_{M_{i,j}}, Y_{M_{i,j}})$, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$, from the model (1) for each strata. Specifically, assume that the observables $(X_{M_{i,j}}, Y_{M_{i,j}})$ can be written as

$$\begin{aligned} X_{M_{i,j}} &= \theta_1 X_{i,j} = \theta_1 Z_{i,j} U_{i,j}, \\ Y_{M_{i,j}} &= \theta_2 Y_{i,j} = \theta_2 Z_{i,j}(1 - U_{i,j}), \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, m, \end{aligned} \tag{2}$$

for unspecified constants θ_1 and θ_2, where $Z_{i,1}, Z_{i,2}, \ldots, Z_{i,n_i}$ are unobserved samples from the distribution F_i defined over $[0, \infty)$, $U_{i,1}, U_{i,2}, \ldots, U_{i,n_i}$ are unobserved samples from the distribution G_i defined over $[0, 1]$, $i = 1, \ldots, m$, and all the sets of unobserved samples are independent of one another. Let D_n be an empirical measure of dependence, computed from paired samples of size n. We assume that values of D_n close to zero indicate lack of dependence, and values far away from zero indicate strong dependence. We also assume that the measure does not change if all the paired data values are multiplied by a common scale factor. The two common nonparametric measures of dependence, Spearman's rho and Kendall's tau (see Examples 1 and 2 below), satisfy these two properties. Suppose, for $\theta \in \Theta \subseteq \mathbb{R}^+$ and $i = 1, \ldots, m$, the quantity $d_{i,n_i}(\theta)$ denote the value of D_{n_i}, when the latter is computed by regarding $\big((V^*_{i,1}(\theta), W^*_{i,1}(\theta)), \ldots, (V^*_{i,n_i}(\theta), W^*_{i,n_i}(\theta))\big)$ as the underlying data, where

$$V_{ij}^*(\theta) = X_{M_{i,j}} + \theta Y_{M_{i,j}}$$

$$\text{and} \qquad W_{ij}^*(\theta) = \tfrac{X_{M_{i,j}}}{X_{M_{i,j}}+\theta Y_{M_{i,j}}},\ j = 1(1)n_i,\ i = 1(1)m. \tag{3}$$

Note that, the assumed scale-invariance property of D_n implies that the value of $d_{i,n_i}(\theta)$ would remain unchanged if one uses $((V_{i,1}(\theta), W_{i,1}(\theta)), \ldots, (V_{i,n_i}(\theta), W_{i,n_i}(\theta)))$ as the data, where

$$V_{i,j}(\theta) = \tfrac{V_{i,j}^*(\theta)}{\theta_1} = \tfrac{Z_{i,j}}{\theta_0}\left(\theta_0 + (\theta-\theta_0)(1-U_{i,j})\right)$$

$$\text{and} \qquad W_{i,j}(\theta) = W_{i,j}^*(\theta) = \tfrac{\theta_0 U_{i,j}}{\theta_0+(\theta-\theta_0)(1-U_{i,j})},\ j = 1(1)n_i,\ i = 1(1)m. \tag{4}$$

When $\theta = \theta_0$, the above 'data' reduces to $((Z_{i,1}, U_{i,1}), \ldots, (Z_{i,n_i}, U_{i,n_i}))$. Independence of the $Z_{i,j}$'s and the $U_{i,j}$'s implies that $d_{i,n_i}^2(\theta)$ should be small for values of θ near the true value θ_0. This property should hold for each stratum. Accordingly, we define an estimator of θ_0 as

$$\widehat{\theta} = \arg\min_{\theta\in\Theta} \sum_{i=1}^{m} d_{i,n_i}^2(\theta). \tag{5}$$

Use of various measures of dependence in (5) would produce different estimators. This class of estimators can be studied together.

Now let us consider some examples. Assume that $\{(V_k, W_k) : k = 1, \ldots, n\}$ is a set of paired random sample of size n drawn from the joint distribution of (V, W).

Example 1: Spearman's rank correlation coefficient, rho, (Spearman, 1904) between V and W is defined as

$$D_n((V_1, W_1), \ldots, (V_n, W_n)) = 1 - \frac{6\sum_{k=1}^n S_k^2}{n(n^2-1)}, \tag{6}$$

where S_k is the difference between the ranks of V_k and W_k, $k = 1, \ldots, n$.

Example 2: Kendall's rank correlation coefficient, tau, (Kendall, 1938) between V and W is defined as

$$D_n((V_1, W_1), \ldots, (V_n, W_n)) = \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{S_{k,l}}{n(n-1)}, \tag{7}$$

where

$$S_{k,l} = sgn(V_l - V_k)sgn(W_l - W_k),\ k, l = 1, \ldots, n,$$

and

$$sgn(u) = \begin{cases} -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0, \\ 1 & \text{if } u > 0. \end{cases}$$

3 Consistency of the estimator

Let the observations $(X_{M_{i,j}}, Y_{M_{i,j}})$, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$ follow the model (2). For $i = 1, \ldots, m$, let $H_{i\theta}$ be the bivariate distribution of $V(\theta) = (Z/\theta_0)\,(\theta_0 + (\theta - \theta_0)(1-U))$ and $W(\theta) = (\theta_0 U)/\,(\theta_0 + (\theta - \theta_0)(1-U))$, where $Z \sim F_i$ and $U \sim G_i$ are independent. In particular, we write $H_{i\theta_0}$ as H_i, which is the product of F_i and G_i. Note that $(V(\theta), W(\theta))$ takes values in $\mathbb{A} = [0,\infty) \times [0,1]$. Let $\mathbb{H}$ be the space of all bivariate distribution functions that are continuous almost everywhere, equipped with the metric ρ induced by the supremum norm. Let $D : \mathbb{H} \to \mathbb{R}$ be a measure of dependence such that $D(H) = 0$ whenever H is the product of its marginal distributions. Let $D_n(\mathbf{Z}, \mathbf{U})$ be a sample version of $D(H)$, computed from the samples $(Z_1, U_1), \ldots, (Z_n, U_n)$ of H, the samples being represented through the vectors $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ and $\mathbf{U} = (U_1, \ldots, U_n)^T$. Let this function be scale invariant, i.e., $D_n(a\mathbf{Z}, a\mathbf{U}) = D_n(\mathbf{Z}, \mathbf{U})$ for any real a. With $\mathbf{Z}$ and $\mathbf{U}$ defined as above, let us denote by $\mathbf{V}(\theta)$ and $\mathbf{W}(\theta)$ the vectors with elements

$$\begin{aligned} & V_i(\theta) = \tfrac{Z_i}{\theta_0}\,(\theta_0 + (\theta - \theta_0)(1 - U_i)) \\ \text{and} \qquad & W_i(\theta) = \tfrac{\theta_0 U_i}{\theta_0 + (\theta - \theta_0)(1 - U_i)},\ i = 1(1)n. \end{aligned} \tag{8}$$

respectively. Note that a similar construction had been used in (4).

We now list a number of conditions for proving a series of results culminating in the consistency of $\widehat{\theta}$.

A. The set Θ is compact.

B. The dependence measure satisfies the following conditions.

   (i) $D : \mathbb{H} \to \mathbb{R}$ is continuous with respect to the metric space $(\mathbb{H}, \rho)$.

   (ii) D is bounded and $D(H_{i\theta}) \neq 0, \quad \forall \theta \in \Theta \backslash \{\theta_0\}$, for $i = 1, \ldots, m$.

C. The function $D_n : \mathbb{R}^{2n} \to \mathbb{R}$ satisfies the following conditions.

   (i) The statistic $D_n(\mathbf{Z}, \mathbf{U})$ can be written as $a_n D(H^{(n)}) - b_n$ where $H^{(n)}$ is the bivariate empirical distribution function based on the data $(\mathbf{Z}, \mathbf{U})$, and a_n and b_n are real sequences such that $a_n \to 1$ and $b_n \to 0$ as $n \to \infty$.

