Estimation of System Reliability for Dynamic Stress-Strength Modeling with Cumulative Stress and Strength Degradation

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Abstract

In many real life scenarios, system reliability depends on dynamic stress-strength interference where strength degrades and stress accumulates concurrently over time. In this paper, we consider the problem of estimating reliability of a system under deterministic strength degradation and cumulative damage due to shocks arriving according to a point process. Maximum likelihood estimation under two different sampling plans has been considered. Large sample properties in general are discussed. The method is illustrated through simulation and real life data analysis.

Keywords: Mechanistic model, Samapling plan, Identifiability, Parameteric bootstrap, Kaplan-Meier estimate.

1 Introduction

Traditionally, estimation of reliability of a stochastic system, with random strength (Y) and subject to random stress (X), has been addressed as the problem of estimating $P(Y > X)$ [1]. In many important applications, the stress is accumulation $X(t)$ of random damages due to shocks arriving at random time points according to a point process [2, p-192]. For example, a vehicle axle fails when the depth of a crack has exceeded a critical level [3, p-2]. Also, the strength of a system degrades possibly by corrosion, fatigue, ageing, etc., which may be reasonably described by a deterministic curve, $s(t)$, say [4, Ch-3]. As a result, both stress and strength may be time dependent. This phenomenon of both strength degradation and stress accumulation has not so far been concurrently considered in reliability analysis, although this phenomena occurs often in real life scenarios. For example, in the area of disaster risk management, especially in seismic risk analysis of civic structures, power plants, etc., proper assessment of seismic risk requires cumulative information regarding seismicity (i.e., frequency and magnitude of earthquake) of the location and the ability of the structure to resist earthquakes of various sizes [5]. Also, the electric power of a cell-phone battery, initially stored by chemical energy, is weakened by normal functionalities of a cell-phone and is subject to frequent calls leading to accumulated damage or energy loss. Similarly, as a result of accumulation of emails over time in an email account, the corresponding mailbox becomes full and the account fails to receive any further email. See Bhuyan and Dewanji [6] for details regarding such a stochastic system in general and calculation of corresponding reliability.

There has not been much work on estimation of system reliability in general when both stress and strength are time dependent. Ebrahimi and Ramallingmam [7] assumed both stress and strength to be independent Brownian motions and, based on observations on the difference process, addressed the issue of estimation of reliability function under two sampling plans. In one of the plans, the difference between strength and stress is observed till the time of failure, while, in the other, only the failure time is observed. In this case, reliability depends on the parameters of the difference process and those are the only estimable quantities. In this work, we discuss estimation of reliability for a system under deterministic strength degradation and stochastic stress accumulation due to damages caused by shocks arriving according to a point process. We define failure time of the system as the time when the accumulated stress equals or exceeds the strength. Then a unit fails when its strength reduces to zero even if no shock arrives by that time. Let $N(t)$ denote the point process representing the number of shocks arriving by time t. Also, let the damages due to the successive shocks be denoted by $X_1, X_2, X_3, \cdots$. The reliability function $R(t)$ at time t is then formally defined as (See Bhuyan and Dewanji, [6])

$$R(t) = P\,[\,T > t] = P\left[\sum_{i=1}^{N(t)} X_i < s(t)\right], \tag{1}$$

where T denotes the failure time and $s(t)$ is the strength at time t. Let $t_0 = \inf\{t\colon s(t) = 0\}$. Note that, since strength is always non-negative, t_0 may not exist. If it does, i.e. $t_0 < \infty$, then $R(t) = 0$ for $t \geq t_0$, with a jump at t_0. We assume (i) $s(t)$ to be non-increasing, (ii) $s(t) \geq 0,$ for all t, with $s(0) > 0$, and (iii) $s(t)$ is right-continuous. One popular choice is the exponential degradation model given by $s(t) = a\exp(-bt)$. Nevertheless, the definition of $R(t)$, for $t < t_0$, can be written in general, using (1) and assuming that the X_i's are independent of the shock arrival process, as

$$R(t) \;\; = [N(t) = 0] + \sum_{n=1}^{\infty} P\,[N(t) = n]\, P\left[\sum_{i=1}^{n} X_i < s(t)\right]. \tag{2}$$

We also assume the X_i's to be independent and identically distributed (iid) with distribution function denoted by $F(\cdot)$. Then, the reliability function, given in (2), can be written as

$$R(t) = \sum_{n=0}^{\infty} F^{(n)}(s(t)-)\,[G_n(t) - G_{n+1}(t)], \tag{3}$$

where $F^{(n)}(s(t)-) = P[\sum_{i=1}^{n} X_i < s(t)]$ and $G_n(\cdot)$ is the distribution for T_n, the arrival time of the nth shock. Note that, when $N(t)$ is a homogeneous Poisson process with rate λ, T_n is a Gamma random variable with scale λ and shape parameter n having mean n/λ. Then, $R(t)$ reduces to

$$R(t) = e^{-\lambda t} + \sum_{n=1}^{\infty} F^{(n)}(s(t)-)e^{-\lambda t}(\lambda t)^n/n!. \tag{4}$$

This equation (4) has been known in the context of cumulative damage model with a constant strength or threshold (that is, $s(t)$ is independent of t) [8, p-92].

Often, while observing a system subject to the stress-strength interference as discussed before, there is extra information in addition to the system failure time. This extra information may be useful for efficient estimation of the model parameters. In this work, we consider two different sampling plans with some such extra information for estimation of the associated parameters involved in this stress-strength interference, and hence, the reliability function $R(t)$ for fixed values of t. We first describe the two different sampling plans in Section 2. Next, in Section 3, we derive the density functions corresponding to the random vectors obtained from the two sampling plans, defined on a suitable probability space, and discuss the associated identifiability issues. In Section 4, we write the likelihood functions for the parameters associated with the damage distribution $F(\cdot)$, the strength function $s(t)$ and the shock generating process $N(t)$ based on data from the two sampling plans for the purpose of maximum likelihood estimation. Large sample properties of the estimates are also discussed. Assuming the shock arrival process to be Poisson and the successive damages to be independent and identically distributed (iid) Gamma variates, numerical illustration of the maximum likelihood method, including its properties, through a simulation study and an illustrative data analysis are reported in Sections 5 and 6, respectively. We end with some concluding remarks in Section 7.

2 Sampling Plans

Continuous monitoring provides maximum possible information for a system under dynamic stress-strength interference. However, it is not feasible to observe a system under operation continuously due to various practical constraints. In addition to that, it also requires trained resources, adequate data storage capacity and proper database maintenance facility. In practice, one of the major challenges is that some of the key metrics (for example, measurements on damages), required for easy estimation of system reliability, may not be readily observable [9]. Generally, in industrial environment, data collection is not done with statistical advice and, as a result of unscientific data collection, complexity of estimation procedure increases. Nevertheless, in different problems, data collection is carried out following certain sampling plans depending on the situation and records on the different features of the dynamic stress-strength system are kept accordingly. Usually, a sampling plan is equipped with a suitable censoring scheme for various reasons. In the following, we describe two different sampling plans for systems under cumulative stress and strength degradation, as discussed in Section 1.

2.1 Sampling Plan I

In the first sampling plan, we observe the failure time of the system along with number of shocks arriving up to the failure time. Note that, under the present stress-strength interface, there are two different types of failure modes, either due to strength degradation below the existing level of accumulated stress ($\Delta = 0$), or due to arrival of a shock resulting in the increased stress exceeding the strength at that time ($\Delta = 1$). We keep record of this mode of system failure as well. In this sampling plan, it is interesting to note that no information on the amount of damages due to successive shocks is required. This feature of this sampling plan makes it widely applicable in many real life scenarios. Also, we adopt Type-I censoring scheme to restrict our observation period within a reasonable time limit τ. If the system fails within time τ, we observe failure time T, the number of shocks arriving up to the failure $N(T)$, and mode of failure Δ. Otherwise, we only observe $N(\tau)$, the number of shocks arriving up to the failure time τ. Note that Δ is an indicator variable taking value 1 if failure of the system is due to the damage of an arriving shock, and 0 otherwise. Let us denote the censoring indicator by E, which takes value 1 if the system fails within the observation period τ, and 0 otherwise. So, one observes the vector $\{T \wedge \tau, N(T \wedge \tau), \Delta, E\}$ for each system. Note that, when $E = 0$, Δ is not observable and is assigned a value of -1. Defined in this way, Δ carries all the information on E as well. Therefore, we write the observation vector as $O_1 = \{T \wedge \tau, N(T \wedge \tau), \Delta\}$.

Field-tracking studies on a group of units in service provide such additional information along with failure time data. For example, an unprotected electronic component of a large electrical system fails due to accumulation of randomly occurring damages from power line voltage spikes during electric storms. Failure of such systems may also occur due to normal product wear or ageing [10, p-383]. In such field-tracking studies, one can observe the failure time and the failure mode of the units under test. In a database system, garbage is accumulated over time due to randomly occurring updates from several on-line terminals. Garbage accumulation reduces the size of the available memory areas and worsens processing efficiency. The system fails if the total garbage exceeds system specific tolerance level. Such a garbage collection model corresponds to the damage model by replacing shocks with updates and damage with amount of garbage [3, p-131]. One can easily monitor such database system and keep records of the failure time and number of shocks for reliability analysis.

