Reliability Computation under Dynamic Stress-Strength Modeling with Cumulative Stress and Strength Degradation

Technical Report No. ASU/2014/7

Dated: 22 July, 2014

Prajamitra Bhuyan

Applied Statistics Unit,

Indian Statistical Institute,

Kolkata 700108

praja_r@isical.ac.in

Anup Dewanji

Applied Statistics Unit,

Indian Statistical Institute,

Kolkata 700108

dewanjia@isical.ac.in

Reliability Computation under Dynamic Stress-Strength Modeling with Cumulative Stress and Strength Degradation

Prajamitra Bhuyan and Anup Dewanji
Applied Statistics Unit, Indian Statistical Institute, Kolkata 700108.

Abstract

Reliability function is defined under suitable assumptions for dynamic stress-strength scenarios where strength degrades and stress accumulates over time. Methods for numerical evaluation of reliability are suggested under deterministic strength degradation and cumulative damage due to shocks arriving according to a Poisson process using simulation method and inversion theorem. These methods are specifically useful in the scenarios where damage distributions do not possess closure property under convolution. The method is also extended for non-identical, dependent damage distributions as well as for random strength degradation. Results from inversion method is compared with known approximate methods and also verified by simulation. As it turns out, the simulation method seems to have an edge in terms of computational burden and has much wider domain of applicability.

1 Introduction

The stress-strength model is widely used in mechanical engineering [1], aerospace engineering [2], seismic risk assessment [3], medicine [4], psychology [5] and various other allied fields for reliability calculation. An item fails whenever stress on the item is larger than its strength. Ample amount of research work on stress-strength modelling, under the assumption that stress and strength are static random variables, are available in the literature [6]. However, in most real life scenarios, strength degrades over time and stress is accumulation of damage due to shocks arriving at different time points according to a point process [7, p-192]. Strength degradation may be due to corrosion, fatigue, ageing, etc., which may be reasonably described by a deterministic curve $s(t)$, say [8]. One can also find enough material on cumulative damage models [9] and its application in crack growth model [10, 11] in the literature. Quite surprisingly, strength degradation and damage accumulation has not been taken concurrently into consideration in reliability analysis. However, this phenomena occurs often in real life scenarios. For example, the electric power of a dry cell or battery, initially stored by chemical energy, is weakened by continuous oxidation process and is subject to frequent use leading to accumulated damage or energy loss [12, p-35]. Also, in the area of disaster risk management, especially, in seismic risk analysis of civic structures, power plants [13], etc., proper assessment of seismic risk requires cumulative information regarding seismicity (i.e., frequency and magnitude of earthquake) of the location and the ability of the structure to resist earthquakes of various sizes [3]. Probabilistic models for earthquake recurrence and notion of cumulative damage due to consecutive shocks and aftershocks are first introduced by Cornell [14]. There has not been much work on reliability calculation when both stress and strength are time-dependent, as discussed above.

It is interesting to note that, in this stress-strength modeling, there is no concept of natural failure. That is to say that a unit fails only due to stress. Note that, even when the strength degrades up to zero at a time t_0, say, there is a positive probability $P[N(t_0) = 0]$ of failure not occurring by time t_0, if no shock arrives by that time, where $N(t)$ is the point process denoting the number of shocks arriving by time t. In order to include the possibility of natural failure even when no shock arrives by a certain time, we define failure time as the time when the accumulated stress equals or exceeds the strength at that time. Thus an item fails as soon as its strength reduces to zero even if no shock arrives by that time. Denoting the damages due to successive shocks by $X_1, X_2, X_3, \cdots$, the reliability function $R(t)$ at time t is then formally defined as

$$R(t) = P\left[T > t\right] = P\left[\sum_{i=1}^{N(t)} X_i < s(t)\right], \tag{1}$$

where T denotes the failure time. Basu and Ebrahimi [15] give an alternative, and more general, definition as

$$R(t) = P\left[T > t\right] = P\left[\inf_{0<u\leq t}\left\{s(u) - \sum_{i=1}^{N(u)} X_i\right\} > 0\right], \tag{2}$$

in a different context, which is equivalent to (1) in the case of non-increasing strength degradation $s(t)$. It is easy to prove that $R(t)$ satisfies the following four properties justifying the definition in (1) (See Section 2): (i) $R(0) = 1$, (ii) $\lim_{t\to\infty} R(t) = 0$, (iii) $R(t)$ is non-increasing in t and (iv) $R(t)$ is right-continuous.

Write $t_0 = \inf\{t\colon s(t) = 0\}$. Note that, since strength is always non-negative, t_0 may not exist. If it does (i.e., $t_0 < \infty$), then $R(t) = 0$ for $t \geq t_0$, since $P[\sum_{i=1}^{N(t)} X_i < 0] = 0$. In such case, there will be a jump of $R(t)$ at time t_0. In the previous definition, when failure occurs after the accumulated stress exceeds the strength, $R(t) = P[T > t] = P[\sum_{i=1}^{N(t)} X_i \leq s(t)]$. In that case, difficulty arises with finite t_0 as remarked before. In addition, $R(t)$ may not satisfy the right-continuity condition when the damage distribution is discrete. As for example, let us consider $s(t) = \max\{100-t, 0\}$ and $X_1, X_2, \cdots$ are consecutive independent damages from the common distribution having equal mass only at 30 and 50. Then $R(t)$ is not right-continuous at $t = 50$. Note that this difficulty does not arise for constant strength.

If $R(t)$ is defined as $P[T \geq t]$, then there is definitional problem regardless of this being defined as $P[\sum_{i=1}^{N(t)} X_i \leq s(t)]$ or $P[\sum_{i=1}^{N(t)} X_i < s(t)]$. At an instant t', when a shock arrives letting $\sum_{i=1}^{N(t')} X_i$ exceed $s(t')$, we have $T = t'$, i.e. $T \geq t'$, but neither $\{\sum_{i=1}^{N(t')} X_i \leq s(t)\}$ nor $\{\sum_{i=1}^{N(t)} X_i < s(t')\}$ is true. This definitional difficulty persists even with constant strength. Therefore, we take (1) as the definition of reliability $R(t)$.

Let us assume $s(t)$ to be right-continuous beside being non-negative and non-increasing with $s(0) > 0$. One popular choice is the exponential degradation model given by $s(t) = a\exp(-bt)$, or the linear degradation model [8, Ch-3] given by $s(t) = a - bt$ for $t \leq t_0 = a/b$ and $s(t) = 0$ for $t > t_0$. If the condition of right-continuity of $s(t)$ is violated, the right-continuity condition of $R(t)$ may be violated. For an example, let us consider the strength function $s(t)$ as

$$s(t) = \begin{cases} 100 - t & , \text{if } t \leq 50 \\ \max\{80 - t, 0\} & , \text{if } t > 50 \end{cases}$$

which is not right-continuous at $t = 50$. Also, suppose that the successive damages $X_1, X_2, X_3, \cdots$ are independent and identically distributed (iid) Exponential with scale parameter unity. Then, the reliability function $R(t)$, as defined in (1), is not right-continuous at $t = 50$, since $P[X_1 < s(50) = 50] \neq P[X_1 < s(50+) = 30-]$. One can easily construct several such examples considering distribution function of the damages to be strictly increasing. In this context, note that, if $s(t)$ has left-discontinuity at some time point and the damage distribution is strictly increasing, $R(t)$ is not left-continuous at the same time point. Nevertheless, the definition of $R(t)$ in general can be written as, using (refdef) and assuming independence between $N(t)$ and X_i's, for $t < t_0$

$$R(t) = P\left[N(t) = 0\right] + \sum_{n=1}^{\infty} P\left[N(t) = n\right] P\left[\sum_{i=1}^{n} X_i < s(t)\right], \tag{3}$$

In particular, when shocks arrive according to a non-homogeneous Poisson process with intensity $\lambda(t)$ and the successive shocks cause iid damages, denoted by $X_1, X_2, X_3, \cdots$, say, to the system following the common distribution F and independent of the Poisson process $N(t)$ generating the shocks, then the reliability function of the system, for $t < t_0$, is given by

$$R(t) = e^{-\Lambda(t)} + \sum_{n=1}^{\infty} F^{(n)}(s(t)-) e^{-\Lambda(t)} (\Lambda(t))^n / n!, \tag{4}$$

where $\Lambda(t) = \int_0^t \lambda(u)du$ and $F^{(n)}(s(t)-) = P[\sum_{i=1}^{n} X_i < s(t)]$. This expression (4) has been known in the context of cumulative damage model with a constant strength or threshold [17, p-93]. However, there has not been much effort in evaluating this reliability, which involves obtaining the compound distribution of $\sum_{i=1}^{N(t)} X_i$. This compound distribution is of interest in many other contexts, including actuarial science, queuing theory, banking and inventory problems, etc., among some others. Since, in the case of Poisson arrival of shocks, the different moments of $\sum_{i=1}^{N(t)} X_i$ can be easily obtained in terms of moments of $N(t)$ and F, often some moment-based approximation method can be applied. The derivation of the compound distribution is challenging even numerically because of the difficulty involved in obtaining $F^{(n)}(t)$ in general. It becomes more difficult when $N(t)$ is a non-Poisson point process, even for a renewal process, since the moments of $N(t)$ then cannot easily be obtained. In this paper, we discuss some approximation methods for evaluation of this compound distribution in general and hence, of the reliability $R(t)$, assuming deterministic strength degradation and stochastic stress accumulation due to damages caused by shocks arriving according to a point process, and Poissoon proces in particular. This, in turn, gives the distribution of T. The different moments of T, however, cannot be obtained easily without resorting to some numerical method using its distribution.