   (ii) For $i = 1, \ldots, m$, whenever ($\mathbf{Z}$,$\mathbf{U}$) are samples from H_i, there is B_n such that $B_n = O_p(1)$ and the following inequality holds for all $\theta, \theta' \in \Theta$, almost surely in H_i:

$$|D_n(\mathbf{V}(\theta), \mathbf{W}(\theta)) - D_n(\mathbf{V}(\theta'), \mathbf{W}(\theta'))| \leq B_n|\theta - \theta'|.$$

D. For $i = 1, \ldots, m$, the distributions F_i and G_i are absolutely continuous with respect to the Lebesgue measure.

We now prove consistency of the class of estimators (5), defined in terms of a dependence measure D_n that satisfies the above conditions.

Theorem 3.1 *Let Conditions B(i) and C(i) hold. Then for each* $i = 1, \ldots, m$

$$d_{i,n_i}(\theta) \to D(H_{i\theta})$$

almost surely as $n_i \to \infty$.

Theorem 3.2 *Suppose Conditions A, B(i), C and D hold. Then for each* $i = 1, \ldots, m$

$$\sup_{\theta \in \Theta} |d_{i,n_i}(\theta) - D(H_{i\theta})| = o_p(1).$$

Theorem 3.3 *Let the estimator* $\widehat{\theta}$ *defined by (5) be based on data arising from the measurement model (2). Let Conditions A, B, C and D hold. Further, let, for some vector* $(\xi_1, \xi_2, \ldots, \xi_m)^T$ *with positive components,*

$$\left(\frac{n_1}{n}, \frac{n_2}{n}, \ldots, \frac{n_m}{n}\right)^T \to (\xi_1, \xi_2, \ldots, \xi_m)^T \tag{9}$$

in Euclidean norm, as $n = \sum_{i=1}^{m} n_i \to \infty$. *Then*

$$\widehat{\theta} \xrightarrow{P} \theta_0.$$

Theorem 3.3 establishes consistency of the general class of estimators (5) based on different measures of dependence, which satisfy the requisite conditions. In particular, Conditions B and C need to be verified separately for each measure of dependence. The next two theorems show that these conditions hold for the two examples mentioned in Section 2.

Theorem 3.4 *If the underlying parameter space satisfies Condition A and the underlying bivariate distributions* H_i *(with marginal distributions* F_i *and* G_i*) satisfy Condition D, then*

$$\widehat{\theta}_S = \arg\min_{\theta \in \Theta} \sum_{i=1}^{m} D_{n_i}^2 \left((V_{i,1}(\theta), W_{i,1}(\theta)), \ldots, (V_{i,n_i}(\theta), W_{i,n_i}(\theta)) \right), \tag{10}$$

where $V_{i,j}(\theta)$ *and* $W_{i,j}(\theta)$, $j = 1(1)n_i$, $i = 1(1)m$, *are defined as in (4) and* D_{n_i} *has the form as in (6), converges in probability to* θ_0.

Theorem 3.5 *If the underlying parameter space satisfies Condition A and the underlying bivariate distributions* H_i *(with marginal distributions* F_i *and* G_i*) satisfy Condition D, then*

$$\widehat{\theta}_K = \arg\min_{\theta \in \Theta} \sum_{i=1}^{m} D_{n_i}^2 \left((V_{i,1}(\theta), W_{i,1}(\theta)), \ldots, (V_{i,n_i}(\theta), W_{i,n_i}(\theta)) \right), \tag{11}$$

where $V_{i,j}(\theta)$ *and* $W_{i,j}(\theta)$, $j = 1(1)n_i$, $i = 1(1)m$, *are defined as in (4) and* D_{n_i} *has the form as in (7), converges in probability to* θ_0.

4 Simulation study

For the purpose of simulation, we assume that the aggregate quantity (Z) has the Gamma distribution with shape and scale parameters p and 1, respectively, and the fraction of bifurcation (X/Z) has the Beta distribution with shape parameters α and β. Further, we assume that the true value of θ is θ_0 and the number of strata is m with sample size n_i, subject to $n = \sum_{i=1}^{m} n_i$, n being the total sample size.

Even though the estimators $\widehat{\theta}_S$ and $\widehat{\theta}_K$ are non-parametric and could possibly be biased for small sample sizes, their performances in terms of the Mean Squared Error (MSE) may be compared with the benchmark provided by the Cramer Rao Lower Bound (CRLB) for θ in the special case of the assumed distribution. A closed form expression of the latter is given in the following lemma.

Lemma 4.1 *Under the model (2) with $\theta = \theta_1/\theta_2$, consider a paired random sample $(Z_{i,j}, U_{i,j})$, where $Z_{i,j} \sim Gamma(p, 1)$ and $U_{i,j} \sim Beta(\alpha, \beta)$, $Z_{i,j}$ and $U_{i,j}$ are independent random variables, $j = 1, \ldots, n_i$ and $i = 1, \ldots, m$ with $n = \sum_{i=1}^{m} n_i$. Then the CRLB for the variance of an unbiased estimator of θ based on the above data, where θ_2 is regarded as an unknown nuisance parameter and p, α and β are regarded as known, is given by*

$$CRLB(\theta) = \frac{n(1+\alpha+\beta)(\alpha+\beta)^2\theta^2}{\alpha\beta\left[p+(\alpha+\beta)^2\right]}. \tag{12}$$

The optimization needed for computing the estimates $\widehat{\theta}_S$ and $\widehat{\theta}_K$ through (10) and (11), respectively, is done through a grid search.

We run 1000 simulations with $m = 1$, $n_1 = 20$, for five different values of the shape parameter ($p = 1, 5, 10, 15, 20$), $\theta_0 = 0.5$ and for the following sets of values of the other parameters:

(i) $\alpha = 1,$ and $\beta = 1;$

(ii) $\alpha = 0.5,$ and $\beta = 0.5.$

Table 1 shows the CRLB for the parameter θ and the simulated MSE of the two estimators of θ, for the specified combination of parameter values. The number within the parentheses represents the ratio of simulated MSE and CRLB.

It can be seen that the ratio of simulated MSE and CRLB for both estimators can be large in some cases, but it comes close to 1 as the shape parameter of distribution of Z becomes large, for both choices of parameter values ($\alpha = 1$, $\beta = 1$) and ($\alpha = 0.5$, $\beta = 0.5$). The calculated values of CRLB would be inflated somewhat (and the ratios in the last two columns of Table 1 would be closer to 1) if the parameters of the gamma and the beta distributions are regarded as unknown. In contrast, the nonparametric estimators are constructed by treating the two distributions as *completely* unknown functions.

In order to check the robustness of the proposed estimators against the sizes of the strata, we use $m = 1$, 2 and 5 strata of equal size making up a total of $n = 100$ pairs

Table 1: CRLB, MSE, (MSE/CRLB) of estimated parameters in case (i) $\theta_0 = 0.5,\ \alpha = 1,\ \beta = 1$, and $p = (1, 5, 10, 15, 20)$, and case (ii) $\theta_0 = 0.5,\ \alpha = 0.5,\ \beta = 0.5$, and $p = (1, 5, 10, 15, 20)$, with $m = 1$, $n_1 = 20$.

(α, β)	CRLB(θ)	MSE (MSE/CRLB)	
		$\widehat{\theta}_S$	$\widehat{\theta}_K$
1	0.0300	0.3405 (11.3)	0.3425 (11.4)
5	0.0167	0.0489 (2.9)	0.0522 (3.1)
10	0.0107	0.0242 (2.3)	0.0248 (2.3)
15	0.0079	0.0133 (1.7)	0.0135 (1.7)
20	0.0062	0.0092 (1.5)	0.0096 (1.5)
1	0.0050	0.2744 (54.9)	0.2698 (53.4)
5	0.0167	0.0362 (2.2)	0.0367 (2.2)
10	0.0091	0.0153 (1.7)	0.0156 (1.7)
15	0.0062	0.0098 (1.6)	0.0100 (1.6)
20	0.0047	0.0061 (1.3)	0.0063 (1.3)

of data, with the parameter values $\theta_0 = 0.5,\ p = 10,\ \alpha = 1$ and $\beta = 1$. We run 1000 simulations for each case. Table 2 shows the CRLB for the parameter θ and the simulated MSE of the two estimators of θ, for the specified combination of parameter values. The number within the parentheses represents the ratio of simulated MSE and CRLB.

It is found that performance of the estimators improves as the stratum size (n_i) increases.