2.2 Sampling Plan II

In the second sampling plan, we monitor each system at prefixed time points $\tau_1, \tau_2, \cdots \tau_m$, say. At each of these monitoring time points, we observe number of shocks arriving up to that time and the operating status of the system; that is, equivalently, we observe whether accumulated damage on the system is less than its strength at the time of monitoring. We continue collecting data until the system fails, or observation time reaches τ_m, whichever is earlier. Let $N(\tau_l)$ be the number of shocks arriving up to time τ_l and D_l the operating status at time τ_l, for $l = 1, 2, \cdots, m$, where $D_l = 1$ means "operating" and $D_l = 0$ means "failed". Write $\tau_0 = 0$ and $D_0 = 1$ to mean that the system is operating at time 0. Note that, for $j \geq 1$, $D_j = 1$ means $D_l = 1$ for $l = 1, \cdots, j-1$. Also, if $D_j = 0$ and $D_l = 1$, for $l = 1, \cdots, j-1$, then $\tau_{j-1} < T \leq \tau_j$ and the observation stops at τ_j. Let J denote the interval in which the system fails, that is, $J = j$ if $\tau_{j-1} < T \leq \tau_j$, for $j = 1, \cdots, m$. Also, if $D_1 = \cdots = D_m = 1$ (that is, failure does not take place by τ_m), then we write $J = m + 1$. Here, we additionally assume that the arrival of shocks continues and is recorded even after the system failure so that one can observe $N(\tau_j)$ even for $J = j$ with $\tau_{j-1} < T \leq \tau_j$. So, one observes the vector $\{J, N(\tau_l), l = 1, \cdots, J \wedge m\}$. Note that the value of J has all the information on the D_l's. In order to have a fixed dimension for the observation vector, we define $N_1 = N(\tau_1)$ and $N_j = N_{j-1} + [N(\tau_j) - N(\tau_{j-1})]I(T > \tau_{j-1})$, for $j = 2, \cdots, m$, so that, for $J = j$ with $\tau_{j-1} < T \leq \tau_j$, we have $N_1 \leq \cdots \leq N_j = N_{j+1} = \cdots = N_m$. For $J = m + 1$, we have $N_j = N(\tau_j)$, for $j = 1, \cdots, m$. Then, the observation vector can be written as $O_2 = \{J, N_1, \cdots, N_m\}$. Similar to our first sampling plan, here also, we do not require any direct measurement on the amount of successive damages. Since we do not observe the exact failure time, the information on the failure mode is not observed either, in contrast with the first plan. This sampling plan relates to a more general set-up in which it is difficult to identify actual failure time and mode of failure.

Engineers interested in higher-level auto-mobile system reliability conduct laboratory experiments keeping units under simulated service environment. As for example, a vehicle axle fails when the depth of a crack has exceeded a critical level [3, p-2]. Similar experiments are also conducted on shock absorber of a automotive system. In such planned experiments, systems are inspected at prefixed time points and current status of the systems under test are recorded. Sobczyk and Trebicki [11] emphasized the importance of statistically designed experiments for estimation of the model parameters for analysing fatigue crack data, modelled as cumulative random process with an appropriate underlying counting process $N(t)$, characterizing the number of crack increments.

3 Identifiability

In this context of dynamic stress-strength modeling of Section 1, it is of interest to investigate the identifiability issue of the model parameters when data arises from either of the two sampling plans discussed in the previous section. Bhuyan et al. [12] have considered such identifiability problems under different data configurations assuming Poisson arrival of shocks. In this section, we assume that the shocks arrive according to a point process $N(t)$ and the successive iid damages, denoted by $X_1, X_2, X_3, \cdots$, follow the common distribution $F(\cdot)$, which is independent of the point process generating the shocks. We also assume that strength degradation of the system is given by a deterministic continuous curve $s(t)$, which is always positive. Note that the system reliability $R(t)$, at a fixed time t, is a function of the unknown damage distribution $F(\cdot)$, the strength function $s(t)$ and the parameters associated with the shock arrival process $N(t)$. Also note that the dynamic stress-strength model, given by equation (4), is not identifiable with only failure time data even if $s(t)$ is known (See Bhuyan et al. [12] for more details).

In order to investigate the identifiability issue of the model parameters under the two sampling plans, we consider O_1 (and O_2) as a random vector defined on a suitable probability space. Note that the dominating probability measure corresponding to the density of $O_1 = \{T \wedge \tau, N(T \wedge \tau), \Delta\}$ is $\nu_1 = \upsilon_{11} \times \upsilon_{12} \times \upsilon_{13}$, where both υ_{12} and υ_{13} are counting measures and υ_{11} is the sum of counting and the Lebesgue measures [13]. Here we assume that the damage distribution $F(x)$ is continuous for all $x > 0$, with corresponding density $f(x)$, and the strength function $s(t)$ is differentiable for all $t > 0$.

Theorem 3.1: The density of $O_1 = \{T \wedge \tau, N(T \wedge \tau), \Delta\}$ with respect to the measure ν_1 is

$$f^*(t,n,\delta) = \begin{cases} F^{(n)}(s(\tau))G^*(n+1,\tau) & , t=\tau,\ \delta=-1,\ n=0,1,\cdots \\ F^*(n,s(t))g_n(t) & , 0<t\le\tau,\ \delta=1,\ n=1,2,\cdots \\ |s'(t)|f^{(n)}(s(t))G^*(n+1,t) & , 0<t\le\tau,\ \delta=0,\ n=1,2,\cdots \\ 0 & , \text{otherwise,} \end{cases}$$

where $F^*(n,x) = F^{(n-1)}(x) - F^{(n)}(x) = P[\sum_{i=1}^{n-1} X_i \le x < \sum_{i=1}^{n} X_i]$ and $G^*(n,x) = G_{n-1}(x) - G_n(x) = P[N(x) = n-1]$, for $n = 1, 2, \cdots$, with $F^{(0)}(x) = G_0(x) = 1$. Here, $f^{(n)}(\cdot)$ and $g_n(\cdot)$ are the probability densities corresponding to the distribution functions $F^{(n)}(\cdot)$ and $G_n(\cdot)$, respectively.

Proof: For $t = \tau$, $\delta = -1$,

$$\begin{aligned} f^*(t,n,-1) &= P\left[T \wedge \tau = \tau, N(T) = n, \Delta = -1\right] \\ &= P\left[T > \tau, N(\tau) = n\right] \\ &= P\left[\sum_{i=1}^{n} X_i < s(\tau)\right] P\left[N(\tau) = n\right] \\ &= F^{(n)}(s(\tau))G^*(n+1,\tau) \end{aligned}$$

For $0 < t \leq \tau$, $\delta = 1$,

$$\begin{aligned}
f^*(t,n,1) &= \lim_{h\to 0} \frac{P\left[t-h < T \leq t, N(T)=n, \Delta=1\right]}{h} \\
&= \lim_{h\to 0} \frac{P\left[\sum_{i=1}^{N(t-h)} X_i < s(t) \leq \sum_{i=1}^{N(t)} X_i, t-h < T \leq t, N(T)=n\right]}{h} \\
&= \lim_{h\to 0} \frac{P\left[\sum_{i=1}^{n-1} X_i < s(t) \leq \sum_{i=1}^{n} X_i, t-h < T_n \leq t\right]}{h} \\
&= \left\{P\left[\sum_{i=1}^{n-1} X_i < s(t)\right] - P\left[\sum_{i=1}^{n} X_i < s(t)\right]\right\} \lim_{h\to 0} \frac{P\left[t-h < T_n \leq t\right]}{h} \\
&= F^*(n, s(t)) g_n(t)
\end{aligned}$$

Then, for $0 < t \leq \tau$, $\delta = 0$,

$$\begin{aligned}
f^*(t,n,0) &= \lim_{h\to 0} \frac{P\left[t-h < T \leq t, N(T)=n, \Delta=0\right]}{h} \\
&= \lim_{h\to 0} \frac{P\left[s(t) \leq \sum_{i=1}^{n} X_i < s(t-h), N(t)=n\right]}{h} \\
&= \lim_{h\to 0} \frac{P\left[\sum_{i=1}^{n} X_i < s(t-h)\right] - P\left[\sum_{i=1}^{n} X_i < s(t)\right]}{h} P\left[N(t)=n\right] \\
&= |s'(t)| f^{(n)}(s(t)) G^*(n+1, t).
\end{aligned}$$

Now we verify that $\int f^*(t,n,\delta) d\nu_1 = 1$.

$$
\begin{aligned}
\int f^*(t,n,\delta)d\nu_1 &= \sum_{n=0}^{\infty} F^{(n)}(s(\tau))G^*(n+1,\tau) + \int_0^{\tau} \sum_{n=1}^{\infty} F^*(n,s(t))g_n(t)dt \\
&\quad + \int_0^{\tau} \sum_{n=1}^{\infty} |s'(t)| f^{(n)}(s(t))G^*(n+1,t)dt \\
&= \int_0^{\tau} \sum_{n=1}^{\infty} F^{(n-1)}(s(t))g_n(t)dt - \int_0^{\tau} \sum_{n=1}^{\infty} F^{(n)}(s(t))g_n(t)dt \\
&\quad + \int_0^{\tau} \sum_{n=1}^{\infty} |s'(t)| f^{(n)}(s(t))G^*(n+1,t)dt + R(\tau) \\
&= \int_0^{\tau} d\left[1 - R(t)\right] + R(\tau) \\
&= 1
\end{aligned}
$$

Note that the density of O_1 remains the same even if the damage distribution $F(\cdot)$ has some positive mass at 0.

Similarly, the dominating probability measure corresponding to the density of O_2 is $\nu_2 = \upsilon_{21} \times \upsilon_{22} \times \cdots \times \upsilon_{2,m+1}$, where υ_{2j}, $j = 1, \cdots, m+1$, are the counting measures for J and $N_1, \cdots, N_m$, respectively. The density of O_2 is difficult to write in simple form in general. However, when shock generating process $N(t)$ possesses the independent increment property, then we have the following theorem.

Theorem 3.2: The density of $O_2 = \{J, N_1, \cdots, N_m\}$ with respect to the measure ν_2 is

$$
g^*(j, n_1, \cdots n_m) = \begin{cases}
F^*(n_{j-1}, n_j, \tau_{j-1}, \tau_j) \prod_{l=1}^{j} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1}) & , j = 1, 2, \cdots, m,\ n_j \geq 1, \\
 & \quad n_1 \leq \cdots \leq n_j = \cdots = n_m \\
F^{(n_m)}(s(\tau_m)) \prod_{l=1}^{m} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1}) & , j = m+1, \\
 & \quad 0 \leq n_1 \leq n_2 \leq \cdots \leq n_m \\
0 & , \text{otherwise}
\end{cases}
$$

where $F^*(n_{j-1}, n_j, \tau_{j-1}, \tau_j) = F^{(n_{j-1})}(s(\tau_{j-1})) - F^{(n_j)}(s(\tau_j)) = P[\sum_{i=1}^{n_{j-1}} X_i < s(\tau_{j-1}), \sum_{i=1}^{n_j} X_i \geq s(\tau_j)]$, for all $j = 1, 2, \cdots m$.