Normal approximation and use of approximation methods analogous to Pearsonian family of distributions [18] are commonly used techniques to approximate the distribution of a random variable based on its moments. These methods however do not seem to perform well in our context. We suggest a method based on inversion formula to obtain the distribution function $F^{(n)}(\cdot)$ from the corresponding characteristic function (See Gil-Pelaez, 1951 [19]) and then use (4). We also suggest a simulation based method to estimate the reliability function, which is known to give consistent estimates by the Strong Law of Large Numbers. The method based on inversion formula, if it can be used, gives comparable results.

We first examine the four properties of $R(t)$ in Section 2. In Section 3, we consider the successive damages to be independent and identically distributed and discuss different methods including the inversion method and a simulation based method. In Section 4, we provide numerical illustration of the different methods for calculation of reliability. Section 5 considers some generalizations including non-identical and dependent damage distributions and some random strength degradation. It turns out that the simulation based method requires less computational effort and is the most flexible in considering general damage distributions and random arrival mechanism of the shocks. Section 6 ends with some concluding remarks.

2 Properties of Reliability Function

In this section we show that $R(t)$ in (3) satisfies all the four conditions of reliability function mentioned in the previous section. For this purpose, we first rewrite $R(t)$, using $P[N(t)=n] = P\left[T_n \le t\right] - P\left[T_{n+1} \le t\right]$ [16], as

$$R(t) = P\left[T_1 > t\right] + \sum_{n=1}^{\infty} \left\{P\left[T_n \le t\right] - P\left[T_{n+1} \le t\right]\right\} P\left[\sum_{i=1}^{n} X_i < s(t)\right], \tag{5}$$

where T_n denotes the occurrence time of the nth shock, for $n = 1, 2, \cdots$. In order to prove these properties of $R(t)$, we use the following two Lemmas.

Lemma 2.1: The infinite sum $\sum_{n=0}^{\infty} P[\sum_{i=1}^{n} X_i < s(t)]P[N(t) = n]$ is uniformly convergent with respect to t.

Proof: Note that $P[\sum_{i=1}^{n} X_i < s(t)]P[N(t) = n] \le P[\sum_{i=1}^{n} X_i < s(0)]$, for all $t \ge 0$. Now it is enough to show, from [20, Therorem 7.10, p-148] that $\sum_{n=0}^{\infty} P[\sum_{i=1}^{n} X_i < s(0)]$ is convergent, which does not depend on t. Note that, when, X_i's are non-negative iid random variables, we can find an m such that $P[\sum_{i=1}^{m} X_i < s(0)] < 1$, assuming that X_i's are not degenerate at 0. Therefore, $\{P[\sum_{i=1}^{n} X_i < s(0)]\}^{\frac{1}{n}} < 1$ for all $n \ge m$ and it is also bounded below. Using Lemma 3.7 of [17, p-94], we have $\{P[\sum_{i=1}^{n} X_i < s(0)]\}^{\frac{1}{n}}$ to be non-increasing in n, for $n = 1, 2, \cdots$. Therefore, $\lim_{n\to\infty}\{P[\sum_{i=1}^{n} X_i < s(0)]\}^{\frac{1}{n}} < 1$ and $\sum_{n=0}^{\infty} P[\sum_{i=1}^{n} X_i < s(0)]$ converges by Cauchy's Root Test [20, Theorem 3.33, p-65]. Hence, $\sum_{n=0}^{\infty} P[\sum_{i=1}^{n} X_i < s(t)]P[N(t) = n]$ is uniformly convergent with respect to t.

For a sequence of non-iid random variables, one can manufacture an example where $P[\sum_{i=1}^{n} X_i < s(0)] = 1$, for all $n = 1, 2, \cdots$. Suppose the X_i's are independent Uniform random variables over $(0, \frac{1}{i^p})$, $p > 1$, for $i = 1, 2, \cdots$, such that $\sum_{i=1}^{\infty} \frac{1}{i^p} < s(0)$. However, this is not a realistic situation, since this example results in $R(t) = 1$ for all t under the fixed strength (i.e., $s(t) = s(0)$, for all t) model. In general, it is unlikely that a distribution from the large class of life distributions as a model for the damage distributions will lead to such situation. Therefore, one can reasonably assume that $P[\sum_{i=1}^{m} X_i < s(0)] < 1$, for some m. In many real life scenarios, successive damages are stochastically increasing, although they are independent. In such situations, we have $\{P[\sum_{i=1}^{n} X_i < s(0)]\}^{\frac{1}{n}}$ to be non-increasing in n [17, Lemma 3.10, p-95]. One can further extend this Lemma for the case when consecutive damages are dependent satisfying the following three conditions: (i) $P[X_k > u|X_1, \cdots, X_{k-1}]$ depends on $X_1, \cdots, X_{k-1}$ only through the sum $\sum_{i=1}^{k-1} X_i$, (ii) $P[X_k > u|\sum_{i=1}^{k-1} X_i = z]$ is increasing in $z \geq 0$ and (iii) $P[X_k > u|\sum_{i=1}^{k-1} X_i = z] \leq P[X_{k+1} > u|\sum_{i=1}^{k} X_i = z]$ for all $k = 1, 2, \cdots$. Under these conditions, we have $\{P[\sum_{i=1}^{n} X_i < s(0)]\}^{\frac{1}{n}}$ to be non-increasing in n [17, Lemma 3.13, p-96]. Rest of the proof is similar to the iid set-up.

Lemma 2.2: If f is left-continuous function and g is non-increasing right-continuous function, then $f \circ g$ is right-continuous.

Proof: Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ and $x_n \geq x$, for all $n = 1, 2, \cdots$. Then, since g is right-continuous, $g(x_n) \to g(x)$. Also, since g is a non-increasing function, $g(x_n) \leq g(x)$, for all $n = 1, 2, \cdots$. Therefore, $f(g(x_n)) \to f(g(x))$, since f is a left-continuous function. From sequential characterization of continuity [20, Theorem 4.2, p-84], we conclude that $f \circ g$ is right-continuous.

Note that, we have $P(T_n \leq 0) = 0$, $P(T_n > 0) = 1$ and $\lim_{t\to\infty} P(T_n \leq t) = 1$, for all $n = 1, 2, \cdots$. Also, the right hand side of (5) is uniformly convergent with respect to t (see Lemma 2.1). Hence, $R(0) = 1$ and $\lim_{t\to\infty} R(t) = 0$. Note that, for $t_1 < t_2$, $s(t_2) \leq s(t_1)$ and $P[\sum_{i=1}^{n+k} X_i < s(t_2)] \leq P[\sum_{i=1}^{k} X_i < s(t_2)] \leq P[\sum_{i=1}^{k} X_i < s(t_1)]$, for

$n, k = 0, 1, 2, \cdots$, since each X_i is a non-negative random variable. Now

$$\begin{aligned}
R(t_2) &= P\left[\sum_{i=1}^{N(t_2)} X_i < s(t_2)\right] \\
&= \sum_{n=0}^{\infty} P\left[\sum_{i=1}^{n} X_i < s(t_2) | N(t_2) = n\right] P\left[N(t_2) = n\right] \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} P\left[\sum_{i=1}^{n} X_i < s(t_2)\right] P\left[N(t_2) = n | N(t_1) = k\right] P\left[N(t_1) = k\right] \\
&\leq \sum_{k=0}^{\infty} P\left[\sum_{i=1}^{k} X_i < s(t_1)\right] \sum_{n=k}^{\infty} P\left[N(t_2) = n | N(t_1) = k\right] P\left[N(t_1) = k\right] \\
&= P\left[\sum_{i=1}^{N(t_1)} X_i < s(t_1)\right], \text{since} \sum_{n=k}^{\infty} P\left[N(t_2) = n | N(t_1) = k\right] = 1 \\
&= R(t_1)
\end{aligned}$$

By definition, $P(T_n \leq t)$ is right-continuous and $P[\sum_{i=1}^{n} X_i < x]$ is left-continuous, for all $n = 1, 2, \cdots$. Since, by assumption, $s(t)$ is non-increasing and right-continuous, it follows from Lemma 2.2 that $P[\sum_{i=1}^{n} X_i < s(t)]$ is right-continuous in t. Therefore, all the terms in the right hand side of (5) is right-continuous functions of t and it is also uniformly convergent (see Lemma 2.1). Hence, $R(t)$ is right-continuous [20, Theorem 7.11, p-149].