We now investigate how the performance of the proposed estimators changes with the number of strata (m) of fixed size ($n_i = 20$). For this study, we run 1000 simulations for $m = 1$, 5 and 10 and parameters values $\theta_0 = 0.5,\ p = 10,\ \alpha = 1$ and $\beta = 1$.

Table 2: CRLB, MSE and (MSE/CRLB) of estimated parameters for $\theta_0 = 0.5,\ p = 10,\ \alpha = 1$, $\beta = 1$, $n = 100$ and $m = 1, 2, 5$.

m	n_i	CRLB(θ)	MSE (MSE/CRLB)	
			$\widehat{\theta}_S$	$\widehat{\theta}_K$
5	20		0.0047 (2.2)	0.0048 (2.3)
2	50	0.0021	0.0041 (1.9)	0.0040 (1.9)
1	100		0.0034 (1.6)	0.0034 (1.6)

Table 3: CRLB, MSE and MSE/CRLB of estimated parameters for $\theta_0 = 0.5$, $p = 10$, $\alpha = 1$, $\beta = 1$, $m = (1, 2, 5)$ and $n_i = 20$, for $i = 1, \ldots, m$.

m	CRLB(θ)	MSE (MSE/CRLB)	
		$\widehat{\theta}_S$	$\widehat{\theta}_K$
1	0.0107	0.0242 (2.3)	0.0248 (2.3)
5	0.0021	0.0047 (2.2)	0.0048 (2.3)
10	0.0011	0.0025 (2.3)	0.0026 (2.4)

Table 3 shows the CRLB for the parameter θ and the simulated MSE of the two estimators of θ, for the specified combination of parameter values. The number within the parentheses represents the ratio of simulated MSE and CRLB.

We find that performance of estimators remains unaffected with the change in the number of strata.

5 Data analysis

The flow of the river Teesta is recorded regularly at the barrage located at Gajaldoba in the Jalpaiguri district of West Bengal, India. The incoming flow is bifurcated into two parts: flow through a diversion canal and flow through the main stream. There is an additional diversion canal that had not been operational during the period of available data. The flow through these two channels are measured indirectly by a formula based on the dimensions of the lock gates and a constant multiplier. The values of the multiplier, determined through the experiments prior to installation of the gates, is specified in manuals. The gates of the river barrage are larger than those on the canal, and different multipliers are used for the two sets of gates. If wrong multipliers are used, that would result in a discrepancy of measurements, depending on how much water is diverted through the canal. For the purpose of decision making, knowing the correct ratio of the multipliers is more important than knowing the multipliers themselves. This can be determined through a controlled on-site experiment. However, such an experiment with a barrage in use may entail logistical hurdles. An alternative solution based on existing data would be very useful.

The data set, obtained by courtesy of the Department of Irrigation of the Government of West Bengal and the Central Water Commission of the Government of India, consists of measurements of the flow (in cubic meters per second or cumec) of the two channels aggregated over 10-day cycles, from 1998 to 2008. The problem of assessing a correct multiplier for calibrating the two sets of flow measurements amounts to estimation of the parameter θ from these data, under the measurement model (2).

Since decision making is most important for the driest period of the year (November to March), we utilize only the data for these months in each year. Thus, the number of strata is $m = 15$ and size of the i^{th} strata is $n_i = 11$, for $i = 1, \ldots, m$. We use the proposed estimators (10) and (11), to estimate the parameter θ. For standard error, we use nonparametric bootstrap estimates based on 1000 resamples.

The results are given in Table 4. The two estimates have similar values, indicating that a measured flow of one cumec through the canal gates is 40 to 45 per cent larger than a measured flow of one cumec through the barrage gates.

The above estimates rely crucially on the assumption of independence between the aggregate ($Z_{i,j}$) and the fraction of bifurcation ($U_{i,j}$), for all i and j under model (2). A verification of this assumption for the particular data set is needed. We perform a bootstrap test for the hypothesis that $Z_{i,j}$ and $U_{i,j}$ are independent. By regarding the estimated value of θ as the correct value, we compute the values of $Z_{i,j}$ and $U_{i,j}$ in each case. Subsequently, for the i^{th} stratum, we draw 1000 bootstrap samples of size 11 for $Z_{i,j}$ and $U_{i,j}$ independently of one another. Then the distribution of the criterion function given in (5) is the null distribution based on the bootstrap sample. The observed values of the criterion functions in (10) and (11) for Spearman's rho and Kendall's tau, correspond to the bootstrap p-values 0.788 and 0.704, respectively. This finding justifies the assumption of independence of $Z_{i,j}$ and $U_{i,j}$.

The large values of the two estimates of the calibration factor point towards the need for

Table 4: Estimates of θ_0 by using (10) and (11), together with bootstrap estimate of the standard error.

	$\widehat{\theta}_S$	$\widehat{\theta}_K$
Estimates	1.4049	1.4479
Boostrap S.E.	0.1685	0.1673

a considerable amount of adjustment. Further, each estimate is more than two standard errors away from 1. Thus, the apparent flow of one cumec through the canal gates is significantly larger than the apparent flow of one cumec flow through the barrage gates. This conclusion is important in the context of decision making. Unless the measurements from the canal and the barrage gates are calibrated, there would never be a proper control over the flow. Any confusion in respect of controlling the flow of an international river is not desirable, and can have serious consequences.

6 Concluding Remarks

In this article, we have introduced a class of nonparametric estimators for the purpose of calibrating differently measured components of a sum, assuming that there is a correct calibrating factor. Each estimator makes use of a particular measure of bivariate dependence. We have proved consistency of this class of estimators, under general conditions.

Apart from the two measures considered here (Spearman's rho and Kendall's tau), one could possibly use other well known measures of dependence such as the product moment correlation and distance correlation (Székely, Rizzo, & Bakirov, 2007). However, consistency of the corresponding estimator is not guaranteed. In particular, Condition B(i) is not satisfied by the product moment correlation, and we could not prove Condition C(ii) in the case of distance correlation. The estimates of the calibration parameter based on these two measures of dependence, for the data set on Teesta flow through the canal and the barrage at Gajaldoba, happen to be 1.2778 and 1.4495, respectively. These numbers are in line with the finding of Section 5.

The simple method of calibration presented in this article may be extended to the situation where an additive quantity is multifurcated into more than two components, which are measured with different degrees of fixed and multiplicative distortion. This type of problem may arise in analyzing situations like diagnostic and therapeutic medicine, Multi-Channel CCD Spectrometers etc. Trifurcation of a river flow through a barrage with two canals (a status recently attained by the Teesta Barrage Project at Gajaldoba) is another example. This research problem may be considered in future.

A Appendix: Proofs

Proof of Theorem 3.1. Consider any $\theta \in \Theta$. It follows from the definition of $H_{i\theta}$ that the data (4) can be regarded as samples from $H_{i\theta}$. Let $H_i^{(n_i)}$ be the empirical distribution function computed from (4). Note that $\{H_i^{(n_i)}\}_{n_i=1}^{\infty}$ is a sequence of empirical distribution functions based on samples from $H_{i\theta}$. Let $A = \{\|H_i^{(n_i)} - H_{i\theta}\| \to 0 \text{ as } n_i \to \infty\}$.

By the Vapnik-Chervonenkis Theorem ((Shorack & Wellner, 1986), p.828), $P(A) = 1$. Note that $H_i^{(n_i)} \in \mathbb{H}$ for all i and n_i. For $\omega \in A$, it follows from Condition B(i) that

$$D(H_i^{(n_i)}) \to D(H_{i\theta}) \quad \text{as } n_i \to \infty. \tag{13}$$

Further, it follows from the definition of $d_{i,n_i}(\theta)$ and Condition C(i) that $d_{i,n_i}(\theta)$ can be written as $a_{n_i} D(H_i^{(n_i)}) - b_{n_i}$, with $a_{n_i} \to 1$ and $b_{n_i} \to 0$ as $n_i \to \infty$. Thus, from (13), we can conclude that whenever $\omega \in A$, $\{d_{i,n_i}(\theta)\}_{n_i=1}^{\infty}$ converges to $D(H_{i\theta})$. Let $B = \{|d_{i,n_i}(\theta) - D(H_{i\theta})| \to 0\}$. It is clear that $A \subset B$, and therefore $P(B) = 1$. Hence the proof. We now state and prove a lemma that will be useful in proving Theorem 3.2.