Proof: For $j \leq m$,

$$\begin{aligned}
g^*(j, n_1, \cdots n_m) &= P\left[J = j, N_1 = n_1, \cdots, N_j = n_j, N_{j+1} = n_j, \cdots N_m = n_j\right] \\
&= P\left[J = j, N(\tau_1) = n_1, \cdots N(\tau_j) = n_j\right] \\
&= P\left[\tau_{j-1} < T \leq \tau_j, N(\tau_1) = n_1, \cdots N(\tau_j) = n_j\right] \\
&= P\left[\sum_{l=1}^{N(\tau_j)} X_i \geq s(\tau_j), \sum_{l=1}^{N(\tau_{j-1})} X_l < s(\tau_{j-1}), N(\tau_1) = n_1, \cdots N(\tau_j) = n_j\right] \\
&= \left\{P\left[\sum_{l=1}^{n_{j-1}} X_l < s(\tau_{j-1})\right] - P\left[\sum_{l=1}^{n_j} X_l < s(\tau_j)\right]\right\} \\
&\quad \times \prod_{l=1}^{j} P\left[N(\tau_l) - N(\tau_{l-1}) = n_l - n_{l-1}\right] \\
&= F^*(n_{j-1}, n_j, \tau_{j-1}, \tau_j) \prod_{l=1}^{j} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1})
\end{aligned}$$

Then, for $j = m + 1$,

$$\begin{aligned}
g^*(m+1, n_1, \cdots n_m) &= P\left[J = m+1, N_1 = n_1, \cdots, N_m = n_m\right] \\
&= P\left[\sum_{l=1}^{n_m} X_l < s(\tau_m)\right] \\
&\quad \times \prod_{l=1}^{n_m} P\left[N(\tau_l) - N(\tau_{l-1}) = n_l - n_{l-1}\right] \\
&= F^{(n_m)}(s(\tau_m)) \prod_{l=1}^{m} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1})
\end{aligned}$$

Now, we verify that $\int g^*(j, n_1, \cdots n_m) d\nu_2 = 1$.

$$\begin{aligned}
\int g^*(j, n_1, \cdots n_m) d\nu_2 &= \sum_{1 \le j \le m} \sum_{n_1, \cdots, n_m} F_*(n_j, \tau_j) \prod_{l=1}^{j} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1}) \\
&\quad + \sum_{n_1, \cdots, n_m} F^{(n_m)}(s(\tau_m)) \prod_{l=1}^{m} G^*(n_l - n_{l-1} + 1, \tau_l - \tau_{l-1}) \\
&= \sum_{1 \le j \le m} \sum_{n_1, \cdots, n_m} P\left[J = j, N_1 = n_1, \cdots N_m = n_m\right] \\
&\quad + \sum_{n_1, \cdots, n_m} P\left[J = m+1, N_1 = n_1, \cdots N_m = n_m\right] \\
&= \sum_{1 \le j \le m} \sum_{n_1, \cdots, n_m} P\left[J = j, N(\tau_1) = n_1, \cdots N(\tau_j) = n_j\right] \\
&\quad + \sum_{n_1, \cdots, n_m} P\left[J = m+1, N(\tau_1) = n_1, \cdots, N(\tau_m) = n_m\right] \\
&= 1 - R(\tau_1) - \sum_{j=2}^{m} \left[R(\tau_j) - R(\tau_{j-1})\right] + R(\tau_m) \\
&= 1
\end{aligned}$$

Note that the density of O_2 remains the same even if the strength function $s(t)$ is not differentiable everywhere for $t > 0$. However, we need to assume that $s(t)$ is a continuous function of t for identifiability of the damage distribution (See Bhuyan et al. [12] for more details). If $F(x)$ is not continuous for all $x > 0$, then one needs to replace $F(x)$ with $F(x-)$ in the statement of Theorem 3.2 and its proof.

Now we discuss the identifiability issues in this dynamic stress-strength framework with respect to the above mentioned density functions. Let $f_1^*(t, n, \delta)$ and $f_2^*(t, n, \delta)$ be the densities with respect to the pair of parameter choices $\{F_1(\cdot), s_1(\cdot), \eta_1\}$ and $\{F_2(\cdot), s_2(\cdot), \eta_2\}$, respectively, based on the observed random vector under the first sampling plan (See Theorem 3.1), where η_1 and η_2 are the real-valued parameters associated with the shock arrival processes with corresponding damage distributions $F_1(\cdot)$ and $F_2(\cdot)$, and strength functions $s_1(\cdot)$ and $s_2(\cdot)$, respectively. We write $G_n(t)$ as $G_n(t, \eta)$ to indicate its explicit dependence on the parameter η and also assume that $G_1(t, \eta)$ is strictly monotonic function of η, for all $t > 0$. When the shock arrival process is Poisson (See (4)), for example, then η is the λ parameter and $G_1(t) = 1 - e^{-\lambda t}$. Now, we consider the identity

$$f_1^*(t, n, \delta) = f_2^*(t, n, \delta), \tag{5}$$

for all $0 < t \leq \tau$, $\delta = -1, 0, 1$, and $n = 0, 1, \cdots$, and attempt to prove that $F_1(\cdot) \equiv F_1(\cdot)$, $s_1(\cdot) \equiv s_2(\cdot)$ and $\eta_1 = \eta_2$. Putting $t = \tau$, $\delta = -1$ and $n = 0$ in both sides of (5), we get $G_1(\tau, \eta_1) = G_1(\tau, \eta_2)$ leading to $\eta_1 = \eta_2$. Now we put $\delta = 1$ and $\eta_1 = \eta_2$ in (5) and get $F_1^{(n)}(s_1(t)) = F_2^{(n)}(s_2(t))$, for all $n = 1, 2, \cdots$, and for all $0 < t \leq \tau$. If $s_1(t) = s_2(t)$, for all $t > 0$, then $F_1^{(n)}(x) = F_2^{(n)}(x)$, for $s_1(\tau) \leq x < s_1(0)$. If $F_1(x) = F_2(x)$, for all $x \geq 0$, and $F_1(\cdot)$ is strictly increasing, then $s_1(t) = s_2(t)$, for all $t > 0$. Therefore, the model is identifiable if either of the damage distribution $F(\cdot)$ (with strictly increasing property) or the deterministic strength function $s(\cdot)$ is known. Note that the damage distribution is clearly not identifiable outside the range $[s(\tau), s(0))$.

Now, for the second sampling plan, let $g_1^*(j, n_1, \cdots n_m)$ and $g_2^*(j, n_1, \cdots n_m)$ be the two densities with respect to the pair of choices $\{F_1(\cdot), s_1(\cdot), \eta_1\}$ and $\{F_2(\cdot), s_2(\cdot), \eta_2\}$, respectively (See Theorem 3.2). Now we consider the identity

$$g_1^*(j, n_1, \cdots n_m) = g_2^*(j, n_1, \cdots n_m), \tag{6}$$

for all $j = 1, \cdots m+1$ and $0 \leq n_1 \leq n_2 \leq n_m$. Putting $j = m+1$ and $n_1 = n_2 = \cdots n_m = 0$ in both sides of (6), we get, as before, $\eta_1 = \eta_2$. Now, putting $\eta_1 = \eta_2$ and $n_j = n$, for $j = 1, \cdots m$, we get $F_1^{(n)}(s_1(\tau_j)) = F_2^{(n)}(s_2(\tau_j))$, for all $n = 1, 2, \cdots$, and for all $j = 1, \cdots, m$. If $s_1(t) = s_2(t)$ for $t > 0$, then $F_1^{(n)}(x) = F_2^{(n)}(x)$, for all $x = s_1(\tau_1), \cdots, s_1(\tau_m)$. If $F_1(x) = F_2(x)$, for all $x > 0$, and $F_1(\cdot)$ is strictly increasing, then $s_1(\tau_j) = s_2(\tau_j)$, for $j = 1, \cdots, m$. Similar to the previous case, the model is identifiable if either of F or $s(t)$ is known. Note that the damage distribution and its successive convolutions are identifiable at $x = s_1(\tau_1), \cdots, s_1(\tau_m)$. In practice, it is, however, quite unlikely to find two different distributions F_1 and F_2 belonging to standard parametric family of continuous life distributions (for example, Exponential, Weibull, Gamma, Lognormal, Pareto, etc.) satisfying the identity $F_1^{(n)}(x) = F_2^{(n)}(x)$, for all $x = s_1(\tau_1), \cdots, s_1(\tau_m)$, and for all $n = 1, 2, \cdots$.

In view of the above non-identifiability between the damage distribution $F(\cdot)$ and the strength function $s(\cdot)$ (See Bhuyan et al. [12], for more details), let θ be the vector of independent parameters associated with $F(\cdot)$ and $s(\cdot)$, so that the reliability function $R(t)$ in (3) or (4) can be denoted by $R_{\theta,\eta}(t)$ to express explicitly its dependence on θ and η. Then, clearly, both θ and η are identifiable from the data arising from either of the two sampling plans of Section 2.

4 Maximum Likelihood Estimation

Based on data from K independent systems under either of the two sampling plans, as described in Section 2, the objective is to obtain the maximum likelihood estimates of θ and η and hence of the reliability function, given in equation (3). Under the first sampling plan, suppose $(t_i \wedge \tau, n_i, \delta_i)$ is the observed value of O_1, for $i = 1, \cdots, K$. Note that the likelihood contribution of $(t_i \wedge \tau, \delta_i)$, given n_i, for $i = 1, \cdots, K$, is

$$L_{11}(\theta) \propto \prod_{i=1}^{K} \left[\left[|s'(t_i)| f^{(n_i)}(s(t_i)) \right]^{1-\delta_i} \left[F^*(n_i, s(t_i)) \right]^{\delta_i} \right]^{\epsilon_i} \left[F^{(n_i)}(s(\tau)) \right]^{1-\epsilon_i}, \tag{7}$$

involving only θ, where $\epsilon_i = I(\delta_i \neq -1)$. The likelihood contribution for n_i, $i = 1, \cdots, K$, involves only η and is given by

$$L_{12}(\eta) \propto \prod_{i=1}^{K} \left[\left[G^*(n_i + 1, t_i) \right]^{1-\delta_i} \left[g_{n_i}(t_i) \right]^{\delta_i} \right]^{\epsilon_i} \left[G^*(n_i + 1, \tau) \right]^{1-\epsilon_i}. \tag{8}$$

Consider now K independent systems under the second sampling plan with observations $\{J = j_i, N(\tau_l) = n_{il}, l = 1, \cdots, j_i \wedge m\}$, for $i = 1, \cdots, K$. Then the likelihood function for θ based on observations on the j_i's, given the n_{il}'s, is

$$L_{21}(\theta) \propto \prod_{i=1}^{K} \left[F^{(n_{i(j_i-1)})}(s(\tau_{j_i-1})) - F^{(n_{ij_i})}(s(\tau_{j_i})) \right]^{\nu_i} \left[F^{(n_{im})}(s(\tau_m)) \right]^{1-\nu_i}, \tag{9}$$

involving only θ, as before, where $\tau_0 = 0$, $n_{i0} = 0$, and $\nu_i = I(j_i \leq m)$. Using independent increment property of the shock generating process $N(t)$, the likelihood function for η based on the n_{il}'s is given by

$$L_{22}(\eta) \propto \prod_{i=1}^{K} \prod_{l=1}^{j_i} G^*(n_{il} + n_{i(l-1)} + 1, \tau_l - \tau_{l-1}) \tag{10}$$

involving η alone. Note that, from (7) and (8), or (9) and (10), estimation of θ and η can be carried out independently in both the sampling plans. Using invariance property of maximum likelihood estimates, one can plug-in $\hat{\theta}$ and $\hat{\eta}$, maximum likelihood estimates (MLEs) of θ and η, in the expression $R_{\theta,\eta}(t)$ in (3) or (4) to obtain its maximum likelihood estimate (MLE), $R_{\hat{\theta},\hat{\eta}}(t)$, at any given point of time t. The estimated asymptotic variance-covariance matrix of the parameter estimates can be obtained from the Fisher information matrix $I(\theta, \eta)$ based on the likelihood functions. One needs to use the delta method for obtaining the asymptotic standard error of the reliability estimate. Partial derivatives of the reliability function with respect to the parameters may not possess closed form expressions, but one may still be able to apply the delta method by employing numerical methods.