3 Independent and Identically Distributed Damages

When the damage distribution is closed under convolution in the sense that $F^{(n)}$, for all n , belongs to the same class of distribution as F, the evaluation of $R(t)$ using (3) is straightforward requiring some numerical calculation. For example, when F is a Normal distribution with mean μ and variance σ^2, $F^{(n)}$ is also Normal with mean $n\mu$ and variance $n\sigma^2$. Then, assuming that $N(t)$ follows a Poisson process, the reliability is given by

$$R(t) = e^{-\Lambda(t)} + \sum_{n=1}^{\infty} \Phi\left(\frac{s(t) - n\mu}{\sqrt{n}\sigma}\right) e^{-\Lambda(t)} \left(\Lambda(t)\right)^n / n!,$$

for $t < t_0$, where $\Phi(\cdot)$ denotes the c.d.f. of the standard Normal distribution. Similarly, if X_i's are exponentially distributed with mean $1/\lambda$, then $F^{(n)}(\cdot)$ corresponds to a Gamma distribution with mean n/λ and

$$R(t) = e^{-\Lambda(t)} + \sum_{n=1}^{\infty} \Gamma\left(s(t), n, \lambda\right) e^{-\Lambda(t)} \left(\Lambda(t)\right)^n /n!,$$

where $\Gamma(\cdot, n, \lambda)$ denotes the c.d.f. of Gamma distribution with scale parameter λ and shape parameter n. We now discuss some methods for evaluation of $R(t)$ when F does not have such closure property.

3.1 Normal Approximation

Approximating the compound distribution of $X(t) = \sum_{i=1}^{N(t)} X_i$ by a Normal distribution with corresponding mean $\mu(t) = \Lambda(t)E(X_1)$ and variance $\sigma^2(t) = \Lambda(t)E(X_1^2)$ is common [21, p-142]. Although there is a positive probability at $X(t) = 0$, (that is, when $N(t) = 0$), this approximation may be reasonable when there are frequent shocks, or $\Lambda(t)$ is large enough. Nevertheless, using this approximation, we have

$$R(t) \approx \Phi\left(\tfrac{s(t)-\mu(t)}{\sigma(t)}\right).$$

3.2 Panjer Recursion

Panjer recursion formula [21, p-161] can be used to evaluate compound distribution for discrete damages arriving according to some frequency distribution belonging to Panjer class, which satisfies $\frac{P[X=k]}{P[X=k-1]} = a + \frac{b}{k}$, for $k = 1, 2, \cdots$. As for example, if we assume shocks arrive accroding to a Poisson process, then $P[N(t) = k]$ satisfies the above relation with $a = 0$ and $b = \lambda t$ and hence it belongs to the Panjer class. Thus, to apply the method for cumulative damage models, where damages are typically continuous, the continuous damages needs to be discretized. Suppose the damage distribution is defined on $0, 1, \cdots, m$ representing multiples of some convenient measuring unit. The number m represents the largest possible damage and could be finite. Then the probability distribution of $X(t)$ is given by the recursion relation (See [21, p-162])

$$P\left[X(t)=x\right]=\frac{\sum_{y=1}^{x\wedge m}\left(a+\frac{by}{x}\right)P\left[X_1=y\right]P\left[X(t)=x-y\right]}{1-aP\left[X_1=0\right]},x=1,2,\cdots,$$

and $P[X(t)=0]=P[N(t)=0]$. Apart from discretization error, this method suffers from numerical instability due to rounding off errors propagated at each stage of recursion. Analogue of Panjer recursion for continuous damages is provided by a recursive integral equation [21, p-167] which can be solved using complicated methods like hybrid MCMC. Apart from computational challenges, it has restricted application due to inability to accommodate various kinds of shock arrival processes arising in real life scenarios.

3.3 Johnson Distribution

Since the moments of $X(t)=\sum_{i=1}^{N(t)}X_i$ can be easily obtained in terms of the moments of the damage distribution $F(\cdot)$ and those of the process $N(t)$, one can think of approximating the distribution of $X(t)$ using its moments by identifying it with a family of Johnson distribution. According to Johnson [18], for given β_1 and β_2 obtained from the first four central moments of $X(t)$, its distribution can be identified with one of the following three transformations as given by

(i) $S_L distribution: Z=\gamma+\delta\log(X-\xi), X\geq\xi,$

(ii) $S_B distribution: Z=\gamma+\delta\log(\frac{X-\xi}{\xi+\lambda-X}),\xi\leq X\leq\xi+\lambda,$

(iii) $S_U distribution: Z=\gamma+\delta\sinh^{-1}(\frac{X-\xi}{\lambda}),$

where Z denotes the standard Normal variate, $X=X(t)$ is the variable of interest and $(\lambda,\gamma,\delta,\xi)$ are the associated parameters which can be obtained from the first four central moments of $X(t)$ [22]. As with the normal approximation of Section 3.1, the fact that $X(t)$ has positive mass at 0 may create some difficulty with the Johnson approximation specially when shocks are infrequent.

3.4 Gil-Pelaez Inversion Formula

Gil-Pelaez [19] derived a version of inversion formula to obtain the distribution function $G(x)$ of a random variable X from its characteristic function $\phi(u) = \int_{-\infty}^{\infty} e^{iux} dG(x)$ as given by

$$\begin{aligned} G(x) &= \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\left(\frac{e^{-iux}\phi(u) - e^{iux}\phi(-u)}{2iu}\right)du \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} Im\left(\frac{e^{-iux}\phi(u)}{u}\right)du, \end{aligned} \tag{6}$$

for a continuity point x of $G(x)$, where $Im(z)$ denotes the imaginary part of a complex number z. This result is useful for numerical evaluation of the distribution function of $Y = \sum_{i=1}^{n} X_i$ by numerical integration [23], where $X_1, \cdots, X_n$ are independent and identically distributed. Note that the characteristic function of Y is given by $\phi_Y(u) = \{\phi_X(u)\}^n$, where $\phi_X(\cdot)$ denotes the characteristic function of X_1 with c.d.f. $F(\cdot)$. Then, using (6), the distribution function $F^{(n)}(\cdot)$ can be written as

$$F^{(n)}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} Im\left(\frac{e^{-iux}\phi_Y(u)}{u}\right)du. \tag{7}$$

One can then use (3) to calculate reliability $R(t)$. Inversion of the characteristic function is done by numerical integration based on uni-dimensional adaptive 15-point Gauss-Kronrod quadrature formula using the Fortran functions DQAGE, and DQAGIE, available from Netlib. Alternatively, any standard software package equipped with numerical integration (e.g., *integrate* in the package R) can be used. The details of the algorithm and its precision are discussed in Piessens et al. (1983) [24]. Computation of $R(t)$ using this approach is computationally challenging. Complexity of computation increases if there is no closed form for the characteristic function of damage distribution (e.g., Weibull, Log-normal, etc.). Apart from that, the behaviour of the integrand in the inversion formula is extremely fluctuating for certain choices of distribution and associated parameters. Despite all these challenges, it provides excellent results under different choices of damage distributions.

3.5 Simulation Method

This involves generation of realizations of the lifetime variable T by simulating the whole process of shock arrivals and accumulation of damages and following it up with the degradation of strength. The lifetime distribution is estimated from a large number of, say 10000, realizations of T along with the estimate of reliability $R(t)$ at some chosen points of time. The algorithm for simulating a realization of T is provided by the following steps.

$U_0 = 0$, $X_0 = 0$; For $i = 1, 2, \cdots,$

(I) Simulate $U_i \sim G(\cdot)$ and $X_i \sim F(\cdot)$;

(II) Calculate $t_i = \sum_{l=0}^{i} U_l$ and $X(t_i) = \sum_{l=0}^{i} X_l$;

(III) If $X(t_i) < s(t_i)$, then next i;
else, if $X(t_{(i-1)}) < s(t_i) \leq X(t_i)$, then $T = t_i$;
else, find $T = t$ by solving $S(t) = X(t_{(i-1)})$, for $i > 1$, and $S(t) = 0$, for $i = 1$.

Note that the programming and associated computation is much simpler than those required for implementation of the inversion method of Section 3.4. Interestingly, the computational burden also appears to be much less (See the comments on computing time after Table 2). More importantly, the domain of application is much wider than that of the inversion method (See Table 7 and Section 6).