Lemma 3.1 *Under Conditions B(i) and D, $D(H_{i\theta})$ is a continuous function of θ.*

Proof : For $i = 1, \ldots, m$, let the function $f_{F_i,G_i} : \Theta \to \mathbb{H}$ be defined by $f_{F_i,G_i}(\theta) = H_{i\theta}$. We will show the continuity of the function $D \circ f_{F_i,G_i} : \Theta \to \mathbb{R}$ by using Condition B(i) and showing the continuity of f_{F_i,G_i} over Θ.

For $\theta \in \Theta$ and $(v, w) \in \mathbb{A}$,

$$\begin{aligned} H_{i\theta}(v,w) &= P\left((Z/\theta_0)(\theta_0 + (\theta - \theta_0)(1-U)) \le v, \frac{U\theta_0}{\theta_0 + (\theta-\theta_0)(1-U)} \le w\right), \\ &= \int_0^{\frac{w\theta}{\theta_0 + (\theta-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0 + (\theta-\theta_0)(1-u)}\right) dG_i(u). \end{aligned}$$

Consider any sequence $\{\theta_{n_i}\}_{n_i=1}^{\infty}$ converging to θ and $\theta_{n_i} \in \Theta$ for all n_i.

Note that

$$\begin{aligned} H_{i\theta_{n_i}}(v,w) &= \int_0^{\frac{w\theta_{n_i}}{\theta_0 + (\theta_{n_i}-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0 + (\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u), \\ &= \int_0^{\frac{w\theta}{\theta_0 + (\theta-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0 + (\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u), \\ &\quad + \int_{\frac{w\theta}{\theta_0 + (\theta-\theta_0)w}}^{\frac{w\theta_{n_i}}{\theta_0 + (\theta_{n_i}-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0 + (\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u). \end{aligned} \tag{14}$$

The last term on the right side of (14) can be bounded as follows.

$$\begin{aligned}
&\Big|\int_{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}}^{\frac{w\theta_n}{\theta_0+(\theta_{n_i}-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0+(\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u)\Big| \\
\leq\ & \Big|\int_{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}}^{\frac{w\theta_{n_i}}{\theta_0+(\theta_{n_i}-\theta_0)w}} dG_i(u)\Big| \\
=\ & \Big|G_i\left(\frac{w\theta_{n_i}}{\theta_0+(\theta_{n_i}-\theta_0)w}\right) - G_i\left(\frac{w\theta}{\theta_0+(\theta-\theta_0)w}\right)\Big|.
\end{aligned}$$

By the continuity of G_i (ensured by Condition D), and the fact that $\{\theta_{n_i}\}_{n_i=1}^{\infty}$ converges to θ, we can conclude that $\left\{G_i\left(\frac{w\theta_{n_i}}{\theta_0+(\theta_{n_i}-\theta_0)w}\right)\right\}_{n_i=1}^{\infty}$ converges to $G_i\left(\frac{w\theta}{\theta_0+(\theta-\theta_0)w}\right)$. Therefore,

$$\Big|\int_{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}}^{\frac{w\theta_{n_i}}{\theta_0+(\theta_{n_i}-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0+(\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u)du\Big| \to 0 \quad \text{as } n_i\to\infty. \tag{15}$$

As for the first term on the right side of (14), it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{n_i\to\infty}\int_0^{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0+(\theta_{n_i}-\theta_0)(1-u)}\right) dG_i(u) \\
=\ & \int_0^{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}} \left[\lim_{n_i\to\infty} F_i\left(\frac{v\theta_0}{\theta_0+(\theta_{n_i}-\theta_0)(1-u)}\right)\right] dG_i(u) \\
=\ & \int_0^{\frac{w\theta}{\theta_0+(\theta-\theta_0)w}} F_i\left(\frac{v\theta_0}{\theta_0+(\theta-\theta_0)(1-u)}\right) dG_i(u),
\end{aligned} \tag{16}$$

by the continuity of F_i.

By using (15) and (16) in (14), for any $\{\theta_{n_i}\}_{n_i=1}^{\infty}$ converging to θ, we have

$$H_{i\theta_{n_i}} \to H_{i\theta} \quad \text{as } n_i\to\infty, \tag{17}$$

which proves the continuity of f_{F_i,G_i}. This completes the proof.

Proof of Theorem 3.2. Let $\epsilon,\nu>0$. Since $B_{n_i}=O_p(1)$ for $i=1,\dots,m$, there is $M>0$ such that the inequality $P[B_{n_i}>\epsilon M/4]<\nu/2$ holds for all n_i. We define $\Delta_{n_i}(\epsilon,\nu)=\frac{B_{n_i}}{M}$, then $P[\Delta_{n_i}(\epsilon,\nu)>\epsilon/4]<\nu/2$. Choose δ' smaller than $\frac{1}{M}$. By Condition A and Lemma 3.1, $D(H_{i\theta})$ is uniformly continuous over Θ. It follows that there exists $\delta''>0$ such that for any $\theta,\theta'\in\Theta$

$$|\theta-\theta'|<\delta'' \Rightarrow |D(H_{i\theta})-D(H_{i\theta'})|<\epsilon/4. \tag{18}$$

Let $\delta_0 = \min(\delta', \delta'')$ and $\mathcal{N}(\theta, \epsilon, \nu) = \{\theta' \in \Theta\colon |\theta' - \theta| < \delta_0\}$.

From Condition C(ii), we have, for any $\theta \in \Theta$,

$$\sup_{\theta' \in \mathcal{N}(\theta,\epsilon,\nu)} |d_{i,n_i}(\theta') - d_{i,n_i}(\theta)| \leq B_{n_i}\delta_0 < \frac{B_{n_i}}{M} = \Delta_{n_i}(\epsilon, \nu) \quad \text{almost surely.} \tag{19}$$

Hence, it follows that

$$P\left(\sup_{\theta' \in \mathcal{N}(\theta,\epsilon,\nu)} \Big| d_{i,n_i}(\theta') - d_{i,n_i}(\theta) \Big| > \epsilon/4 \right) < \nu/2.$$

Since Θ is a compact set, it has an open cover of the form $\cup_{k=1}^{K} \mathcal{N}(\theta_k, \epsilon, \nu)$ for some finite K. Then by triangle inequality,

$$\begin{aligned}
&\sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta) - D(H_{i\theta})\Big| \\
&= \max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|d_{i,n_i}(\theta) - d_{i,n_i}(\theta_k) + D(H_{i\theta_k}) - D(H_{i\theta}) + d_{i,n_i}(\theta_k) - D(H_{i\theta_k})\Big|, \\
&\leq \max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|d_{i,n_i}(\theta) - d_{i,n_i}(\theta_k)\Big| + \max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|D(H_{i\theta_k}) - D(H_{i\theta})\Big| \\
&\quad + \max_{1\leq k\leq K}\Big|d_{i,n_i}(\theta_k) - D(H_{i\theta_k})\Big|,
\end{aligned} \tag{20}$$

which implies that

$$\begin{aligned}
&P\Big(\sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta) - D(H_{i\theta})\Big| > \epsilon\Big) \\
&\leq P\Big(\Big(\max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|d_{i,n_i}(\theta) - d_{i,n_i}(\theta_k)\Big| + \max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|D(H_{i\theta_k}) - D(H_{i\theta})\Big| \\
&\quad + \max_{1\leq k\leq K}\Big|d_{i,n_i}(\theta_k) - D(H_{i\theta_k})\Big|\Big) > \epsilon\Big), \\
&\leq P\left(\Big(\max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|d_{i,n_i}(\theta) - d_{i,n_i}(\theta_k)\Big| + \max_{1\leq k\leq K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|D(H_{i\theta_k}) - D(H_{i\theta})\Big|\Big) > \epsilon/2\right) \\
&\quad + P\Big(\max_{1\leq k\leq K}\Big|d_{i,n_i}(\theta_k) - D(H_{i\theta_k})\Big| > \epsilon/2\Big).
\end{aligned} \tag{21}$$

From Theorem 3.1 it follows that, for each $\theta \in \Theta$

$$d_{i,n_i}(\theta) \xrightarrow{P} D(H_{i\theta}) \quad \text{as } n_i \to \infty.$$