Let us write $\psi = (\theta, \eta)$ to represent the vector of model parameters. Note that the likelihoods $L_1(\psi) = L_1(\theta, \eta) = L_{11}(\theta) \times L_{12}(\eta)$ and $L_2(\psi) = L_2(\theta, \eta) = L_{21}(\theta) \times L_{22}(\eta)$, based on data from K independent systems under the two sampling plans of Section 2, can be written as product of K densities corresponding to K iid random vectors with the common densities $f^*(t, n, \delta)$ and $g^*(j, n_1, \cdots, n_m)$ of Theorem 3.1 and 3.2, respectively. To make the dependence on ψ explicit, let us denote the two densities by $f_\psi^*(v)$ and $g_\psi^*(w)$, respectively, where $v = (t, n, \delta)$ and $w = (j, n_1, \cdots, n_m)$. Once the likelihood is identified as a product of densities, standard results for consistency and asymptotic normality of the MLE of the model parameters apply. In particular, under the following sufficient conditions, we argue that the maximum likelihood estimates of the model parameters are consistent (See Lehmann [14, Theorem 7.5.2, p-501]).

(C1) The parameter θ is identifiable with respect to the family of densities $p_\theta = |s'(\cdot)|f(s(\cdot))$, and the parameter η is identifiable with respect to the family of densities $p_{\eta,n} = g_n(\cdot)$ for all n. In other words, $\theta_1 \neq \theta_2$ implies $p_{\theta_1} \neq p_{\theta_2}$ and $\eta_1 \neq \eta_2$ implies $p_{\eta_1,n} \neq p_{\eta_2,n}$ for all $n = 1, 2, \cdots$.

(C2) The parameter spaces for θ and η are open.

(C3) $N(t)$ is independent of $X_1, X_2, \cdots$, for all $t > 0$.

(C4) The sets $A_1 = \{u : p_\theta(u) > 0\}$ and $B_n = \{u : p_{\eta,n}(u) > 0\}$, $n = 1, 2, \cdots$, are independent of θ and η, respectively.

(C5) For all u in A_1, the partial derivative $\frac{\delta}{\delta\theta} p_\theta(u)$ exists and is a continuous function of θ. Similarly, for all u in B_n, the partial derivative $\frac{\delta}{\delta\eta} p_{\eta,n}(u)$ exists and is a continuous function of η, for all $n = 1, 2, \cdots$.

(C6) For all v in $A_{f_\psi^*} = \{v : f_\psi^*(v) > 0\}$, the partial derivative $\frac{\delta^3}{\delta\psi_i\psi_j\psi_k} f_\psi^*(v)$ exists and are continuous such that the corresponding derivative of $\int f_\psi^*(v) d\nu_1(v)$ can be obtained by differentiating under the integral sign. Similarly, for all w in $A_{g_\psi^*} = \{w : g_\psi^*(w) > 0\}$, the partial derivative $\frac{\delta^3}{\delta\psi_i\psi_j\psi_k} g_\psi^*(w)$ exists and are continuous such that the corresponding derivative of $\int g_\psi^*(w) d\nu_2(w)$ can be obtained by differentiating under the integral sign.

(C7) There exists a positive number $c(\psi_o)$ such that $|\frac{\delta^3}{\delta\psi_i\psi_j\psi_k}\log f^*_\psi(v)| \leq M^{f^*}_{ijk}(v)$ and $E_{\psi_o}[M^{f^*}_{ijk}(V)] < \infty$ for all ψ with $(\psi-\psi_o)'(\psi-\psi_o) < c(\psi_o)$, and for all i,j,k, where ψ_o is the true value of ψ. Similarly, there exists a positive number $d(\psi_o)$ such that $|\frac{\delta^3}{\delta\psi_i\psi_j\psi_k}\log g^*_\psi(w)| \leq M^{g^*}_{ijk}(w)$ and $E_{\psi_o}[M^{g^*}_{ijk}(W)] < \infty$ for all ψ with $(\psi-\psi_o)'(\psi-\psi_o) < d(\psi_o)$, and for all i,j,k.

(C8) The Fisher information matrix $I(\psi)$ is positive definite.

It can be seen that the conditions C1-C4 imply conditions M1-M4 of Theorem 7.5.2 of Lehman [14, p-501] for the present case. The condition C5 implies that $\frac{\delta}{\delta\theta}F^{(n)}_\theta(\cdot)$ and $\frac{\delta}{\delta\eta}G_n(\cdot,\eta)$ are well-defined and, hence, it implies condition M5 of Theorem 7.5.2 of Lehman [14, p-501]. Conditions C6-C8 are same as conditions M6$''$, M7, and M8 of Theorem 7.5.2 of Lehman [14, p-501], respectively. Under the conditions C1-C8, Theorem 7.5.2 of Lehman [14, p-501] implies that the maximum likelihood estimate $\hat{\psi}$ of ψ is consistent and asymptotically normally distributed with mean $\psi_0 = (\theta_0, \eta_0)$ and covariance matrix $I^{-1}(\psi_0)$. We now assume that η is a real-valued parameter and prove that $R_{\hat{\theta},\hat{\eta}}(t)$ is a consistent estimate of the reliability function $R_{\theta,\eta}(t)$.

Theorem 4.1: The maximum likelihood estimate of the reliability function $R_\psi(t) = \sum_{n=0}^{\infty} F^{(n)}_\theta(s(t)-)[G_n(t,\eta) - G_{n+1}(t,\eta)]$, given by $R_{\hat{\psi}}(t) = R_{\hat{\theta},\hat{\eta}}(t)$, is consistent for all $t > 0$ under the following additional conditions.

(D1) $G_n(t,\eta)$ is strictly monotonic function of η, for all $t > 0$ and for all $n = 1, 2, \cdots$.

(D2) $E_\eta[N(t)]$ exists for all $t > 0$ and for all η.

Proof: Since the maximum likelihood estimates of ψ is consistent under the conditions C1-C8, it is enough to show that $R_\psi(t)$ is a continuous function of ψ, for all $t > 0$ [14, Theorem 2.1.4, p-51].

Note that, by C5, $F^{(n)}_\theta(s(t)-)[G_n(t,\eta) - G_{n+1}(t,\eta)]$ is a continuous function of ψ. Without loss of generality, we assume that $G_n(t,\eta)$ is strictly increasing function of η, for all $t > 0$ and for all $n = 1, 2, \cdots$. Then, for fixed $\psi = \psi^* = (\theta^*, \eta^*)$, we have

$$F^{(n)}_{\theta^*}(s(t)-)[G_n(t,\eta^*) - G_{n+1}(t,\eta^*)] \leq G_n(t,\eta^*) - G_{n+1}(t,\eta^*) \leq \quad G_n(t,\eta^*) \leq G_n(t,\eta^{**})$$

for some $\eta^{**} \geq \eta^*$, and for all $n = 0, 1, \cdots$. The infinite sum $\sum_{n=0}^{\infty} G_n(t, \eta^{**}) = \sum_{n=0}^{\infty} P[N^{**}(t) \geq n] = E_{\eta^{**}}[N^{**}(t)] < \infty$, where $N^{**}(t)$ is the point process with parameter η^{**}. Therefore, $R_\psi(t)$ is uniformly convergent with respect to ψ with $0 < \eta < \eta^{**}$ [15, Theorem 7.10, p-148] and each term is a continuous function of ψ. Hence, $R_\psi(t)$ is continuous at the fixed $\psi = \psi^*$ (See [15], Theorem 7.11, p-149). Since η^* is chosen arbitrarily, $R_\psi(t)$ is a continuous function of ψ over the entire parameter space and, hence, the maximum likelihood estimate $R_{\hat{\psi}}(t)$ of the system reliability $R_\psi(t)$ is consistent for all $t > 0$.

Remark 4.1: If we further assume that $\frac{\delta}{\delta\theta} R_{\theta,\eta}(t)$ and $\frac{\delta}{\delta\eta} R_{\theta,\eta}(t)$ exist (See [15], Theorem 7.17, p-152) and at least one of the components is non-zero, then one can apply delta method (See [14], Theorem 5.4.6, p-315) to establish asymptotic normality of the MLE of the reliability function with mean $R_{\theta,\eta}(t)$ and variance $\vartheta' I(\psi)^{-1}\vartheta$ for a specified time $t > 0$, where $\vartheta = (\frac{\delta}{\delta\theta} R_{\theta,\eta}(t), \frac{\delta}{\delta\eta} R_{\theta,\eta}(t))$.

In particular, if we assume Poisson arrival of shocks with intensity λ, then the distribution of T_n is Gamma with shape parameter n and scale parameter λ, denoted by $G_n(t, \lambda) = \int_0^t \frac{x^{n-1}}{(n-1)!} e^{-\lambda x} x^{n-1} dx = \int_0^{\lambda t} \frac{x^{n-1}}{(n-1)!} e^{-x} dx$, which is strictly increasing function of λ for all $t > 0$, and for all $n = 1, 2, \cdots$, and $\sum_{n=0}^{\infty} G_n(t, \lambda) \leq \sum_{n=0}^{\infty} G_n(t, \lambda^*) = \lambda^* t < \infty$, for all $\lambda^* \geq \lambda$. Therefore, the maximum likelihood estimate of the reliability $R_{\theta,\eta}(t)$ is consistent and asymptotically normal for any choice of $F(\cdot)$ and $s(\cdot)$ satisfying C1-C8.