4 Numerical Illustration

As remarked before, numerical evaluation of reliability is straightforward if distributions used to model damages and inter-arrival times possess closure property under convolution, or their convolutions have some known distributions. In other situations, one can use the different approximations mentioned in Section 3. In this section, we present some numerical calculation of reliability $R(t)$ under different distributional assumptions with several sets of values for the associated parameters using the methods of Section 3.4 and 3.5. In our calculations, we have verified convergence of these two numerical methods by varying number of ordinates in numerical integration and number of observations in simulation. Precision of computed values of reliability is assured up to third decimal places.

For our presentation of the results in Table 1, for each chosen set of distributional assumptions, we first estimate the pth quantile of the distribution of T by the method of simulation, for $p = 0.1, 0.3, 0.5, 0.7$ and 0.9, as reported in parentheses. Then, the reliabilities at these quantiles are calculated using the inversion method, which are expected to be close to $1-p$. This is also evident from the results in Table 1. This kind of illustration indicates the closeness between the two methods of reliability calculation. For the illustration, we consider two models for the shock arrival process $N(t)$, namely, (i) a Poisson process with rate λ, and (ii) a renewal process with inter-arrival time distribution G. Note that the choice (i) is a special case of choice (ii) in which inter-arrival distribution is Exponential with mean $1/\lambda$, denoted by $Exp(\lambda)$. We also consider three models for the iid damage distribution F, namely, (i) a $Exp(\lambda)$ distribution, (ii) a Weibull distribution with scale parameter α and shape parameter p, denoted by $Wei(\alpha, p)$, with mean damage being $\alpha\Gamma(1+\frac{1}{p})$, and (iii) a Log-normal distribution with Normal parameters μ and σ, denoted by $LN(\mu, \sigma)$, with mean damage being $\exp(\mu + \frac{1}{2}\sigma^2)$. Note that no closed form expression for the characteristic functions of Weibull and Log-normal distributions are available. For the purpose of calculating the integral in equation (7), the integrand is evaluated at different abscissae, which involves evaluation of the characteristic function $\phi_Y(u) = \{\phi_X(u)\}^n$ of the nth order convolution of the respective damage distribution. If the characteristic function $\phi_X(u)$ is not available in closed form, one can use numerical integration to evaluate the characteristic function only at those abscissae required for the evaluation of convolution of the damage distribution, using $\phi_Y(u) = \{\phi_X(u)\}^n = \left\{\int_0^\infty \exp(-iux)f(x)dx\right\}^n$, where $f(\cdot)$ is the density corresponding to the distribution function $F(\cdot)$ with characteristic function $\phi_X(\cdot)$. Similar evaluation of convolutions of the inter-arrival time distribution will be required while computing reliability $R(t)$ (See (5)) if the corresponding closed form expression is not available. In Table 1, we consider different combinations for inter-arrival time distribution and iid damage distribution with several sets of values for the associated model parameters with the corresponding means in parentheses, and the deterministic curve $s(t)$ for strength degradation. These simulation models are chosen to reflect different patterns of stress accumulation involving frequency of shocks and size of corresponding damages, along with exponential or linear strength degradation. For the sake of comparison, we also compute reliability $R(t)$ by the normal approximation (Section 3.1) and Johnson approximation (Section 3.3). When both F and G are Exponential, then one can compute $R(t)$ directly (as remarked at the beginning of Section 3) and the method is named 'Direct' in Table 1. From the results in Table 1, it is evident that the normal approximation and the Johnson approximation often fail to provide good approximation to the distribution of $\sum_{i=1}^{N(t)} X_i$, whereas the other two methods, namely the inversion method and the simulation method, seem to give reasonably close values. Note that, for the last three simulation models in Table 1, when the shock arrival process $N(t)$ is a renewal process with Log-normal renewal distribution, the moments of $\sum_{i=1}^{N(t)} X_i$ are difficult to obtain and so the two approximation methods cannot be used. This is one more reason why the two approximation methods may not be preferred.

We now investigate the effect of different distributional assumptions with associated model parameters chosen so as to keep the mean inter-arrival time and mean damage fixed and also with a fixed strength curve $s(t)$. We fix these two means at 5 and 2, respectively, and take $s(t) = max\{80 - 0.5t, 0\}$. The results on the reliability calculation are presented in Table 2, as in Table 1, with corresponding means and quantiles in parentheses. Here we use only the inversion method since the normal and Johnson approximations do not often give good results, as is evident from Table 1. However, the inversion method and the simulation method produce similar results regardless of the distributional assumptions (See both Tables 1 and 2). Since the simulation method is known to give consistent estimate, both the methods can be useful. It is to be noted that, while the inversion method is theoretically more satisfying involving only approximation error, the simulation method gives only an estimate involving sampling error. Both are, however, computationally intensive and the simulation method appears to have an edge over the inversion method in this regard. On the other hand, the simulation method has wider domain of application as will be discussed in the following section.

Table 1: Reliability calculation by different methods with different damage and inter-arrival time distributions with means and estimated quantiles in parentheses.

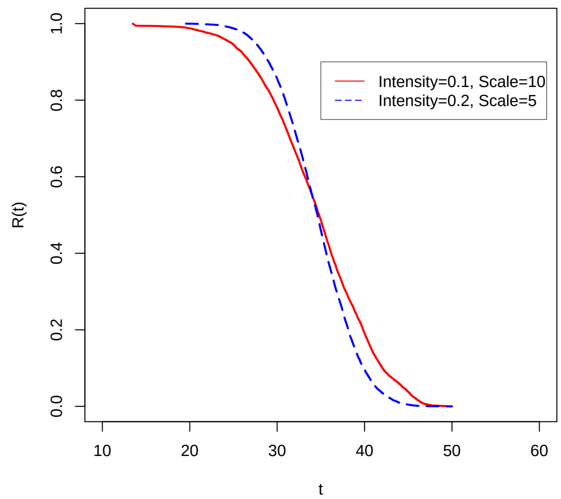
						$1-p$		
G	F	$s(t)$	Method	0.9	0.7	0.5	0.3	0.1
			Normal	0.911	0.668	0.451	0.278	0.121
$Exp(0.4)$	$Exp(0.04)$	$700\exp(-0.05t)$	Johnson	0.900	0.704	0.499	0.303	0.106
(2.5)	(25)		Direct	0.900	0.702	0.498	0.304	0.107
				(17.170)	(20.598)	(23.234)	(25.886)	(29.986)
			Normal	0.907	0.702	0.477	0.276	0.105
$Exp(0.2)$	$Wei(5,20)$	$max\{100-t,0\}$	Johnson	0.902	0.716	0.497	0.286	0.099
(5)	(4.87)		Inversion	0.901	0.708	0.500	0.301	0.094
				(41.289)	(46.664)	(51.132)	(55.584)	(61.516)
			Normal	0.897	0.607	0.414	0.276	0.155
$Exp(0.1)$	$LN(0.10,1)$	$850\exp(-0.1t)$	Johnson	0.860	0.527	0.245	0.203	0.024
(10)	(1.82)		Inversion	0.903	0.701	0.502	0.304	0.102
				(40.387)	(44.588)	(47.633)	(50.900)	(56.354)
$LN(-0.31,1.41)$	$Exp(0.04)$	$550\exp(-0.01t)$	Inversion	0.902	0.704	0.507	0.306	0.100
(1.98)	(25)			(17.288)	(23.905)	(29.529)	(36.521)	(50.317)
$LN(1.80,1)$	$LN(0.19,1)$	$max\{100-0.7t,0\}$	Inversion	0.894	0.695	0.505	0.307	0.100
(9.97)	(1.99)			(95.464)	(105.835)	(112.073)	(118.047)	(126.279)
$LN(1.00,1)$	$Wei(10,15)$	$max\{150-1.2t,0\}$	Inversion	0.899	0.702	0.505	0.300	0.101
(4.48)	(9.66)			(32.671)	(38.493)	(43.762)	(49.863)	(59.892)

Table 2: Reliability calculation by inversion method for different damage and inter-arrival time distributions with fixed means 2 and 5, respectively, and fixed strength $s(t) = max\{80-0.5t, 0\}$. The estimated quantiles are in parentheses.