Therefore, for the given $\epsilon, \nu > 0$, $\exists n_{i0}(\epsilon, \nu)$, such that $n_i \geq n_{i0}(\epsilon, \nu)$ implies

$$P\Big(\max_{1\leq k\leq K}\Big|d_{i,n_i}(\theta_k) - D(H_{i\theta_k})\Big| > \epsilon/2\Big) < \nu/2. \tag{22}$$

Using (18) and (19) we get,

$$\max_{1\le k\le K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|d_{i,n_i}(\theta)-d_{i,n_i}(\theta_k)\Big| + \max_{1\le k\le K}\sup_{\theta\in\mathcal{N}(\theta_k,\epsilon,\nu)}\Big|D(H_{i\theta_k})-D(H_{i\theta})\Big|$$
$$< \Delta_{n_i}(\epsilon,\nu)+\epsilon/4 \quad \text{almost surely.}$$

Hence,

$$P\left(\left(\max_{1\le k\le K}\sup_{\theta\in\mathcal{N}(\theta_k)}\Big|d_{i,n_i}(\theta)-d_{i,n_i}(\theta_k)\Big| + \max_{1\le k\le K}\sup_{\theta\in\mathcal{N}(\theta_k)}\Big|D(H_{i\theta_k})-D(H_{i\theta})\Big|\right) > \epsilon/2\right)$$
$$\le P\left(\left(\Delta_{n_i}(\epsilon,\nu)+\epsilon/4\right) > \epsilon/2\right) = P\left(\Delta_{n_i}(\epsilon,\nu) > \epsilon/4\right) < \nu/2. \tag{23}$$

By using the inequalities (22) and (23) in (21), we have

$$n_i \ge n_{0i}(\epsilon,\nu) \;\Rightarrow\; P\Big(\sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta)-D(H_{i\theta})\Big| > \epsilon\Big) < \nu. \tag{24}$$

Hence the proof.

Another lemma is needed for proving Theorem 3.3.

Lemma 3.2 *Let conditions of Theorem 3.2 hold, together with Condition B(ii). Then for each* $i = 1,\ldots,m$

$$\sup_{\theta\in\Theta}\left|d^2_{i,n_i}(\theta)-D^2(H_{i\theta})\right| = o_p(1).$$

PROOF: Note that,

$$\sup_{\theta\in\Theta}\Big|d^2_{i,n_i}(\theta)-D^2(H_{i\theta})\Big|$$
$$\le \sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta)+D(H_{i\theta})\Big| \times \sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta)-D(H_{i\theta})\Big|. \tag{25}$$

By Condition C(i), the first term of (25) can be written as

$$\sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta)+D(H_{i\theta})\Big| = \sup_{\theta\in\Theta}\Big|a_{n_i}D(H_i^{(n_i)})-b_{n_i}+D(H_{i\theta})\Big|.$$

By Condition B(ii), $|D(H_i^{(n_i)})| < C$ and $|D(H_{i\theta})| < C$ where C is a positive constant. Therefore

$$\sup_{\theta\in\Theta}\Big|d_{i,n_i}(\theta)+D(H_{i\theta})\Big| \le \Big(|a_{n_i}|C+|b_{n_i}|+C\Big) \le R,$$

for some positive constant R. The last inequality follows from the fact that $\{a_{n_i}\}_{n_i=1}^{\infty}$ and $\{b_{n_i}\}_{n_i=1}^{\infty}$, being convergent sequences of real numbers, are both bounded.

By (25) and Theorem 3.2, it follows that

$$\sup_{\theta\in\Theta}\Big|d^2_{i,n_i}(\theta) - D^2(H_{i\theta})\Big| = o_p(1).$$

Hence the proof.

Proof of Theorem 3.3. Let us denote,

$$S(\theta) = \sum_{i=1}^{m} D^2(H_{i\theta}) \quad \text{and} \quad S_n(\theta) = \sum_{i=1}^{m} d^2_{i,n_i}(\theta).$$

By Theorem 3.1 and the condition (9), it follows that

$$S_n(\theta_0) \to 0, \quad \text{in probability.} \tag{26}$$

As $\widehat{\theta}$ is the minimizer of S_n over Θ, we have

$$0 \le S_n(\widehat{\theta}) \le S_n(\theta_0). \tag{27}$$

Using (26) and (27) we obtain

$$S_n(\widehat{\theta}) \to 0, \quad \text{in probability.} \tag{28}$$

Suppose, if possible, $\widehat{\theta} \nrightarrow \theta_0$ in probability, that is $\exists \epsilon_1, \delta_1 > 0$ such that,

$$P(|\widehat{\theta} - \theta_0| > \epsilon_1) > \delta_1, \quad \text{infinitely often.} \tag{29}$$

From (28), it follows that $\exists \epsilon_2, \delta_2 > 0$ such that,

$$P(|S_n(\widehat{\theta})| > \epsilon_2/2) < \delta_2, \quad \text{infinitely often.} \tag{30}$$

Let us define, for any $\epsilon > 0$

$$B_\epsilon(\theta_0) = \{\theta \in \Theta : |\theta - \theta_0| \ge \epsilon\} \quad \text{and} \quad m_\epsilon(\theta_0) = \inf_{\theta\in B_\epsilon(\theta_0)} S(\theta).$$

It follows from Condition B(ii) and Lemma 3.1 that $m_\epsilon(\theta_0) > 0 \; \forall \epsilon > 0$. Let $\epsilon_2 = m_{\epsilon_1}(\theta_0)(> 0)$ and $\delta_2 = \delta_1/2(> 0)$. Then

$$\{|\widehat{\theta} - \theta_0| > \epsilon_1\} \Rightarrow \{S(\widehat{\theta}) \ge \epsilon_2\}. \tag{31}$$

Note that,

$$\Big\{\sup_{\theta\in\Theta}|S_n(\theta)-S(\theta)|\le\epsilon_2/2\Big\}\Rightarrow\Big\{|S_n(\widehat{\theta})-S(\widehat{\theta})|\le\epsilon_2/2\Big\}. \tag{32}$$

From (31) and (32),

$$\begin{aligned}&\Big\{|\widehat{\theta}-\theta_0|>\epsilon_1\cap\sup_{\theta\in\Theta}|S_n(\theta)-S(\theta)|\le\epsilon_2/2\Big\},\\ \Rightarrow\ &\Big\{S(\widehat{\theta})\ge\epsilon_2\cap|S_n(\widehat{\theta})-S(\widehat{\theta})|\le\epsilon_2/2\Big\},\\ \Rightarrow\ &\Big\{S_n(\widehat{\theta})\ge\epsilon_2/2\Big\},\end{aligned}$$

because of the fact that $S_n(\widehat{\theta})\ge S(\widehat{\theta})-|S_n(\widehat{\theta})-S(\widehat{\theta})|$. Hence,

$$\begin{aligned}P\Big\{S_n(\widehat{\theta})\ge\epsilon_2/2\Big\}&\ge P\Big\{|\widehat{\theta}-\theta_0|>\epsilon_1\cap\sup_{\theta\in\Theta}|S_n(\theta)-S(\theta)|\le\epsilon_2/2\Big\}\\&\ge P\Big\{|\widehat{\theta}-\theta_0|>\epsilon_1\Big\}+P\Big\{\sup_{\theta\in\Theta}|S_n(\theta)-S(\theta)|\le\epsilon_2/2\Big\}-1.\end{aligned}$$

The first term in the last expression is greater than δ_1 infinitely often as presumed, and from Lemma 3.2 the second term is greater than $1-\frac{\delta_1}{2}$ infinitely often. It follows that

$$P\Big\{|S_n(\widehat{\theta})|\ge\frac{\epsilon_2}{2}\Big\}>\delta_2,\quad\text{infinitely often.} \tag{33}$$

Inequality (33) contradicts (30). This completes the proof.

Proof of Theorem 3.4. By the assumptions of the Theorem 3.4, the underlying parameter space satisfies Condition A and underlying bivariate distributions H_i (whose marginal distributions are F_i and G_i) satisfy Condition D of Theorem 3.3. We will now show that all the remaining conditions, B(i) is satisfied by D, B(ii) is satisfied by $D(H_{i\theta})$ and C is satisfied by D_n.