Now, we describe the estimation procedure assuming the successive damages to be iid Gamma random variables with shape parameter α and scale parameter μ. We also assume that the successive shocks arrive according to a homogeneous Poisson process with intensity λ, independent of the damage distribution. Let us consider the deterministic strength function to be $s(t) = AB^t$, with $A > 0$, $0 < B < 1$. As discussed at the end of Section 3, we have $\theta = (\alpha, \beta, B)$ with $\beta = \mu/A$ and $\eta = \lambda$. We first obtain the likelihood functions based on data from both the sampling plans. For the first sampling plan, the likelihood function (7) is written as

$$\prod_{i=1}^{K} \left[\left[|\log(B)| AB^{t_i} \gamma(AB^{t_i}, n_i, \alpha, \mu) \right]^{1-\delta_i} \left[\Gamma^*(AB^{t_i}, n_i, \alpha, \mu) \right]^{\delta_i} \right]^{\epsilon_i} \left[\Gamma(AB^{\tau}, n_i\alpha, \mu) \right]^{1-\epsilon_i},$$

where $\gamma(x, n, \alpha, \mu) = \frac{\mu^{-n\alpha}}{\Gamma(n\alpha)} x^{n\alpha-1} e^{-\frac{x}{\mu}}$, $\Gamma^*(x, n, \alpha, \mu) = \Gamma(x, (n-1)\alpha, \mu) - \Gamma(x, n\alpha, \mu)$ and $\Gamma(x, \alpha, \mu) = \int_0^x \gamma(u, 1, \alpha, \mu) du$. Now, using the re-parametrization $\frac{\mu}{A} = \beta$, the likelihood function for $\theta = (\alpha, \beta, B)'$ is given by

$$L_{11}(\theta) \propto \prod_{i=1}^{K} \left[\left[|\log(B)| B^{t_i} \gamma(B^{t_i}, n_i, \alpha, \beta) \right]^{1-\delta_i} \left[\Gamma^*(B^{t_i}, n_i, \alpha, \beta) \right]^{\delta_i} \right]^{\epsilon_i} \left[\Gamma(B^{\tau}, n_i\alpha, \beta) \right]^{1-\epsilon_i}. \quad (11)$$

The likelihood function (8), with $\eta = \lambda$, can be written as

$$L_{12}(\eta) \propto \prod_{i=1}^{k} \left[\left[\gamma(t_i, n_i, 1, \frac{1}{\lambda}) \right]^{\delta_i} \left[\Gamma^*(t_i, n_i, 1, \frac{1}{\lambda}) \right]^{1-\delta_i} \right]^{\epsilon_i} \left[\left[\Gamma^*(\tau, n_i + 1, 1, \frac{1}{\lambda}) \right] \right]^{1-\epsilon_i}. \quad (12)$$

Similarly, we re-parametrize the likelihood functions based on the observations from the second sampling plan and rewrite the likelihood functions (9) and (10) as

$$L_{21}(\theta) \propto \prod_{i=1}^{K} \left[\Gamma(B^{\tau_{j_i-1}}, n_{ij_i-1}\alpha, \beta) - \Gamma(B^{\tau_{j_i}}, n_{ij_i}\alpha, \beta) \right]^{\nu_i} \left[\Gamma(B^{\tau_m}, n_{im}\alpha, \beta) \right]^{1-\nu_i} \quad (13)$$

and

$$L_{22}(\eta) \propto \prod_{i=1}^{K} \prod_{l=1}^{j_i} \frac{\{\lambda(\tau_l - \tau_{l-1})\}^{(n_{il} - n_{i(l-1)})}}{(n_{il} - n_{i(l-1)})!} exp\left\{ -\lambda(\tau_l - \tau_{l-1}) \right\}, \quad (14)$$

respectively. Therefore, the likelihood function $L(\psi)$ with $\psi = (\alpha, \beta, B, \lambda)$ is the product of equations (11) and (12), or equations (13) and (14), for data obtained from the first and the second sampling plan, respectively. The Newton-Raphson method can be used to maximize the log-likelihoods in order to estimate θ and η. Alternatively, any standard software package equipped with general purpose optimization (e.g., *optim* in the package R) can be used.

5 Simulation Study

For the purpose of simulation, we assume that the shocks appear according to a Poisson process with intensity λ and the damages follow iid Gamma distribution with shape parameter α and scale parameter μ. We assume that strength degrades with the function $s(t) = AB^t$, $A > 0$, and $0 < B < 1$. We use the following values of the parameters.

(i) $\alpha = 5,\ \mu = 3,\ \lambda = 0.1,\ A = 550,\ B = 0.8,$

(ii) $\alpha = 0.5,\ \mu = 1,\ \lambda = 0.3,\ A = 200,\ B = 0.9.$

These two choices of parameter values represent two different patterns of stress accumulation with the mean damages being 15 and 0.5 units, respectively, and the shocks arriving, on the average, three times more frequently in the second case, than those in the first case. On the other hand, initial strength of the system for the first choice is higher than that of the second one, whereas strength deteriorates more rapidly in the first case. These two contrasting choices of parameter values provide interesting comparison of maximum likelihood estimates of reliability function with theoretical values for median lifetime being 15.10 and 35.25 time units, respectively. For the second sampling plan, we consider periodic inspections at times $5, 10, 15, 20, 25, 30, 35, 40$ and $28, 31, 34, 37, 40, 43$ for the two choices, respectively, and the censoring times for the first sampling plan are 40 and 43, respectively.

For each simulated data in each sampling plan, the maximum likelihood estimates of $\alpha, \beta, B, \lambda$ are obtained by maximizing the product of (11) and (12) for the first sampling plan, and (13) and (14) for the second one. Corresponding standard errors are obtained by using the observed information matrix along with the asymptotic 95% confidence intervals based on normal approximation for both the sampling plans. Reliability functions are also estimated at 1st, 3rd, 5th, 7th and 9th deciles of the life distribution and the corresponding standard errors are obtained by using delta method along with the asymptotic 95% confidence interval. This is repeated 1000 times and mean and standard deviation of the estimates over the 1000 simulations are obtained with sample sizes $k = 50, 100, 200, 500$. The coverage probabilities of the asymptotic 95% confidence intervals are estimated based on the 1000 simulations. Results of this simulation study are presented in Table 1-4 and Table 5-8 for the choices (i) and (ii), respectively, for the model parameters. We report both the average of the asymptotic standard errors (ASE) and the standard deviation (SSE) of the corresponding estimates over 1000 simulations. These two measures of standard error are similar, as expected, specially for large k. The results indicate consistency of the estimates as the bias and the standard error (SSE or ASE) decrease with k. Also, the estimated coverage probability is closer to 95% for larger k indicating evidence in favour of asymptotic normality. Interestingly, the evidence in favour of asymptotic normality through the estimated coverage probability of the asymptotic 95% confidence interval for the reliability function seems to be stronger than those for the model parameters. Comparing results from the two sampling plans, we find that the standard errors of estimates for the second sampling plan are comparatively larger than those of the first sampling plan. The additional uncertainty in the estimates from the second sampling plan is possibly due to the interval censoring of failure time data in between the inspection times and lack of information on mode of failure. Interestingly, this difference decreases as k increases for all the model parameters and it is almost negligible for reliability estimates.

Table 1: Results of the simulation study for parameter choice (i) with $k = 50$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	5.00	5.79	190.92	183.19	96.50	5.00	7.70	660.43	1022.22	95.40
$\beta \times 10^3$	5.45	5.12	150.10	151.35	88.70	5.45	4.16	281.60	257.68	77.80
B	0.80	0.80	2.39	2.30	92.00	0.80	0.77	10.49	9.28	88.90
λ	0.10	0.10	1.07	1.10	94.60	0.10	0.10	1.40	1.02	95.50
$R(11.56)$	0.90	0.90	2.81	2.85	93.90	0.90	0.90	10.24	6.77	84.40
$R(13.44)$	0.70	0.70	4.80	4.85	94.10	0.70	0.71	10.42	10.24	91.50
$R(15.10)$	0.50	0.50	5.58	5.59	94.40	0.50	0.50	7.79	5.93	94.60
$R(17.10)$	0.30	0.30	5.06	4.99	93.40	0.30	0.29	7.07	6.25	88.90
$R(23.12)$	0.10	0.10	2.51	2.59	94.30	0.10	0.10	2.51	2.41	94.20

Table 2: Results of the simulation study for parameter choice (i) with $k = 100$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	5.00	5.41	124.43	119.13	95.10	5.00	5.69	291.71	499.22	94.30
$\beta \times 10^3$	5.45	5.29	106.03	106.96	91.10	5.45	4.93	183.09	179.16	88.40
B	0.80	0.80	1.64	1.58	93.20	0.80	0.79	5.23	3.18	93.40
λ	0.10	0.10	0.77	0.78	95.60	0.10	0.10	0.86	0.73	95.20
$R(11.56)$	0.90	0.90	2.01	2.00	93.50	0.90	0.90	5.13	2.77	92.70
$R(13.44)$	0.70	0.70	3.42	3.43	94.50	0.70	0.70	5.64	4.20	94.00
$R(15.10)$	0.50	0.50	3.95	3.97	94.80	0.50	0.50	4.89	4.22	94.30
$R(17.10)$	0.30	0.30	3.54	3.54	95.20	0.30	0.30	4.28	3.81	92.30
$R(23.12)$	0.10	0.10	1.79	1.82	96.10	0.10	0.10	1.77	1.69	94.80

Table 3: Results of the simulation study for parameter choice (i) with $k = 200$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	5.00	5.18	83.00	80.22	95.10	5.00	5.36	133.46	134.61	96.10
$\beta \times 10^3$	5.45	5.35	75.67	75.52	92.60	5.45	5.14	128.21	123.75	90.00
B	0.80	0.80	1.14	1.11	93.10	0.80	0.80	1.96	1.76	93.90
λ	0.10	0.10	0.55	0.55	94.50	0.10	0.10	0.53	0.51	93.50
$R(11.56)$	0.90	0.90	1.41	1.41	94.10	0.90	0.90	2.03	1.92	93.80
$R(13.44)$	0.70	0.70	2.42	2.43	94.10	0.70	0.70	2.88	2.76	94.20
$R(15.10)$	0.50	0.50	2.83	2.81	94.10	0.50	0.50	3.07	2.99	94.90
$R(17.10)$	0.30	0.30	2.52	2.50	94.60	0.30	0.30	2.80	2.67	93.70
$R(23.12)$	0.10	0.10	1.29	1.28	95.20	0.10	0.10	1.23	1.19	94.20