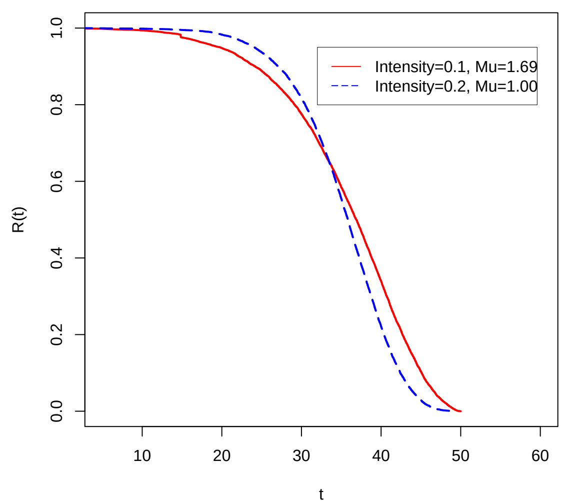
				$1-p$		
G	F	0.9	0.7	0.5	0.3	0.1
$Exp(0.2)$	$Exp(0.5)$	0.903	0.699	0.496	0.294	0.099
(5)	(2)	(72.729)	(83.061)	(90.126)	(97.132)	(106.745)
$Exp(0.2)$	$Wei(1,0.5)$	0.902	0.703	0.497	0.297	0.101
(5)	(2)	(63.607)	(81.854)	(93.664)	(104.216)	(117.431)
$Exp(0.2)$	$LN(0.19,1)$	0.901	0.701	0.507	0.305	0.102
(5)	(2)	(70.992)	(83.170)	(90.776)	(98.250)	(108.383)
$LN(1.11,1)$	$Exp(0.5)$	0.899	0.699	0.503	0.304	0.106
(5)	(2)	(70.960)	(81.467)	(88.892)	(96.602)	(108.118)
$LN(1.11,1)$	$LN(0.19,1)$	0.904	0.699	0.498	0.294	0.098
(5)	(2)	(68.809)	(81.695)	(90.128)	(98.484)	(110.380)
$LN(1.11,1)$	$Wei(1,0.5)$	0.898	0.701	0.507	0.304	0.098
(5)	(2)	(62.530)	(80.710)	(92.308)	(103.635)	(118.908)

The computing time varies with the number of time points and their positions in the domain of the life distribution, at which reliability function is to be computed, keeping all the parameters of the stress-strength modeling fixed. The following computing times correspond to the values of the parameters associated with the deterministic strength function, inter-arrival time distribution and damage distribution as specified in Table 1, with a computer equipped with 3 GB RAM and 2.0 GHz Intel Core(TM) 2 Duo processor. Reliability computation at a particular time point by any of the Direct, Johnson approximation and Normal approximation methods takes only a fraction of a second for one set of parameters. Inversion method, in general, takes more time compared to the other methods as mentioned above. The Inversion method takes just about less than a minute for reliability calculation with Exponential inter-arrival times and Lognormal damages at any of the specified quantiles. However, for Weibull damages with Exponential inter-arrival times, it takes on an average 1.5 minutes of computing time. It takes on an average 3 minutes of time for Lognormal inter-arrival times with Weibull damages. However, the computation time ranges from 3 minutes to 8 minutes for the case of Lognormal inter-arrival times with Lognormal damages. Simulation method provides estimates of the entire life distribution in contrast with the Inversion method which gives reliability value for a specified quantile. Simulating the 10000 realizations takes only 4 to 8 seconds for one set of parameters.

It is of interest to investigate the effect of different patterns of stress accumulation while keeping the expected cumulative stress fixed. For this purpose, we consider $N(t)$ to be a Poisson process with rate λ so that the expected cumulative stress at time t is $\lambda t E(X_1)$. Specifically, we like to investigate the effect of small damages arriving frequently as against rarely arriving heavy damages on the reliability curve. So, the choices of λ and the damage distribution reflect different such patterns of stress accumulation so that $\lambda E(X_1)$ remains the same. We carry out this investigation by means of some example of reliability calculation with $s(t) = \max\{100 - 2t, 0\}$ and some choices of λ and damage distribution such that $\lambda E(X_1)$ is the same. We consider $\lambda = 0.1$ and 0.2 and $Weibull(\alpha, p)$ damage distribution with $p = 5$ and $\alpha = 10$ and 5, respectively, with corresponding mean cumulative damage remaining at $0.918t$ at time t. Figure $1(a)$ gives the reliability curves for these two scenarios obtained by the method of simulation. Figure $1(b)$ gives the reliability curves with the same $s(t)$ and same two values of λ, but with $LN(\mu, \sigma)$ damage distribution with $\sigma = 1$ and $\mu = log(2) + 1 = 1.69$ and 1, respectively. Here, the mean cumulative damage at time t remains at $0.896t$. We see from both Figures $1(a)$ and $1(b)$



(a) Reliability Curves with iid Weibull Damages



(b) Reliability Curvess with iid Log-normal Damages

Figure 1: Comparison of Reliability Curves with Fixed Expected Cumulative Stress

that, although the mean cumulative damage remains the same for all t, the reliability curves for the two scenarios under study are different and crossing at a time point. It is also observed that the reliability is initially smaller for the case of larger damages occurring less frequently. Intuitively, this is because, as initially the strength curve is higher, the system with larger damages is more likely to exceed the strength than the system with frequently occurring smaller damages. On the other hand, at later time when the strength is diminished, the system with frequently occurring smaller damages is more likely to exceed the diminished strength than the less frequent damages.

5 Some Generalizations

In this section, we will consider some generalizations where both the inversion formula and the simulation method can be applied. For all the numerical illustrations in this section, as in Tables 1 and 2, the quantiles in parentheses are obtained by the simulation method and then the reliabilities at these quantiles are obtained by the inversion method. The mean inter-arrival time and mean damage are also given in parentheses.

5.1 Independent but Non-iid Damage Distributions

Sometimes it is of interest for engineers to study life distribution or reliability curve under some stochastic ordering of successive damages coming from the same family of distributions. For example, the successive damages may be stochastically larger. In such situations, the successive damages may be independent but not identically distributed. In general, the non-identical damage distributions need not be from the same family. Nevertheless, if $X_1, X_2, \cdots X_n$ are independent but non-identically distributed, then the characteristic function of $Y = \sum_{k=1}^{n} X_k$ is given by $\phi_Y(u) = \prod_{k=1}^{n} \phi_{X_k}(u)$, where $\phi_{X_k}(u)$ is the characteristic function of X_k. This characteristic function $\phi_Y(u)$ is very useful for reliability computation using inversion method, in particular (3) and (7), for shocks causing damages with non-identical distributions. One can also use the simulation method of Section 3.5 for such reliability calculation.

For illustration, let us first assume that the damage corresponding to ith shock follows a Gamma distribution with scale parameter β_i and shape parameter α, denoted by $Ga(\beta_i, \alpha)$, with mean being $\alpha\beta_i$. Also, the shocks are assumed to arrive according to a renewal process with distribution of inter-arrival time being $LN(1.8, 1)$ and strength function $s(t) = 950 \exp(-0.05t)$. Computational results with $\alpha = 2, \beta_i = (1.2)^{i-1}$ are presented in the top panel of Table 3. Here, F_i denotes the damage distribution due to the ith shock and the corresponding means are in parentheses. In another illustration, in the bottom panel of the Table 3, we assume that the ith damage distribution follows $LN(\mu_i, \sigma)$ with $\mu_i = 3 + (i-1) \times 0.4$ and $\sigma = 0.5$. Also, the shocks arrive according to a Poisson process with rate $\lambda = 0.07$. Here we consider the strength to degrade linearly with $s(t) = \max\{80 - 0.75t, 0\}$. Compared to identically distributed damages, computation by inversion method in this case is time consuming, while the simulation method seems to be unaffected by the choice of identical or non-identical damage distributions. However, from Table 3, both the methods seem to give similar reliability values.

Table 3: Reliability calculation by inversion method for non-iid damage distribution.