The population version of (6) can written as

$$D(H)=12\int\int_{\mathbb{A}}F(v)G(w)dH(v,w)-3, \tag{34}$$

where H is a generic member of $\mathbb{H}$ with marginals F and G. By Sklar's Theorem ((Schweizer & Wolff, 1981), (Nelsen, 2006)), there exists a copula C such that for all v,w in $\mathbb{A}$,

$$H(v,w)=C(F(v),G(w)).$$

Let $x = F(v)$, $y = G(w)$ and $\mathbb{I}^2 = [0,1] \times [0,1]$. Then $D(H)$ can be written as

$$\begin{aligned} D(H) &= 12 \iint\limits_{\mathbb{I}^2} xy dC(x,y) - 3 \\ &= 12 \iint\limits_{\mathbb{I}^2} C(x,y) dx dy - 3. \end{aligned}$$

Note that for any positive constant δ, and $H, H' \in \mathbb{H}$ with corresponding copula functions C and C' respectively, $\|H - H'\| < \delta \Rightarrow \|C - C'\| < \delta$. For given $\epsilon > 0$, choose $\delta = \epsilon/12$. Then for $H, H' \in \mathbb{H}$, $\|H - H'\| < \delta$ implies

$$\begin{aligned} |D(H) - D(H')| &= 12 \left| \iint\limits_{\mathbb{I}^2} C(x,y) dx dy - \iint\limits_{\mathbb{I}^2} C'(x,y) dx dy \right| \\ &\leq 12 \iint\limits_{\mathbb{I}^2} \left| C(x,y) - C'(x,y) \right| dx dy \\ &\leq 12\delta \iint\limits_{\mathbb{I}^2} dx dy = \epsilon. \end{aligned}$$

It follows that D is continuous with respect to the metric space $(\mathbb{H}, \rho)$. Hence, (34) satisfies Condition B(i).

In order to verify Condition B(ii), we use an alternative interpretation of the population version of Spearman's rho. Let $(V_1(\theta), W_1((\theta)))$, $(V_2(\theta), W_2(\theta))$ and $(V_3(\theta), W_3(\theta))$ be three independent random vectors with the common bivariate distribution function $H_{i\theta}$. The population version of Spearman's rho (Kruskal, 1958) for $H_{i\theta}$ can be written as

$$\begin{aligned} D(H_{i\theta}) &= 3\big(P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0] \\ &\quad - P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) < 0]\big). \end{aligned}$$

Since the random variables are continuous,

$$\begin{aligned} &P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0] \\ &\qquad = 1 - P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) < 0] \end{aligned}$$

and hence

$$D(H_{i\theta}) = 6P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0] - 3. \tag{35}$$

Boundedness of D follows immediately. Let us denote

$$\Omega_1 = \{W_1(\theta) < W_2(\theta) < W_3(\theta)\},\ \Omega_2 = \{W_1(\theta) < W_3(\theta) < W_2(\theta)\},$$
$$\Omega_3 = \{W_2(\theta) < W_1(\theta) < W_3(\theta)\},\ \Omega_4 = \{W_2(\theta) < W_3(\theta) < W_1(\theta)\},$$
$$\Omega_5 = \{W_3(\theta) < W_1(\theta) < W_2(\theta)\},\ \Omega_6 = \{W_3(\theta) < W_2(\theta) < W_1(\theta)\}.$$

Then

$$\begin{aligned}
&P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0] \\
&= P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_1] + P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_2] \\
&\quad + P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_3] + P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_4] \\
&\quad + P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_5] + P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_6] \\
&= P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_3(\theta)) > 0, \Omega_1] + P[(V_1(\theta) - V_3(\theta))(W_1(\theta) - W_2(\theta)) > 0, \Omega_1] \\
&\quad + P[(V_2(\theta) - V_1(\theta))(W_2(\theta) - W_3(\theta)) > 0, \Omega_1] + P[(V_3(\theta) - V_1(\theta))(W_3(\theta) - W_2(\theta)) > 0, \Omega_1] \\
&\quad + P[(V_2(\theta) - V_3(\theta))(W_2(\theta) - W_1(\theta)) > 0, \Omega_1] + P[(V_3(\theta) - V_2(\theta))(W_3(\theta) - W_1(\theta)) > 0, \Omega_1] \\
&= P[(V_1(\theta) < V_2(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_1(\theta) < V_3(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] \\
&\quad + P[(V_2(\theta) < V_1(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_1(\theta) < V_3(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] \\
&\quad + P[(V_2(\theta) > V_3(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_3(\theta) > V_2(\theta)), W_1(\theta) < W_2(\theta) < W_3(\theta)] \\
&= \frac{1}{6}\Big\{P[(V_1(\theta) < V_2(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_1(\theta) < V_3(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)] \\
&\quad + P[(V_2(\theta) < V_1(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_1(\theta) < V_3(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)] \\
&\quad + P[(V_2(\theta) > V_3(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)] + P[(V_3(\theta) > V_2(\theta))|W_1(\theta) < W_2(\theta) < W_3(\theta)]\Big\} \\
&= \frac{1}{3}\Big\{P[V_1(\theta) < V_3(\theta)|W_1(\theta) < W_2(\theta) < W_3(\theta)] + 1\Big\}.
\end{aligned}$$

It follows from (35) that

$$D(H_{i\theta}) = 2P[V_1(\theta) < V_3(\theta)|W_1(\theta) < W_2(\theta) < W_3(\theta)] - 1.$$

Since $W_1(\theta) < W_2(\theta) < W_3(\theta) \Leftrightarrow U_1 < U_2 < U_3$ and $P(U_1 < U_2 < U_3) = 1/6$, it follows that

$$\begin{aligned}
D(H_{i\theta}) &= 2P\left[Z_1\{\theta_0 + (\theta - \theta_0)(1 - U_1)\} < Z_3\{\theta_0 + (\theta - \theta_0)(1 - U_3)|U_1 < U_2 < U_3\}\right] - 1 \\
&= 12P\left[Z_1\{\theta_0 + (\theta - \theta_0)(1 - U_1)\} < Z_3\{\theta_0 + (\theta - \theta_0)(1 - U_3), U_1 < U_2 < U_3\}\right] - 1.
\end{aligned}$$

Let $\theta > \theta_0$. Then

$$
\begin{aligned}
&P\left[Z_1\{\theta_0 + (\theta-\theta_0)(1-U_1)\} < Z_3\{\theta_0 + (\theta-\theta_0)(1-U_3)\}, U_1 < U_2 < U_3\right] \\
&= \int_0^1 \int_0^{u_3} \int_0^{u_2} P\Big[Z_1\{\theta_0 + (\theta-\theta_0)(1-u_1)\} < Z_3\{\theta_0 + (\theta-\theta_0)(1-u_3)\}\Big] \\
&\qquad g(u_1)g(u_2)g(u_3)du_1du_2du_3 \\
&< \int_0^1 \int_0^{u_3} \int_0^{u_2} P\Big[Z_1\{\theta_0 + (\theta-\theta_0)(1-u_3)\} < Z_3\{\theta_0 + (\theta-\theta_0)(1-u_3)\}\Big] \\
&\qquad g(u_1)g(u_2)g(u_3)du_1du_2du_3 \\
&= \int_0^1 \int_0^{u_3} \int_0^{u_2} P\Big[Z_1 < Z_3\Big] g(u_1)g(u_2)g(u_2)du_1du_2du_3 \\
&= \frac{1}{2}\int_0^1 \int_0^{u_3} \int_0^{u_2} g(u_1)g(u_2)g(u_2)du_1du_2 = 1/12.
\end{aligned}
$$

Similar arguments shows that if $\theta < \theta_0$,

$$P\left[Z_1\{\theta_0 + (\theta-\theta_0)(1-U_1)\} < Z_3\{\theta_0 + (\theta-\theta_0)(1-U_3)\}, U_1 < U_2 < U_3\right] > 1/12$$

Therefore $D(H_{i\theta}) \neq 0$ whenever $\theta \neq \theta_0$. Thus, (34) satisfies Condition B(ii).