Table 4: Results of the simulation study for parameter choice (i) with $k = 500$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	5.00	5.05	48.47	49.19	95.80	5.00	5.09	78.19	79.53	94.50
$\beta \times 10^3$	5.45	5.42	48.34	47.84	94.70	5.45	5.35	77.13	78.55	93.00
B	0.80	0.80	0.70	0.70	94.20	0.80	0.80	1.09	1.06	95.00
λ	0.10	0.10	0.35	0.35	95.70	0.10	0.10	0.32	0.32	95.30
$R(11.56)$	0.90	0.90	0.93	0.89	94.60	0.90	0.90	1.18	1.20	95.30
$R(13.44)$	0.70	0.70	1.58	1.54	94.70	0.70	0.70	1.69	1.72	95.70
$R(15.10)$	0.50	0.50	1.81	1.78	95.70	0.50	0.50	1.86	1.89	95.60
$R(17.10)$	0.30	0.30	1.59	1.58	95.50	0.30	0.30	1.66	1.68	95.60
$R(23.12)$	0.10	0.10	0.80	0.80	95.50	0.10	0.10	0.74	0.75	95.10

Table 5: Results of the simulation study for parameter choice (ii) with $k = 50$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	0.50	0.54	23.47	22.15	93.20	0.50	0.55	27.83	26.62	92.00
$\beta \times 10^3$	5.00	4.68	234.69	241.20	88.00	5.00	4.71	282.95	292.02	85.30
B	0.90	0.89	2.73	2.62	92.80	0.90	0.89	3.57	3.35	89.60
λ	0.30	0.30	1.31	1.30	95.00	0.30	0.30	1.28	1.28	95.10
$R(30.43)$	0.90	0.90	3.18	3.17	92.90	0.90	0.90	3.40	3.46	91.30
$R(33.17)$	0.70	0.70	5.22	5.10	93.40	0.70	0.70	5.36	5.29	93.60
$R(35.25)$	0.50	0.50	5.84	5.68	94.00	0.50	0.50	6.01	5.81	93.90
$R(37.51)$	0.30	0.30	5.37	5.23	93.20	0.30	0.30	5.63	5.37	92.80
$R(41.10)$	0.10	0.10	3.13	3.10	92.60	0.10	0.10	3.38	2.26	91.80

Table 6: Results of the simulation study for parameter choice (ii) with $k = 100$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	0.50	0.52	14.60	14.73	94.30	0.50	0.53	18.62	17.99	93.60
$\beta \times 10^3$	5.00	4.76	169.04	171.77	91.70	5.00	4.71	206.34	210.44	91.40
B	0.90	0.90	1.66	1.66	96.40	0.90	0.90	2.13	2.16	94.20
λ	0.30	0.30	0.92	0.92	95.20	0.30	0.30	0.89	0.90	94.70
$R(30.43)$	0.90	0.90	2.17	2.24	94.60	0.90	0.90	2.41	2.44	93.30
$R(33.17)$	0.70	0.70	3.58	3.61	94.80	0.70	0.70	3.79	3.72	94.20
$R(35.25)$	0.50	0.50	4.08	4.03	93.90	0.50	0.50	4.21	4.11	94.20
$R(37.51)$	0.30	0.30	3.81	3.72	93.40	0.30	0.30	3.92	3.84	94.60
$R(41.10)$	0.10	0.10	2.22	2.18	92.70	0.10	0.10	2.33	2.32	93.80

Table 7: Results of the simulation study for parameter choice (ii) with $k = 200$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	0.50	0.51	10.45	10.21	94.40	0.50	0.51	12.30	12.12	94.30
$\beta \times 10^3$	5.00	4.83	119.89	121.73	93.20	5.00	4.81	150.65	149.54	92.10
B	0.90	0.90	1.10	1.11	94.50	0.90	0.90	1.43	1.42	95.30
λ	0.30	0.30	0.64	0.65	94.50	0.30	0.30	0.64	0.64	93.70
$R(30.43)$	0.90	0.90	1.57	1.58	94.20	0.90	0.90	1.73	1.73	93.90
$R(33.17)$	0.70	0.70	2.59	2.55	94.90	0.70	0.70	2.68	2.64	93.30
$R(35.25)$	0.50	0.50	2.91	2.85	95.90	0.50	0.50	2.97	2.91	94.30
$R(37.51)$	0.30	0.30	2.68	2.64	94.10	0.30	0.30	2.76	2.71	94.30
$R(41.10)$	0.10	0.10	1.55	1.54	93.60	0.10	0.10	1.64	1.63	93.70

Table 8: Results of the simulation study for parameter choice (ii) with $k = 500$

	Plan I					Plan II				
Parameter	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$	True	Estimate	$SSE \times 100$	$ASE \times 100$	$Coverage\%$
α	0.50	0.51	6.15	6.35	96.10	0.50	0.51	7.47	7.55	94.20
$\beta \times 10^3$	5.00	4.89	73.52	77.36	95.50	5.00	4.95	95.30	95.63	93.40
B	0.90	0.90	0.65	0.68	96.20	0.90	0.90	0.85	0.85	95.30
λ	0.30	0.30	0.40	0.41	96.10	0.30	0.30	0.42	0.40	94.40
$R(30.43)$	0.90	0.90	0.96	1.00	95.30	0.90	0.90	1.10	1.10	94.50
$R(33.17)$	0.70	0.70	1.56	1.61	95.70	0.70	0.70	1.69	1.67	94.60
$R(35.25)$	0.50	0.50	1.75	1.81	95.80	0.50	0.50	1.88	1.84	94.70
$R(37.51)$	0.30	0.30	1.62	1.67	95.90	0.30	0.30	1.75	1.72	95.00
$R(41.10)$	0.10	0.10	0.94	0.97	95.70	0.10	0.10	1.04	1.03	94.10

6 Data Analysis

In Section 2, we have discussed, some application in database management systems for its efficient operation. It is a common practise among email users to forward emails automatically from various email accounts to a preferred email account for ease of operation. In this process, users normally do not clean the mailbox of the secondary email accounts. As a result of accumulation of emails over time, the secondary mailbox becomes full and the account fails to receive any further email. For illustration, we have collected data on 22 such identical systems according to the first sampling plan. Here, the mailbox limit (that is, the strength of the system $s(t)$) is kept fixed at 5 MB and the censoring time is taken as $\tau = 672$ hours. The observed failure time (in hours) data, the number of emails received up to the time of failure and the failure mode for the 22 systems are presented in Table 9. From the same experiment, the data collected according to the sampling plan II with regular inspection at $96, 192, 288, 384, 480, 576, 672$ hours, is presented in Table 10. Note that, since the strength $s(t)$ of the system is fixed over time, failure occurs only when a shock (or, email) arrives with $\Delta = 1$. There is no observation with $\Delta = 0$. In a preliminary data analysis, the average number of arriving shocks seems to increase with time in a linear fashion. Therefore, we assume that emails arrive according to a homogeneous Poisson process and the size (in MB) of the successive emails follow iid Gamma distribution, as in the previous section. We obtain the maximum likelihood estimates of the associated parameters and the reliability function at some specified time points along with the corresponding asymptotic standard errors (ASE) and 95% confidence intervals (ACI) assuming asymptotic normality.

The results of our analysis for sampling plan I and II are reported in Tables 11 and 12, respectively. We also carried out parametric bootstrap with 10000 replications. The corresponding mean (BMean), standard deviation (BSE) and 95% confidence interval (BCI) based on the corresponding bootstrap distribution are also reported. We see that the reliability estimates at different time points are very similar to those of the bootstrap means upto two decimal places. However, bootstrap confidence intervals of these reliability values are shorter compared to the corresponding asymptotic confidence intervals. The comparison of the parametric estimate of the reliability function and the Kaplan-Meier estimate, along with the asymptotic 95% confidence interval, indicates reasonably good fit of the model (See Figures 1 and 2 for plan I and plan II, respectively).

Note the advantage of assuming Gamma damage distribution which permits closed form expression for the convolution $F^{(n)}(\cdot)$, required for likelihood evaluation. Otherwise, the analysis becomes very difficult. We have carried out an approximate analysis of the mailbox data under both the sampling plans, assuming Log-normal and Weibull damage distributions with no closed form expression for $F^{(n)}(\cdot)$, using simulation method. As before, we assume that the emails arrive according to a homogeneous Poisson process. Note that the likelihood functions $L_{12}(\eta)$ and $L_{22}(\eta)$ remain the same; however, $L_{11}(\theta)$ and $L_{21}(\theta)$ involve convolutions of damage distribution $F(\cdot)$, which has no closed form expression for Log-normal and Weibull distributions, resulting in difficulty in their evaluation. However, for a particular choice of parameters, one can approximately evaluate $L_{11}(\theta)$ and $L_{21}(\theta)$ numerically through the method of simulation. In such cases, $F^{(n)}(s(t))$ is approximated by $\#\{i\colon \sum_{j=1}^{n} Z_{ij} < s(t), i = 1, \cdots, 10000\}/10000$, where Z_{ij}'s are simulated independently from a Log-normal or Weibull distribution with the specified choice of parameters. This approximate likelihood function may not be reasonably smooth and the conventional optimization procedure like Newton-Raphson method does not perform well in order to estimate the parameters associated with the damage distribution. In this case, we employ grid search method to find out maximum likelihood estimates of the parameters of the Log-normal or Weibull distribution. Since any simulation based method produces different output at each different run, we perform the grid search method multiple times and consider the mean of all these estimates of a particular parameter as the maximum likelihood estimate. The maximum likelihood estimates of the parameters corresponding to the Gamma size distribution, suitably utilized through method of moments, can provide some initial estimates of the parameters associated with Log-normal or Weibull size distributions. This initial guess reduces the computational efforts during the grid search procedure to some extent. Once the maximum likelihood estimates of the model parameters are available, we estimate the reliability function using the method of simulation. See Bhuyan and Dewanji [6] for details regarding computation of the reliability function under Weibull and Log-normal size distributions.

For the first sampling plan, the AIC values corresponding to modeling assumptions of Gamma, Log-normal and Weibull size distributions are $564.00, 563.11$ and 563.53, respectively, while the same for the second sampling plan are $9311.51, 9310.46$ and 9310.75, respectively. It is evident from the AIC values that all the three models fit the data equally well. The reliability estimates both the sampling plans corresponding to Log-normal and Weibull size distributions are provided in Table 13. Although the likelihood functions $L_{11}(\theta)$ and $L_{21}(\theta)$ seem to be well-approximated by the simulation method, ASE obtained by second order numerical differentiation of these estimated likelihood functions may not be reliable. In such cases, one can calculate BSE as a measure of standard error for the estimates of model parameters and the reliability function. Estimation of the reliability function with Log-normal or Weibull model is itself a computationally intensive procedure and hence evaluation of BSE in this case is time consuming. We have not provided the ASE and BSE figures in Table 13 due to the computational difficulties as mentioned above.