					$1-p$		
G	F_i	$s(t)$	0.9	0.7	0.5	0.3	0.1
$LN(1.8, 1)$	$Ga((1.2)^{i-1}, 2)$	$950 \exp(-0.05t)$	0.900	0.695	0.499	0.304	0.099
(9.97)	$(2 \times (1.2)^{i-1})$		(58.880)	(65.775)	(70.955)	(76.737)	(87.356)
$Exp(0.07)$	$LN(3 + (i-1) \times 0.4, 0.5)$	$max\{80 - 0.75t, 0\}$	0.898	0.698	0.498	0.301	0.101
(14.23)	$(\exp(3 + (i-1) \times 0.4 + 0.125))$		(13.250)	(23.177)	(31.855)	(41.745)	(57.973)

5.2 Dependent Damage Distributions

It is not always realistic to assume the successive damages to be independently distributed [11]. In such case, a multivariate life distribution needs to be used to incorporate inherent dependency among the damages. In particular situations, the nature of dependence between successive damages may be simply modeled using information on the underlying mechanism. Nevertheless, if the joint characteristic function $\psi_{X_1,\cdots,X_n}(\mathbf{s})$ of $X_1, \cdots, X_n$, with $N(t) = n$, is available in closed form, then the characteristic function $\phi_Y(u)$ of $Y = \sum_{i=1}^n X_i$ is given by $\psi_{X_1,\cdots,X_n}(u.\mathbf{1})$, which can be used for numerical evaluation of reliability by inversion method. As before, one can also use the simulation method. As an example, suppose that the damage X corresponding to a shock is modeled as the sum of a basic damage Z_0 and an additional damage Z specific to the particular shock. Then the successive shocks $X_1, X_2, \cdots$, where $X_i = Z_0 + Z_i$ with Z_i denoting the additional damage specific to the ith shock, are dependent because of the common basic damage Z_0. Let us assume Z_0 and the Z_i's to be independent $Ga(\theta_i, 1)$ variates, for $i = 0, 1, 2, \cdots$. Then the joint distribution of $X_1, \cdots, X_n$ is known as Cheriyan and Ramabhadran's multivariate Gamma [25, p-454] with characteristic function

$$\psi_{X_1,\cdots,X_n}(\mathbf{s}) = (1 - i\sum_{k=1}^{n} s_k)^{-\theta_0} \prod_{k=1}^{n} (1 - is_k)^{-\theta_k},$$

where $\mathbf{s} = (s_1, \cdots, s_n)$. Although the cumulative damage distribution under this assumption does not belong to any standard class of distributions, the characteristic function of $Y = X_1 + \cdots + X_n$ reduces to

$$\phi_Y(u) = (1 - inu)^{-\theta_0}(1 - iu)^{-\sum_{k=1}^{n}\theta_k},$$

which becomes useful for computing reliability at different points by inversion method using (7). In some other general set-up, conditional distribution of the damage due to the ith shock, $P[X_i > u|X_1, \cdots X_{i-1}]$, depends on the magnitude of the cumulative damage, $\sum_{k=1}^{i-1} X_k$, due to the shocks preceding the ith one [17, p-96]. In such case, the characteristic function of $Y = \sum_{i=1}^n X_i$ is not always readily available in closed form and hence difficulty arises in the implementation of inversion method. Simulation method seems to be the most convenient among all the competing methods due to its flexibility to adopt complicated dependence mechanism.

For illustration, we first assume that the damages follow the Cheriyan and Ramabhadran's multivariate Gamma distribution with parameters $\theta_0 = 0.5$ and $\theta_k = \theta = 4.5$, for all

$k = 1, 2, \cdots$, denoted by $MVGa(\theta_0, \theta)$, with mean damage being $\theta_0 + \theta$. We also assume that the shocks arrive according to a renewal process with distribution of inter-arrival time to be $LN(1.5, 0.5)$ and strength function $s(t) = \max\{75 - t, 0\}$. In the next illustration, we take $\theta_0 = 10$ and $\theta = 1.5$, $s(t) = \max\{150 - 0.5t, 0\}$ and assume the shocks to arrive according to a Poisson process with rate $\lambda = 0.05$. Computational results are presented in Table 4 in which F means the joint distribution of the dependent damages. Again, both the methods give similar reliability values.

Table 4: Reliability calculation by inversion method for dependent damages from multivariate Gamma distribution.

			$1-p$				
G	F	$s(t)$	0.9	0.7	0.5	0.3	0.1
$LN(1.5, 0.5)$	$MVGa(0.5, 4.5)$	$max\{75 - t, 0\}$	0.897	0.705	0.502	0.303	0.098
(5.08)	(5)		(32.579)	(36.402)	(39.145)	(41.818)	(45.816)
$Exp(0.05)$	$MVGa(10, 1.5)$	$max\{150 - 0.5t, 0\}$	0.903	0.706	0.502	0.305	0.101
(20)	(11.5)		(100.470)	(126.194)	(145.012)	(163.076)	(189.542)

5.3 Damages with Mixture Distribution

In many real life scenarios, shocks appear from multiple sources thereby causing damages with different distributions depending on the source of the corresponding shock. One can then ideally model the damage distribution corresponding to a shock to have a mixture distribution. For simplicity, the successive damages may be assumed to be independent and identically distributed with common distribution F being a mixture distribution. Suppose an arriving shock is from one of k sources with corresponding damages being denoted by $Z_1, Z_2, ...Z_k$ with the characteristic functions $\psi_{Z_i}(u)$, for $i = 1, 2, \cdots k$, respectively. Then, the characteristic function of X, the damage from a shock (which is one of $Z_1, \cdots, Z_k$), is given by $\phi_X(u) = \sum_{i=1}^k p_i \psi_{Z_i}(u)$, where p_i's denote the mixing probabilities with $\sum_{i=1}^k p_i = 1$ and $p_i > 0$ for all $i = 1, \cdots, k$. We can now use this characteristic function to evaluate the cumulative distribution function of nth order convolution of F

by using the inversion formula (7). One can then use (4) or (5) to calculate the reliability function at different points of time. As before, one can also use the simulation method to calculate reliability.

To illustrate, we first assume that the shocks arrive according to a renewal process with distribution of inter-arrival time being $LN(1,1)$ producing damages from mixture of $Ga(3,1.5)$ and $Ga(0.5,4)$ distributions with proportions 0.2 and 0.8, respectively. We also assume that the strength $s(t) = 200\exp(-0.05t)$. The results are presented in the top panel of Table 5. In another illustration, we assume that the shocks arrive according to a Poisson process with rate $\lambda = 0.1$ producing damages from mixture of $Ga(2,3)$ and $Ga(1,6)$ distributions with proportions 0.6 and 0.4, respectively. We also assume that the strength $s(t) = 500\exp(-0.1t)$. The corresponding results are presented in the bottom panel of Table 5. As expected, both the methods give similar results.

Table 5: Reliability calculation by inversion method for damages from mixture distribution.

			$1-p$				
G	F	$s(t)$	0.9	0.7	0.5	0.3	0.1
$LN(1,1)$	$0.2Ga(3,1.5)+0.8Ga(0.5,4)$	$200\exp(-0.05t)$	0.901	0.697	0.505	0.301	0.102
(4.48)	(2.5)		(35.493)	(39.665)	(42.631)	(46.116)	(51.927)
$Exp(0.1)$	$0.6Ga(2,3)+0.4Ga(1,6)$	$500\exp(-0.1t)$	0.897	0.699	0.503	0.299	0.099
(10)	(6)		(23.680)	(26.772)	(29.267)	(32.288)	(37.723)

5.4 Random Strength Degradation

In many real life scenarios, initial strength or its path of deterioration over time is random. Sometimes, deterioration of strength over time is due to various environmental causes changing stochastically at every instant. In summary, deterministic degradation of strength may not often be realistic. In this section, we consider some examples of random strength degradation and carry out reliability calculation as before.

Random coefficient strength degradation model is extremely useful to model lifetime of products [26]. Here, the initial strength S_0 at time 0 is assumed to be random, but the path of degradation $g(t)$ is assumed to be deterministic satisfying the condition of $s(t)$ as discussed in Section 1, with $g(0) = 1$. Then the random strength degradation is given by $S(t) = S_0 g(t)$ with the reliability at time t given by

$$R(t) = \int_s R(t|S_0 = s) dF_{S_0}(s), \tag{8}$$

where $F_{S_0}(s)$ denotes the distribution function of S_0 and $R(t|S_0 = s)$, the reliability at time t for given $S_0 = s$, can be calculated for given s using the inversion method or the simulation method. This modeling can be generalized further to have $g(t)$, the deterministic path of degradation, defined through some coefficients θ and then let θ be random.

For illustration, we first assume that the shocks arrive according to a renewal process with distribution of inter-arrival time being $LN(0.5, 0.5)$ producing damages from a $Wei(3, 6)$ distribution. For random coefficient degradation, we assume $g(t) = \exp(-0.02t)$ with S_0 following an Exponential distribution having mean $1/0.01$, denoted by $S(t) = Exp(0.01)\exp(-0.02t)$. The results on $R(t)$, given by (8) and obtained by the inversion method at the quantiles estimated by the simulation method, are presented in the top panel of Table 6. In the next illustration, we assume the shocks to arrive according to a Poisson process with rate $\lambda = 0.1$ producing damages from a mixture of $Ga(1, 5)$ and $Ga(2, 9)$ with proportions 0.3 and 0.7, respectively. For random coefficient degradation, we assume $g(t) = \exp(-0.1t)$ with S_0 following an Exponential distribution having mean $1/0.002$, denoted by $S(t) = Exp(0.002)\exp(-0.1t)$. The corresponding results are presented in the bottom panel of Table 6. Here also, the results indicate similar reliability values by both the methods.