The sample Spearman's rho based on n paired data can be written as

$$D_n(\mathbf{Z}, \mathbf{U}) = a_n D(H^{(n)}) - b_n$$

where $a_n = \frac{n^2}{n^2-1}$, $b_n = \frac{3(2n+1)}{(n^2-1)}$ and

$$D(H^{(n)}) = 12 \int_0^\infty \int_0^1 F^{(n)}(z) G^{(n)}(u) dH^{(n)}(z,u) - 3.$$

Hence (6) satisfies Condition C(i).

Further, for $i = 1, \ldots, m$, $d_{i,n}(\theta)$ can be written as

$$
\begin{aligned}
&\frac{3}{n(n^2-1)} \sum_{i,j=1}^{n} I\left(Z_i\{\theta_0 + (\theta-\theta_0)(1-U_i)\} \geq Z_j\{\theta_0 + (\theta-\theta_0)(1-U_j)\}\right) I\left(U_i \geq U_j\right) \\
&+ \frac{3}{n(n^2-1)} \sum_{\substack{i,j,k=1 \\ k\neq j}}^{n} I\left(Z_i\{\theta_0 + (\theta-\theta_0)(1-U_i)\} \geq Z_j\{\theta_0 + (\theta-\theta_0)(1-U_j)\}\right) I\left(U_i \geq U_k\right).
\end{aligned}
$$

Clearly $d_{i,n}(\theta)$ is a differentiable function with respect to θ and the derivative at $\theta \in \Theta$ is zero except at countably many points. By Lagrange's Mean Value Theorem, for any $\theta, \theta' \in \Theta$, there exists some $\xi \in (\theta, \theta')$ such that

$$d_{i,n}(\theta) - d_{i,n}(\theta') = d'_{i,n}(\xi)(\theta - \theta').$$

Since $d_{i,n}$ is a piecewise constant function, with jump discontinuities at most at $n(n-1)$ locations determined by a sample from the continuous product distribution of the U_i's and Z_i's, it follows that $d'_{i,n} = 0$ almost surely. Hence, we can choose $B_n = 0$, and have $|d_{i,n}(\theta) - d_{i,n}(\theta')| \leq B_n|\theta - \theta'| = 0$, almost surely. Therefore D_n defined in (6) satisfies Condition C(ii). Since all the conditions of Theorem 3.3 holds, it follows that $\widehat{\theta}_S$ converges in probability to θ_0. This completes the proof.

Proof of Theorem 3.5. As in the proof Theorem 3.4, Conditions A and D of Theorem 3.3 are fulfilled by the assumptions of the Theorem 3.5 ; Conditions B and C remain to be checked.

The population version of (7) can be written as

$$D(H) = 4\int\int_{\mathbb{A}} H(v,w)dH(v,w) - 1.$$

Given $\epsilon > 0$, choose $\delta = \epsilon/8$. Let $H, H' \in \mathbb{H}$, then $\|H - H'\| < \delta$ implies

$$\begin{aligned}
&|D(H) - D(H')| \\
&= 4\left|\int\int_{\mathbb{A}} H(v,w)dH(v,w) - \int\int_{\mathbb{A}} H'(v,w)dH'(v,w)\right| \\
&= 4\left|\int\int_{\mathbb{A}} H(v,w)d[H(v,w) - H'(v,w)] + \int\int_{\mathbb{A}}[H(v,w) - H'(v,w)]dH'(v,w)\right| \\
&= 4\Big|\int\int_{\mathbb{A}} H'(v,w)d[H(v,w) - H'(v,w)] + \int\int_{\mathbb{A}}[H(v,w) - H'(v,w)]dH'(v,w) \\
&\quad + \int\int_{\mathbb{A}}[H(v,w) - H'(v,w)]d[H(v,w) - H'(v,w)]\Big|.
\end{aligned}$$

Integration by parts of the first term cancels out second term and we have

$$\begin{aligned}
|D(H) - D(H')| &= 4\Big|\int\int_{\mathbb{A}}[H(v,w) - H'(v,w)]d[H(v,w) - H'(v,w)]\Big| \\
&\leq 4\delta\int\int_{\mathbb{A}} d|H(v,w) - H'(v,w)| \\
&\leq 4\delta\left[\int\int_{\mathbb{A}} dH(v,w) + \int\int_{\mathbb{A}} dH'(v,w)\right] = \epsilon.
\end{aligned}$$

Hence the population version of Kendall's tau satisfies Condition B(i).

For checking Condition B(ii), we use an alternative interpretation of the population version of Kendall's tau. Let $(V_1(\theta), W_1(\theta))$ and $(V_2(\theta), W_2(\theta))$ be two independent random vectors with common bivariate distribution function $H_{i\theta}$. The population version of

Kendall's tau (Kruskal, 1958) for $H_{i\theta}$ can be written as

$$D(H_{i\theta}) = P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_2(\theta)) > 0] - P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_2(\theta)) < 0].$$

Since the random variables are continuous, $P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_2(\theta)) > 0] = 1 - P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_2(\theta)) < 0]$ and hence

$$\begin{aligned} D(H_{i\theta}) &= 2P[(V_1(\theta) - V_2(\theta))(W_1(\theta) - W_2(\theta)) > 0] - 1 \\ &= 2[P(V_1(\theta) > V_2(\theta), W_1(\theta) > W_2(\theta)) + P(V_1(\theta) < V_2(\theta), W_1(\theta) < W_2(\theta))] - 1 \\ &= 4P(V_1(\theta) < V_2(\theta), W_1(\theta) < W_2(\theta)) - 1. \end{aligned}$$

Boundedness of D follows immediately. Note that

$$D(H_{i\theta}) = 4P\left[Z_1\{\theta_0 + (\theta - \theta_0)(1 - U_1)\} < Z_2\{\theta_0 + (\theta - \theta_0)(1 - U_2)\}, U_1 < U_2\right] - 1.$$

Let $\theta > \theta_0$. Then

$$\begin{aligned} &P\left[Z_1\{\theta_0 + (\theta - \theta_0)(1 - U_1)\} < Z_2\{\theta_0 + (\theta - \theta_0)(1 - U_2)\}, U_1 < U_2\right] \\ &= \int_0^1 \int_0^{u_2} P\Big[Z_1\{\theta_0 + (\theta - \theta_0)(1 - u_1)\} < Z_2\{\theta_0 + (\theta - \theta_0)(1 - u_2)\}\Big] \\ &\qquad g(u_1)g(u_2)du_1du_2 \\ &< \int_0^1 \int_0^{u_2} P\Big[Z_1\{\theta_0 + (\theta - \theta_0)(1 - u_2)\} < Z_2\{\theta_0 + (\theta - \theta_0)(1 - u_2)\}\Big] \\ &\qquad g(u_1)g(u_2)du_1du_2 \\ &= 1/2 \int_0^1 \int_0^{u_2} g(u_1)g(u_2)du_1du_2 = 1/4. \end{aligned}$$

A similar argument show that whenever $\theta < \theta_0$, we have

$$P\left[Z_1\{\theta_0 + (\theta - \theta_0)(1 - U_1)\} < Z_2\{\theta_0 + (\theta - \theta_0)(1 - U_2)\}, U_1 < U_2\right] > 1/4$$

Therefore $D(H_{i\theta}) \neq 0$ whenever $\theta \neq \theta_0$. Thus, Condition B(ii) is established.

The sample Kendall's tau based on n paired data can be expressed as

$$D_n(\mathbf{Z}, \mathbf{U}) = a_n D(H^{(n)}) - b_n$$

where $a_n = \frac{n}{n-1}$, $b_n = \frac{3}{n-1}$ and $D(H^{(n)}) = 4\int_0^\infty \int_0^1 H^{(n)}(z, u)dH^{(n)}(z, u) - 1$. Hence D_n defined in (7) satisfies Condition C(i).