The electric power of a dry cell or battery, initially stored by chemical energy, is weakened by continuous oxidation process and is subject to frequent use leading to accumulated damage or energy loss [3]. Similar phenomena happens in cell-phone battery after each and every recharge. Once fully charged, cell-phone battery looses its energy over time due to normal functionalities of the cell-phone in the switch-on mode and frequent incoming and outgoing calls leading to accumulated damage or energy loss. We have data on 11 identical cell-phone batteries according to sampling plan I with censoring time $\tau = 80$ hours. The observed failure time (in hours) data and the number of calls (incoming and outgoing) up to the time of failure is presented in Table 14. Note that the strength degradation function is not known for the cell-phone battery data and one needs to employ adequate re-parametrization for model identifiability (See Sections 3 and 4). We assume that incoming and outgoing calls take place according to a homogeneous Poisson process and the damages due to successive calls follow iid Gamma distribution. We also assume that $s(t) = AB^t$. Note that, with this assumed functional form of $s(t)$, the expression for $F(s(t))$, for the Weibull damage distribution with shape parameter p and scale parameter σ, becomes $F(s(t)) = 1 - \exp(-\sigma AB^t)^p = 1 - \exp(-CD^t)$, where $C = (\sigma A)^p$ and $D = B^p$. This is indistinguishable with the expression for $F(s(t))$ for Exponential damage distribution and the same functional form of $s(t)$. On the other hand, the Exponential distribution is a special case of the Gamma distribution. Hence, the assumption of Gamma distribution encompasses the Weibull distribution as well. Nevertheless, we obtain the maximum likelihood estimates of the associated parameters and the reliability function at some pre-specified time points along with the corresponding asymptotic standard errors (ASE) and 95% confidence intervals (ACI). The results of our analysis are reported in Table 15, considering initial strength fixed at $A = 100$ units. We also carried out a parametric bootstrap with 10000 replications. We see that the reliability estimates at different time points are similar to those of the bootstrap means upto two decimal places. However, the bootstrap confidence intervals are shorter compared to the corresponding asymptotic confidence intervals, as in the first example. The comparison of the parametric estimate of the reliability function and the Kaplan-Meier estimate, along with the asymptotic 95% confidence interval, indicates reasonably good fit of the model (See Figure 3).

It may be of interest to compare the reliability estimate obtained by the analysis of the dynamic stress-strength model (See Section 1) using additional information, as described in Section 2, with that obtained by the analysis of some commonly used failure time models using only failure time data (See Lu and Meeker [16] for such comparison). For this purpose, we consider analysis of Gamma, Log-normal and Weibull models using only the 22 failure times of the mailbox data of Table 9. The estimated reliability curves are presented in Figure 4, along with the same obtained from the analysis of the stress-strength model with iid Gamma damages due to Poisson shock arrivals and the corresponding Kaplan-Meier estimate, as reported in Figure 1. Similarly, we consider the cell-phone battery data (See Table 14) and the comparison of reliability estimates based on the different models, as discussed before, is presented in Figure 5. We perform a goodness-of-fit test based on χ^2 statistic and the corresponding p-values (χ^2_p) are calculated using Monte Carlo simulation (See Hope [17]). We also compute a modified Kolmogorov-Smirnov (MKS) statistic, where the empirical distribution function is replaced by the Kaplan-Meier estimate. So, the statistic MKS is given by $\sup|R_{\hat{\phi}}(t) - \hat{R}_{KM}(t)|$, where $\hat{\phi}$ denotes the MLE of the vector of model parameters ϕ of the assumed model and $\hat{R}_{KM}(t)$ is the Kaplan-Meier estimate. The p-values corresponding to MKS statistic (MKS_p) are also calculated using the same Monte Carlo simulation method involving the following steps. First we estimate ϕ under the assumed parametric model and calculate the χ^2 or MKS statistic based on the observed data. Then we simulate K observations from the assumed model using $\hat{\phi}$ in place of ϕ, and compute the relevant statistic (χ^2 or MKS) based on this simulated data. This step is repeated 10000 times to obtain the mull distribution of the statistic under consideration. The proportion of observations exceeding the observed value of χ^2 or MKS statistic is considered as the corresponding p-value. The results from these tests are summarized in Table 16, which seem to indicate that all the models fit the cell-phone battery data equally well and the mechanistic model fits the mailbox data best. This is because the mechanistic model incorporates the extra information in addition to the failure time data and takes account of the failure mechanism due to stress-strength interference. We also compare ASE×100 (See Section 5) of the reliability estimates from different models at two different time points at both the extreme tails of the life distribution and the results are reported in Table 17. As expected, the ASEs corresponding to the mechanistic model are smaller than those of the other models.

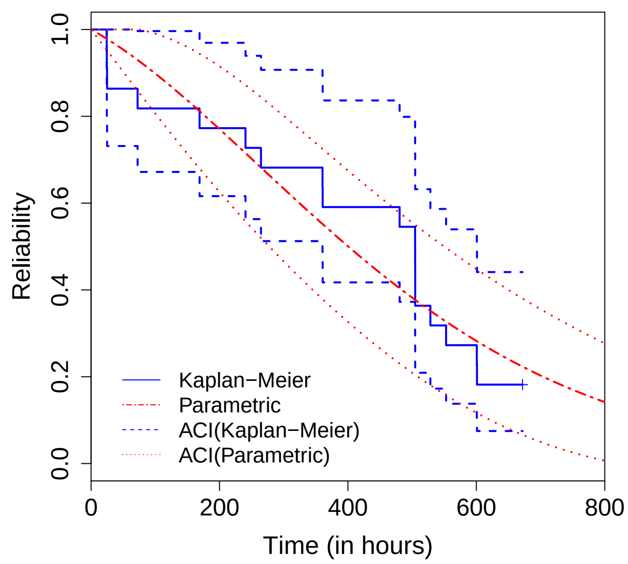

Figure 1: Estimated reliability function and point-wise confidence interval with mailbox failure data for plan I

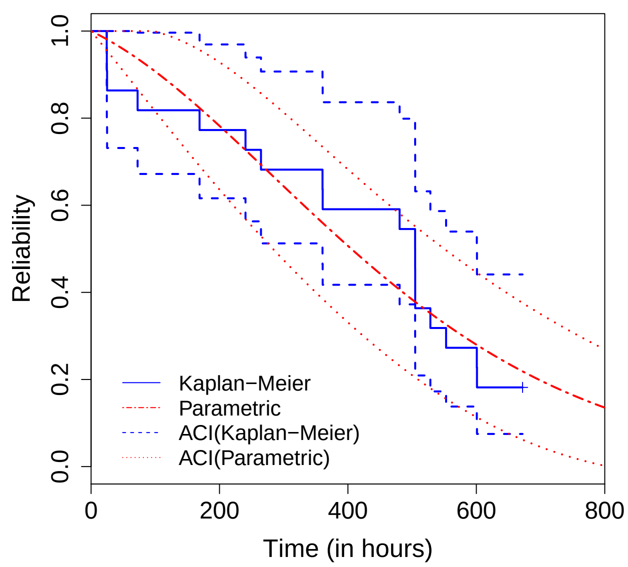

Figure 2: Estimated reliability function and point-wise confidence interval with mailbox failure data for plan II

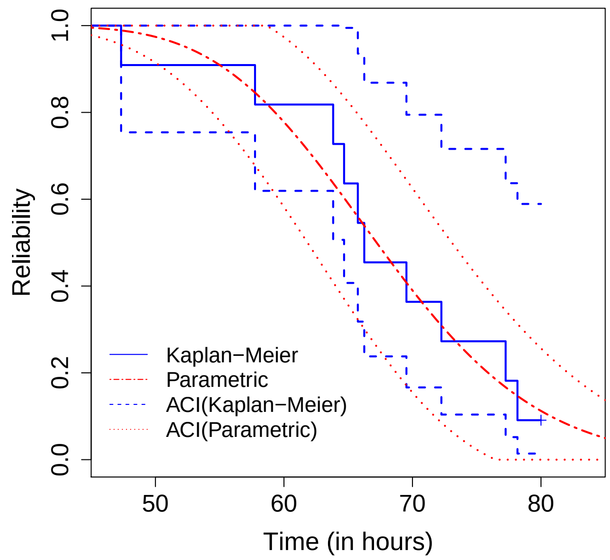

Figure 3: Estimated reliability function and point-wise confidence interval with cell-phone battery data

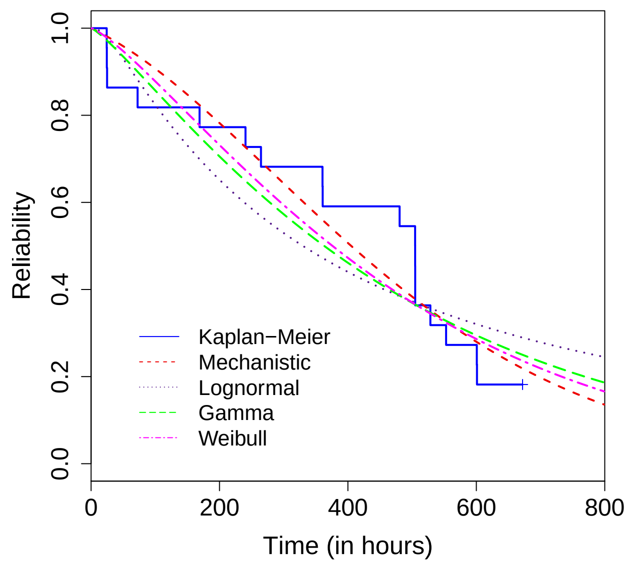

Figure 4: Comparison of reliability estimates from the mechanistic, Weibull, Log-normal and Gamma models with mailbox data (Plan I)

Table 9: Mailbox failure data: Plan I

Sl No.	$T \wedge \tau$	$N(T \wedge \tau)$	Δ	Sl No.	$T \wedge \tau$	$N(T \wedge \tau)$	Δ
1	360.64	145	1	12	24.50	15	1
2	528.42	185	1	13	264.60	57	1
3	600.74	137	1	14	672.00	165	-1
4	504.69	134	1	15	24.91	15	1
5	672.00	144	-1	16	24.31	13	1
6	360.41	101	1	17	672.00	230	-1
7	504.80	152	1	18	168.93	33	1
8	552.88	224	1	19	504.81	152	1
9	480.47	201	1	20	72.42	26	1
10	600.98	175	1	21	672.00	155	-1
11	504.62	107	1	22	240.51	86	1