Random coefficient degradation model fails to capture the random temporal variation associated with the random degradation process over time [27]. As deterioration is generally uncertain and monotonic, Gamma process is very popular in modeling wear, fatigue, crack growth and thinning due to corrosion, etc. [28, 29], as discussed in the following. A Gamma process is a continuous time stochastic process $\{Y(t), t \geq 0\}$ with the following properties:

Table 6: Reliability calculation by inversion method with random coefficient strength degradation.

			$1-p$				
G	F	$S(t)$	0.9	0.7	0.5	0.3	0.1
$LN(0.5, 0.5)$	$Wei(6, 3)$	$Exp(0.01)\exp(-0.02t)$	0.901	0.697	0.498	0.305	0.102
(1.87)	(5.36)		(4.050)	(10.922)	(17.930)	(25.836)	(38.537)
$Exp(0.1)$	$0.3Ga(1, 5) + 0.7Ga(2, 9)$	$Exp(0.002)\exp(-0.1t)$	0.899	0.699	0.499	0.299	0.096
(10)	(14)		(12.046)	(19.850)	(25.014)	(29.946)	(37.368)

(1) $Y(0) = 0$,

(2) $Y(t_2) - Y(t_1) \sim Ga(\gamma, \alpha(t_2) - \alpha(t_1))$, for $t_2 > t_1$ and

(3) $Y(t)$ has independent increments,

where $\alpha(t)$ is non-decreasing, right-continuous real-valued function for $t \geq 0$ with $\alpha(0) = 0$. Let us denote the random strength at time t by $S(t)$ and model it as $S(t) = \max\{s_0 - Y(t), 0\}$, where s_0 is the known initial strength. Here $Y(t)$, the Gamma process, represents the amount of deterioration at time t. For stochastic strength degradation by Gamma process, calculation of reliability $R(t)$ at time t involves integration over all possible values of $Y(t)$ at time t, and hence, the reliability at time t is given by

$$R(t) = \int_0^{s_0} R(t|Y(t) = y)dF_{Y(t)}(y),$$

where $F_{Y(t)}(y)$ denote the distribution function of $Y(t)$, i.e. of $Ga(\gamma, \alpha(t))$, and $R(t|Y(t) = y)$, the reliability at time t for given $Y(t) = y$, or $S(t) = s_0 - y$, can be calculated for given y using the inversion method. On the other hand, a Gamma process can be simulated by generating independent increments from Gamma distributions. For our purpose of simulating T, it is not necessary to simulate the entire degradation path, not even till the failure time T is realized. Since the system fails at the shock arrival times, or between two successive arrival times as degraded strength goes below the accumulated strength, it is enough to simulate realization of the random strength at those arrival time points and then check for the exact failure time T. We suggest two such algorithms for simulating T following above principle. In both the algorithms, as we simulate the successive shock arrival times $t_1, t_2, \cdots$ along with the resulting damages $X_1, X_2, \cdots$, we also simulate the independent increments $Y(t_1), Y(t_2) - Y(t_1), \cdots$ of the Gamma process corresponding to the arrival times $t_1, t_2, \cdots$. On each such occasion, say ith for $i = 1, 2, \cdots$, we check if the degraded strength $S(t_i)$ lies in $[\sum_{l=1}^{i-1} X_i, \sum_{l=1}^{i} X_i]$. If yes, then T is taken as t_i. If $S(t_i)$ exceeds $\sum_{l=1}^{i} X_i$, then we need to simulate t_{i+1}, the next arrival time, and so on. Otherwise, the system fails in between t_{i-1} and t_i, with $t_0 = 0$, when the degraded strength equals $\sum_{l=1}^{i-1} X_l$, for $i > 1$, and 0, for $i = 1$. The two following algorithms suggest simulating T between t_{i-1} and t_i, given the history $H_i = \{t_1, \cdots, t_i\} \cap \{X_1, \cdots, X_i\} \cap \{S(t_1) = u_1, \cdots, S(t_i) = u_i\}$.

In the first, we obtain the conditional distribution of T, given H_i, for $t_{i-1} < t < t_i$, as

$$\begin{aligned}
&P\left[T \le t | H_i\right] \\
&= \frac{P\left[S(t) \le u, S(t_l) = u_l, l = 1, 2, \cdots, i\right]}{P\left[S(t_l) = u_l, i = 1, 2, \cdots, l\right]}, \text{where } u = \sum_{l=1}^{N(t)} X_i = \sum_{l=1}^{i-1} X_i \\
&= \frac{\int_{u_i}^{u} \prod_{l=1}^{i-1} g(u_l, u_{l-1}, t_l, t_{l-1}, \gamma) g(u_{i-1}, w, t, t_{i-1}, \gamma) g(w, u_i, t_i, t, \gamma) dw}{\prod_{l=1}^{i} g(u_l, u_{l-1}, t_l, t_{l-1}, \gamma)} \\
&= \frac{\int_{u_i}^{u} g(u_{i-1}, w, t, t_{i-1}, \gamma) g(w, u_i, t_i, t, \gamma) dw}{g(u_i, u_{i-1}, t_i, t_{i-1}, \gamma)} \\
&= F_{w, t_{i-1}, t_i}(t), \text{ say,}
\end{aligned} \tag{9}$$

where $u_0 = s_0$ and $g(u, v, t_1, t_2, \gamma)$ is the Gamma density at $u - v$ with shape parameter $\alpha(t_1) - \alpha(t_2)$ and scale parameter γ. One can simulate an observation on T from the conditional distribution (9). One needs to use numerical integration and method of bisection for this. Alternatively, in the second algorithm, one can employ the Gamma bridge sampling algorithm [30] to simulate the degraded strength $S(t)$ at time t between t_{i-1}

and t_i. Conditioning on H_i, this $S(t)$ can be simulated as $S(t) = u_{i-1} - \{u_{i-1} - u_i\}Y$, where Y follows a Beta distribution with parameters $\alpha(t) - \alpha(t_{i-1})$ and $\alpha(t_i) - \alpha(t)$. The objective is to solve for that t satisfying $S(t) = u$. For this, mimicking the method of bisection, we first simulate strength at time $t = (t_{i-1} + t_i)/2$ and check whether it falls below or above u. If $S(t) > u$, then replace t_{i-1} by $(t_{i-1} + t_i)/2$, otherwise we replace t_i by $(t_{i-1} + t_i)/2$ and continue until the simulated strength lies within a small neighbourhood of u. The corresponding t is taken as a realization of T. Both the algorithms work equally well, the second one having an edge because of the numerical integration required in the first. However, the first one seems more general with (9) being valid with any density $g(\cdot)$ other than Gamma.

To illustrate, we first assume that the shocks arrive according to a renewal process with distribution of inter-arrival time being $LN(0.5, 0.1)$ and the damages are from iid $Wei(0.6, 5)$ distribution. Strength degradation from initial strength $s_0 = 50$ is modelled by a Gamma process with shape function $\alpha(t) = t$ and scale parameter $\gamma = 5.5$. In the next illustration, we assume the shocks to appear according to a Poisson process with rate $\lambda = 0.2$ and the damages from iid mixture of $Ga(3, 1)$ and $Ga(2, 5)$ with proportions 0.7 and 0.3, respectively. Strength degradation from initial strength $s_0 = 250$ is modelled by a Gamma process with shape function $\alpha(t) = 2t$ and scale parameter $\gamma = 1.5$. The results are presented in Table 7 in which the quantiles are estimated by simulation method using the Gamma bridge sampling algorithm, while the reliability calculations are carried out by (i) the inversion method and the (ii) simulation method using the conditional failure distribution (9). The results indicate that these methods give similar reliability values.

In a different framework, the random strength degradation $S(t)$ can be assumed to follow a Brownian motion with drift so that, although there is an expected pattern of strength degradation, there may be ups and downs in an actual path of strength degradation. Hence, the definition (1) of reliability is no more valid. However, the definition (2) is still valid with deterministic strength degradation $s(t)$ replaced by random strength degradation $S(t)$. Coutin and Dorobantu [31] consider the Brownian motion strength degradation and prove the existence of a density corresponding to the reliability function

(2). However, computation of reliability in this model remains unattempted.

Table 7: Reliability calculation by different methods with strength degrdation by Gamma process.