Again $d_{i,n}(\theta)$ can be written as

$$\frac{2}{n(n-1)} \sum_{i,j=1}^{n} I\left(Z_i\{\theta_0 + (\theta-\theta_0)(1-U_i)\} \geq Z_j\{\theta_0 + (\theta-\theta_0)(1-U_j)\}\right) I\left(U_i \geq U_j\right) - 1.$$

Clearly $d_{i,n}(\theta)$ is a differentiable function with respect to θ and the derivative at $\theta \in \Theta$ is zero except at countably many points. By Lagrange's Mean Value Theorem, for any $\theta, \theta' \in \Theta$, there exists some $\xi \in (\theta, \theta')$ such that

$$d_{i,n}(\theta) - d_{i,n}(\theta') = d'_{i,n}(\xi)(\theta - \theta').$$

Since $d_{i,n}$ is a piecewise constant function, with jump discontinuities at most at $n(n-1)$ locations determined by a sample from the continuous product distribution of the U_i's and Z_i's, it follows that $d'_{i,n} = 0$ almost surely. Hence, we can choose $B_n = 0$, and have $|d_{i,n}(\theta) - d_{i,n}(\theta')| \leq B_n|\theta - \theta'| = 0$, almost surely. Therefore D_n defined in (7) satisfies Condition C(ii). Since all the conditions of Theorem 3.3 holds, it follows that $\widehat{\theta}_K$ converges in probability to θ_0. This completes the proof.

PROOF OF LEMMA 4.1. In general if $Z_{i,j} \sim f$ and $U_{i,j} \sim g$, $j = 1(1)n_i$, $i = 1(1)m$, then under the model (2), the likelihood of $(\theta,\ \theta_2)$ is

$$L(\theta, \theta_2) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f\left(\frac{x_{M_{i,j}} + \theta y_{M_{i,j}}}{\theta\theta_2}\right) g\left(\frac{x_{M_{i,j}}}{x_{M_{i,j}} + \theta y_{M_{i,j}}}\right) \frac{1}{\theta_2(x_{M_{i,j}} + \theta y_{M_{i,j}})}.$$

Since each factor has the same functional form, it should suffice to calculate the information 'per sample'. After dropping subscripts for simplicity, the log-likelihood of θ and θ_2 based on a single observation (x, y) is

$$\begin{aligned}
l(\theta, \theta_2) &= \log f\left(\frac{x+\theta y}{\theta\theta_2}\right) + \log g\left(\frac{x}{x+\theta y}\right) - \log(\theta_2(x+\theta y)) \\
\text{then, } \frac{\partial l(\theta, \theta_2)}{\partial \theta} &= -\frac{g\prime(\frac{x}{x+\theta y})}{g(\frac{x}{x+\theta y})} \frac{xy}{(x+\theta y)^2} - \frac{f\prime(\frac{x+\theta y}{\theta\theta_2})}{f(\frac{x+\theta y}{\theta\theta_2})} \frac{x}{\theta^2\theta_2} - \frac{y}{x+\theta y}, \\
\text{and } \frac{\partial l(\theta, \theta_2)}{\partial \theta_2} &= -\frac{f\prime(\frac{x+\theta y}{\theta\theta_2})}{f(\frac{x+\theta y}{\theta\theta_2})} \frac{(x+\theta y)}{\theta\theta_2^2} - \frac{1}{\theta_2}.
\end{aligned}$$

Further,

$$\frac{\partial^2 l(\theta, \theta_2)}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-\frac{g\prime(\frac{x}{x+\theta y})}{g(\frac{x}{x+\theta y})} \frac{xy}{(x+\theta y)^2} - \frac{f\prime(\frac{x+\theta y}{\theta\theta_2})}{f(\frac{x+\theta y}{\theta\theta_2})} \frac{x}{\theta^2\theta_2} - \frac{y}{x+\theta y}\right). \tag{36}$$

It can be verified that

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left(-\frac{g\prime(\frac{x}{x+\theta y})}{g(\frac{x}{x+\theta y})}\frac{xy}{(x+\theta y)^2}\right) &= \frac{1}{\theta^2}\left[(1-u^2)\left\{u^2\left(\frac{g''(u)}{g(u)}-\left(\frac{g\prime(u)}{g(u)}\right)^2\right)+2\frac{ug\prime(u)}{g(u)}\right\}\right],\\
\frac{\partial}{\partial\theta}\left(-\frac{f\prime(\frac{x+\theta y}{\theta\theta_2})}{f(\frac{x+\theta y}{\theta\theta_2})}\frac{x}{\theta^2\theta_2}\right) &= \frac{1}{\theta^2}\left[z^2u^2\left(\frac{f''(z)}{f(z)}-\left(\frac{f\prime(z)}{f(z)}\right)^2\right)+2zu\frac{f\prime(z)}{f(z)}\right],\\
\text{and } \frac{\partial}{\partial\theta}\left(-\frac{y}{x+\theta y}\right) &= \frac{(1-u)^2}{\theta^2},
\end{aligned}$$

where $z=\frac{x+\theta y}{\theta\theta_2}$ and $u=\frac{x}{x+\theta y}$.

By combining the above three expressions, we get

$$\begin{aligned}
E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta^2}\right) &= \frac{1}{\theta^2}E\Big[(1-U^2)\left\{U^2\left(\frac{g''(U)}{g(U)}-\left(\frac{g\prime(U)}{g(U)}\right)^2\right)+2U\frac{g\prime(U)}{g(U)}+1\right\}\\
&\quad +Z^2U^2\left(\frac{f''(Z)}{f(Z)}-\left(\frac{f\prime(Z)}{f(Z)}\right)^2\right)+2ZU\frac{f\prime(Z)}{f(Z)}\Big]. \qquad (37)
\end{aligned}$$

Similarly, we get

$$E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta_2^2}\right) = \frac{1}{\theta_2^2}E\left[Z^2\left\{\frac{f''(Z)}{f(Z)}-\left(\frac{f\prime(Z)}{f(Z)}\right)^2\right\}+2Z\frac{f\prime(Z)}{f(Z)}+1\right], \qquad (38)$$

and

$$E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta\partial\theta_2}\right) = \frac{1}{\theta\theta_2}E\left[Z^2U\left\{\frac{f''(Z)}{f(Z)}-\left(\frac{f\prime(Z)}{f(Z)}\right)^2\right\}+ZU\frac{f\prime(Z)}{f(Z)}\right]. \qquad (39)$$

Since $Z\sim Gamma(p,1)$, we have

$$f(z) = \frac{\lambda^p}{\Gamma p}z^{p-1}e^{-z},\ \frac{f\prime(z)}{f(z)}=\frac{p-z-1}{z} \text{ and } \frac{f''(z)}{f(z)}-\left(\frac{f\prime(z)}{f(z)}\right)^2=-\frac{p-1}{z^2}.$$

Further, since $U\sim Beta(\alpha,\beta)$, we have

$$\begin{aligned}
g(u)=\frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)},\ \frac{g\prime(u)}{g(u)} &= \frac{u(2-\alpha-\beta)-(1-\alpha)}{u(1-u)}\\
\text{and } \frac{g''(u)}{g(u)}-\left(\frac{g\prime(u)}{g(u)}\right)^2 &= -\frac{\alpha-1}{u^2}-\frac{(\beta-1)}{(1-u)^2}.
\end{aligned}$$

Substitution of these expressions in (37), (38) and (39) produces

$$E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta^2}\right)=-\frac{1}{\theta^2}\frac{\alpha}{(\alpha+\beta+1)}\left[\beta+\frac{p(1+\alpha)}{\alpha+\beta}\right],$$

$$E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta_2^2}\right)=-\frac{p}{\theta_2^2},\ \text{and } E\left(\frac{\partial^2 l(\theta,\theta_2)}{\partial\theta\theta_2}\right)=-\frac{1}{\theta\theta_2}\frac{p\alpha}{\alpha+\beta}.$$

Therefore, the Information matrix based on a single sample is

$$I(\theta, \theta_2) \quad = \quad \begin{bmatrix} \frac{1}{\theta^2} \frac{\alpha}{(1+\alpha+\beta)} \left(\beta + \frac{p(1+\alpha)}{\alpha+\beta} \right) & \frac{1}{\theta\theta_2} \frac{p\alpha}{\alpha+\beta} \\ \frac{1}{\theta\theta_2} \frac{p\alpha}{\alpha+\beta} & \frac{p}{\theta_2^2} \end{bmatrix}.$$

It can be deduced from the above expression that the CRLB of θ based on n observations is given by

$$CRLB(\theta) = \frac{n(1 + \alpha + \beta)(\alpha + \beta)^2\theta^2}{\alpha\beta\,[p + (\alpha + \beta)^2]}. \tag{40}$$

This completes the proof.

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