Table 10: Mailbox failure data: Plan II

Sl No.	J	N_1	N_2	N_3	N_4	N_5	N_6	N_7	Sl No.	J	N_1	N_2	N_3	N_4	N_5	N_6	N_7
1	4	33	57	96	145	145	145	145	12	1	28	28	28	28	28	28	28
2	6	28	49	94	146	174	218	218	13	3	11	30	60	60	60	60	60
3	7	15	30	59	88	114	143	172	14	8	31	55	76	94	133	149	165
4	6	27	55	77	100	121	141	141	15	1	29	29	29	29	29	29	29
5	8	19	36	67	88	108	125	144	16	1	40	40	40	40	40	40	40
6	4	22	41	67	104	104	104	104	17	8	27	59	86	131	172	201	230
7	6	23	53	85	111	133	173	173	18	2	23	68	68	68	68	68	68
8	6	46	79	119	143	192	223	223	19	6	42	66	124	140	186	212	212
9	6	40	80	112	156	194	232	232	20	2	32	83	83	83	83	83	83
10	7	17	33	76	110	138	161	195	21	8	11	34	59	87	104	120	155
11	6	9	31	49	65	93	130	130	22	3	28	56	96	96	96	96	96

Table 11: Results of the mailbox data analysis for plan I

Parameter	Estimate	$ASE \times 100$	ACI	BMean	$BSE \times 100$	BCI
$\alpha \times 100$	0.99	53.34	(0,2.04)	1.16	61.58	(0.46,2.91)
μ	5.83	459.10	(0,14.82)	6.89	472.06	(1.51,20.99)
λ	0.29	0.57	(0.28,0.31)	0.29	0.59	(0.28,0.31)
$R(100)$	0.90	4.70	(0.81,0.99)	0.90	3.73	(0.82,0.98)
$R(200)$	0.77	7.38	(0.63,0.92)	0.77	6.42	(0.64,0.90)
$R(300)$	0.63	8.55	(0.47,0.80)	0.64	8.06	(0.47,0.79)
$R(400)$	0.50	8.89	(0.33,0.68)	0.50	8.90	(0.32,0.67)
$R(500)$	0.38	8.82	(0.21,0.56)	0.38	9.10	(0.21,0.56)
$R(600)$	0.28	8.45	(0.12,0.45)	0.28	8.79	(0.12,0.46)
$R(700)$	0.20	7.80	(0.05,0.36)	0.20	8.04	(0.06,0.38)

Table 12: Results of the mailbox data analysis for plan II

Parameter	Estimate	$ASE \times 100$	ACI	BMean	$BSE \times 100$	BCI
$\alpha \times 100$	1.08	57.68	(0,2.21)	1.32	71.11	(0.47,3.17)
μ	4.96	373.54	(0,12.27)	5.77	422.01	(1.31,18.57)
λ	0.30	0.55	(0.29,0.31)	0.30	0.56	(0.29,0.31)
$R(100)$	0.91	4.68	(0.82,1.00)	0.91	3.88	(0.83,0.98)
$R(200)$	0.78	7.46	(0.64,0.93)	0.79	6.64	(0.65,0.92)
$R(300)$	0.64	8.65	(0.47,0.81)	0.65	8.22	(0.48,0.81)
$R(400)$	0.51	8.96	(0.33,0.68)	0.51	8.98	(0.33,0.69)
$R(500)$	0.38	8.67	(0.21,0.56)	0.38	9.16	(0.21,0.57)
$R(600)$	0.28	8.48	(0.11,0.45)	0.28	8.81	(0.12,0.46)
$R(700)$	0.20	7.79	(0.05,0.35)	0.20	8.03	(0.06,0.37)

Table 13: Reliability estimates for mailbox data: Log-normal and Weibull damages

Sampling Plan	F	$R(100)$	$R(200)$	$R(300)$	$R(400)$	$R(500)$	$R(600)$	$R(700)$
I	Log-normal	0.92	0.81	0.67	0.52	0.38	0.25	0.15
	Weibull	0.93	0.82	0.68	0.53	0.38	0.25	0.15
II	Log-normal	0.91	0.79	0.66	0.52	0.39	0.28	0.18
	Weibull	0.92	0.81	0.67	0.54	0.40	0.28	0.19

Table 14: Cell-phone battery data: Plan I

Sl No.	1	2	3	4	5	6	7	8	9	10	11
$T \wedge \tau$	63.83	66.25	77.25	64.68	80.00	72.25	69.53	65.75	57.75	47.33	78.17
$N(T \wedge \tau)$	22	21	19	18	27	9	20	17	20	22	21
Δ	0	0	0	1	-1	1	0	0	0	0	1

Table 15: Results of the analysis of cell-phone battery data

Parameter	Estimate	$ASE \times 100$	ACI	BMean	$BSE \times 100$	BCI
$\alpha \times 10$	1.93	186.31	(0,5.58)	3.49	450.49	(0.17,14.72)
μ	1.54	125.40	(0,4.00)	1.16	110.52	(0.01,3.95)
B	0.96	2.29	(0.91,1)	0.94	4.45	(0.83,0.99)
λ	0.29	1.97	(0.25,0.33)	0.29	1.99	(0.25,0.33)
$R(50)$	0.97	3.04	(0.91,1)	0.97	2.97	(0.89,1)
$R(55)$	0.91	6.56	(0.78,1)	0.91	6.33	(0.76,1)
$R(60)$	0.78	9.98	(0.58,0.97)	0.78	10.20	(0.57,0.96)
$R(65)$	0.59	11.94	(0.36,0.82)	0.59	12.51	(0.34,0.83)
$R(70)$	0.39	12.00	(0.15,0.63)	0.38	12.31	(0.15,0.63)
$R(75)$	0.22	10.24	(0.02,0.42)	0.22	10.22	(0.05,0.44)
$R(80)$	0.11	7.35	(0,0.26)	0.12	7.50	(0.01,0.29)

Table 16: Results of the Goodness-of-fit tests

	Mailbox Data				Cell-phone Battery Data			
Model	χ^2	χ_p^2	MKS	MKS_p	χ^2	χ_p^2	MKS	MKS_p
Weibull	7.196	0.122	0.202	0.281	0.541	0.742	0.145	0.948
Gamma	8.771	0.063	0.205	0.255	0.344	0.860	0.175	0.816
Log-normal	15.827	0.004	0.209	0.237	0.373	0.860	0.184	0.781
Mechanistic	5.400	0.248	0.187	0.364	0.226	0.863	0.181	0.799

Table 17: Comparison of Asymptotic Standard Errors

	Mailbox Data				Cell-phone Battery Data			
Model	$R(100)$	$R(200)$	$R(600)$	$R(700)$	$R(50)$	$R(55)$	$R(75)$	$R(80)$
Weibull	5.77	8.00	8.56	8.29	4.30	7.03	10.90	7.69
Gamma	6.09	7.77	8.55	8.19	3.46	7.02	10.87	8.42
Log-normal	6.64	8.22	8.68	8.47	3.36	7.13	10.99	8.77
Mechanistic	4.70	7.38	8.45	7.80	3.04	6.56	10.24	7.35

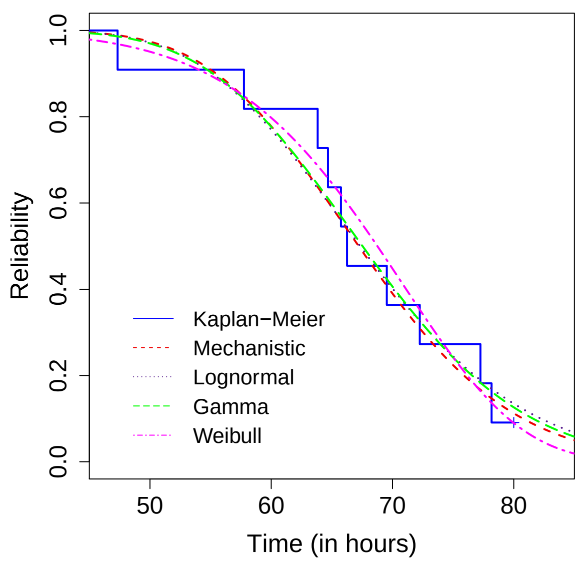

Figure 5: Comparison of reliability estimates from the mechanistic, Weibull, Log-normal and Gamma models with cell-phone battery data (Plan I)

7 Concluding Remark

Dynamic stress-strength model, where stress accumulates over time and strength degrades concurrently, can be used to explain wide range of mechanical and natural phenomena. In this context, estimation of model parameters based on experimental data has been emphasized by several authors [3][11]. Commonly used parametric failure time models can be used to analyse failure time data and estimate the associated parameters and then the reliability function. However, different models may give different reliability estimates, specially in extreme cases when, for example, reliability estimate at a large time point is of interest, even though these different models give reasonably good fit for the observed data (See Figures 4 and 5). In that case, reliability estimate based on a mechanistic model, like the dynamic stress-strength model, incorporating the physical aspect of the failure mechanism may be preferred. This also allows analysis of any additional information like, for example, the number of shocks to make more efficient inference, unlike the commonly used failure time models. Also, since the model parameters have physical interpretations, these and the factors affecting them can be investigated to study their impact on reliability. This paper makes an attempt in that direction by considering, possibly for the first time, sampling plans and estimation methodology accounting for both stress accumulation and deterministic strength degradation in a single model.

Sampling plans are designed to facilitate efficient estimation of model parameters taking technical difficulties as well as practical limitations into consideration. Issues like non-identifiability is dealt carefully by collecting additional information in both the sampling plans. Most importantly, measurements on impact of damages are not considered because of the inherent difficulty in observing such latent impacts of damages on the system under consideration. In terms of the design and the mode of implementation, these two sampling plans are quite different and can be easily implemented in various industrial testing or scientific experiments. One can, in principle, think of many other sampling plans for collection of data that result in meaningful estimate of the model parameters and hence reliability.

Although the primary objective is to obtain estimate of the reliability function under the dynamic stress-strength model, the estimate of the model parameters give specific insights into the process mechanism. This not only helps in the quantitative understanding of the stress-strength interference, but also suggests ways to improve reliability by modifying factors affecting certain parameters suitably.

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