				$1-p$				
G	F	$s(t)$	Method	0.9	0.7	0.5	0.3	0.1
$LN(0.5, 0.1)$	$Wei(0.6, 5)$	$max\{50 - Ga(5.5, t), 0\}$	Inversion	0.893	0.697	0.501	0.297	0.100
(1.66)	(0.55)		Conditional Failure	0.892	0.699	0.502	0.299	0.104
				(5.711)	(7.552)	(8.918)	(10.432)	(12.674)
$Exp(0.2)$	$0.7Ga(3, 1) + 0.3Ga(2, 5)$	$max\{250 - Ga(1.5, 2t), 0\}$	Inversion	0.897	0.701	0.499	0.298	0.100
(5)	(5.1)		Conditional Failure	0.899	0.708	0.501	0.298	0.098
				(53.322)	(58.699)	(62.617)	(66.517)	(72.099)

6 Concluding Remarks

Dynamic stress-strength model where stress accumulates over time and strength degrades concurrently can be used to analyse wide range of mechanical and natural phenomena. Theoretical properties of the life distribution under some suitable assumptions on damage distribution and shock arrival process has been studied in the past for fixed threshold. Importance of such models has been emphasized by several authors in the past. However, no one has addressed the issue of computational difficulty that arises in reliability calculation. This paper makes an attempt in that direction by defining, possibly for the first time, the reliability function accounting for both damage accumulation and deterministic strength degradation in a single model. In this paper, we have also compared different methods of reliability computation under several generalizations of shock arrival process, damage distributions and strength degradation. We have also provided interesting comparison of life distributions of two different systems keeping expected cumulative stress fixed. One can easily compare life distributions of two different systems in terms of stochastic ordering of damage distributions and shock arrival process. For example, lifetime of a system, subjected to stochastically larger shocks, is stochastically smaller compared to that of the system subjected to stochastically smaller shocks. Similarly, lifetime of a system which suffers less frequent shocks is stochastically larger compared to a system which is subjected to more frequent shocks.

After comparing different methods of reliability computation, we conclude that the simulation method is the most convenient among all other available methods. The three approximation methods of Section 3.1-3.3 generally possess some limitations and hence fail to provide good approximation for the cumulative damage distribution in general. The inversion method and the simulation method provide similar results across different quantiles of life distributions. However, computational burden for the inversion method is higher compared to that of the simulation method. Simulation method provides an estimate of the entire life distribution in contrast with the inversion method which gives reliability value for a specified quantile. Inversion method requires numerical evaluation of integrals involving badly behaving integrands and hence it is time consuming. This difficulty increases for non-identical damage distributions and for the cases where there is no closed form expression of the characteristic function. Also, evaluation of distribution by inversion method is restricted for continuity points only and for the cases where characteristic function can be evaluated easily by numerical techniques. On the other hand, simulation method does not involve evaluation of characteristic function and hence it is completely unaffected by these issues. In some dynamic stress-strength mechanism, closed form expression of the reliability function like (1) is not available where stress accumulates or strength degrades in non-monotonic fashion. For example, strength may go through some auto-repairing process while degrading over time thereby resulting in some ups and downs in strength [32]. Also, the accumulated stress may go through some waning process [12, p-31]. In such situations, the definition (1) is no more valid and simulation seems to be the only possible method which can be used for numerical evaluation of the reliability function. The reliability function for the non-cumulative damage model is of much simpler form compared to that of the cumulative damage model. However, it may involve integrals which cannot be expressed in closed form [28]. Therefore, the simulation method seems to be the most suitable one in terms of domain of application and computational ease. It is to be noted that, in practice, the reliability function needs to be calculated only once at the parameter values estimated from additional information other than only on T.

Acknowledgements

The authors are thankful to Dr. Arnab Chakraborty and Dr. G. Jay Kerns for many helpful comments and suggestions.

References

[1] R. C. Gupta, S. Ramakrishnan, X. Zhou, *"Point and Interval Estimation of $P(X < Y)$: The Normal Case with Common Coefficient of Variation", Ann. Inst. Statist. Math., VOL-51,* 1999.

[2] I. Guttman, R. A. Johnson, G. K. Bhattacharayya, B. Reiser, *"Confidence Limits for Stress-strength Models with Explanatory Variables", Technometrics, VOL-30,* 1988.

[3] S. Kaplan, *"On The Method of Discrete Probability Distributions in Risk and Reliability Calculations - Application to Seismic Risk Assessment", Risk Analysis, VOL-1,* 1981.

[4] F. Wilcoxon, *"Individual Comparison by Ranking Methods", Biometrical Bulletin, VOL-1,* 1945.

[5] J. S. Simonoff, Y. Hochberg, B. Reiser, *"Alternative Estimation Procedure for $P(X \leq Y)$ in Categorized Data", Biometrics, VOL-42,* 1986.

[6] S. Kotz, Y. Lumelskii, M. Pensky, *"The Stress-Strength Model and its Generalizations", World Scientific Publishing Co. Pvt. Ltd.,* 2003.

[7] K. C. Kapur, L. R. Lamberson, *"Reliability in Engineering Design", John Wiley and Sons. Inc.,* 1977.

[8] I. B. Gertbsbakh, K. B. Kordonskiy, *"Models of Failure" Springer-Verlag,* 1969.

[9] J. L. Bogdanoff, F. Kozin, *"Probabilistic Models of Cumulative Damage", John Wiley and Sons. Inc.,* 1985.

[10] W. J. Hopp, Y. Kuo, *"An Optimal Structured Policy for Maintenance of Partially Observable Aircraft Engine Components", Naval Research Logistics, VOL-45,* 1998.

[11] K. Sobczyk, J. Trbicki, *"Modelling of Random Fatigue by Cumulative Jump Processes", Engineering Fracture Mechanics, VOL-34,* 1989.

[12] T. Nakagawa, *"Shock and Damage Models in Reliability Theory", Springer-Verlag London Ltd.,* 2007.

[13] R. P. Kennedy, C. A. Cornell, R. D. Campbell, S. Kaplan, H. F. Perla, *"Probabilistic Seismic Safety Study of an Existing Nuclear Power Plant", Nuclear Engineering and Design, VOL-59,* 1980.

[14] C. A. Cornell, *"Engineering Seismic Risk Analysis", Bulletin of the Seismological Society of America, VOL-58,* 1968.

[15] A. P. Basu, N. Ebrahimi, *"On the Reliability of Stochastic System", Statistics and Probability Letters, VOL-1,* 1983.

[16] S. Ross, *"Stochastic Processes", John Wiley & Sons, Inc.,* 1996.

[17] R. E. Barlow, F. Proschan, *"Statistical Theory of Reliability and Life Testing: Probability Models", Holt, Rinehart and Winston, Inc.,* 1975.

[18] N. L. Johnson, *"System of Frequency Curves generated by Method of Translation", Biometrika, VOL-36,* 1949.

[19] J. Gil-Pelaez, *"Note on the Inversion Theorem.", Biometrika, VOL-38,* 1951.

[20] W. Rudin, *"Principles of Mathematical Analysis", MacGraw Hill Education Private Limited,* 1976.

[21] S. A. Kulgman, H. H. Panjer, G. E. Willmot, *"Loss Models: From Data to Decisions", John Wiley and Sons.Inc.,* 2004

[22] I. D. Hill, R. Hill, R. L. Holder, *"Fitting Johnson Curves by Moments", Journal of Royal Statistical Society. Series C, VOL-22,* 1976.

[23] V. Witkovsky, *"On the Exact Computation of Density and on the Quantiles of Linear Combination of t and F Ransom Variables", Journal of Statistical Planing and Inference, VOL-94,* 2001.

[24] R. Piessens, E. De Doncker-Kapenga, C. W. Uberhuber, D. K. Kahaner *"Quadpack: a Subroutine Package for Automatic Integration", Springer-Verlag.,* 1983

[25] S. Kotz, N. Balakrishnan, N. L. Johnson, *"Continuous Multivariate Distributions: Models and Applications", John Wiley and Sons.Inc.,* 2000.

[26] J. Xue, K. Yang, *"Upper and Lower Bounds of Stress-Strength Inference Reliability with Random Strength Degradation", IEEE Trans. Reliability, VOL-46,* 1997.

[27] Y. Wu, L. Xie, N. Wu, J. Li, *"Time-Dependent Reliability Models for Components with Strength Degradation based on Gamma Process", Proceedings of the 2011 9th International Conference on Reliability, Maintainability and Safety,* 2011.

[28] J. M. van Noortwijk, J. A. M. van der Weide, M. J. Kallen, M. D. Pandey, *"Gamma process and Peaks-over-threshold Distributions for Time-dependent Reliability", Reliability Engineering and System Safety, VOL-92,* 2007.

[29] J. M. van Noortwijk, *"A Survey of the Applications of Gamma Process in Maintenance", Reliability Engineering and System Safety, VOL-94,* 2009.

[30] S. Chick, P. J. Sanchez, D. Ferrin, D. J. Morrice, *"Efficient Simulation of Gamma and Variance-gamma Processes", Proceedings of the 2003 Winter Simulation Conference,* 2003.

[31] L. Coutin, D. Dorobantu, *"First Hitting Time Law for Some Jump-diffusion Processes", arXiv:0904.1669v3,* 2012.

[32] N. Ebrahimi, T. Ramallingam, *"Estimation of System Reliability in Brownian Stress-Strength Models Based on Sample Paths", Ann. Inst. Statist. Math., VOL-45,* 1993.