JRF IN MATHEMATICS 2011

TEST CODE RMI, RMII

There will be two tests RMI and RMII of 2 hours duration each in the forenoon and in the afternoon. Topics to be covered in these tests along with an outline of the syllabus and sample questions are given below:

1) Topics for RMI (Forenoon examination) : Real Analysis, Measure and Integration, Complex Analysis, Ordinary Differential Equations and General Topology.

2) Topics for RMII (Afternoon examination) : Algebra, Linear Algebra, Functional Analysis, Elementary Number Theory and Combinatorics.

Candidates will be judged based on their performance in **both** the tests.

OUTLINE OF THE SYLLABUS

1. **General Topology :** Topological spaces, Continuous functions, Connectedness, Compactness, Separation Axioms. Product spaces. Complete metric spaces. Uniform continuity. Baire category theorem.

2. **Functional Analysis :** Normed linear spaces, Banach spaces, Hilbert spaces, Compact operators. Knowledge of some standard examples like $C[0,1], L^p[0,1]$. Continuous linear maps (linear operators). Hahn-Banach Theorem, Open mapping theorem, Closed graph theorem and the uniform boundedness principle.

3. **Real analysis :** Sequences and series, Continuity and differentiability of real valued functions of one and two real variables and applications, uniform convergence, Riemann integration.

4. **Linear algebra :** Vector spaces, linear transformations, characteristic roots and characteristic vectors, systems of linear equations, inner product spaces, diagonalization of symmetric and Hermitian matrices, quadratic forms.

5. **Elementary number theory and Combinatorics:** Divisibility, congruences, standard arithmetic functions, permutations and combinations, and combinatorial probability.

6. **Lebesgue integration :** Lebesgue measure on the line, measurable functions, Lebesgue integral, convergence almost everywhere, monotone and dominated convergence theorems.

7. **Complex analysis :** Analytic functions, Cauchy's theorem and Cauchy integral formula, maximum modulus principle, Laurent series, Singularities, Theory of residues, contour integration.

8. **Abstract algebra :** Groups, homomorphisms, normal subgroups and quotients, isomorphism theorems, finite groups, symmetric and alternating groups, direct product, structure of finite Abelian groups, Sylow theorems. Rings and ideals, quotients, homomorphism and isomorphism theorems, maximal ideals, prime ideals, integral domains, field of fractions; Euclidean rings, principal ideal domains, unique factoristation domains, polynomial rings. Fields, characteristic of a field, algebraic extensions, roots of polynomials, separable and normal extensions, finite fields.

9. **Ordinary differential equations :** First order ODE and their solutions, singular solutions, initial value problems for first order ODE, general theory of homogeneous and nonhomogeneous linear differential equations, and Second order ODE and their solutions.

SAMPLE QUESTIONS

Topology

(1) Let (X, d) be a compact metric space. Suppose that $f : X \to X$ is a function such that

$$d(f(x), f(y)) < d(x, y) \text{ for } x \neq y, x, y \in X.$$

Then show that there exists $x_0 \in X$ such that $f(x_0) = x_0$.

(2) Let X be a Hausdorff space. Let $f : X \to \mathbb{R}$ be such that $\{(x, f(x)) : x \in X\}$ is a compact subset of $X \times \mathbb{R}$. Show that f is continuous.

(3) Let X be a compact Hausdorff space. Assume that the vector space of real-valued continuous functions on X is finite dimensional. Show that X is finite.

(4) Let (X, d) be a complete metric space, $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of closed sets in X such that $\sup\{d(x, y) : x, y \in A_n\}$ tends to zero as n tends to infinity. Let $f : X \to X$ be a continuous map. Show that

$$f(\bigcap_n A_n) = \bigcap_n f(A_n).$$

(5) Show that the set of all rational numbers with the usual topology is not locally compact.

Functional analysis and Linear algebra

(6) Let $y_1, y_2, \ldots$ be a sequence in a Hilbert space. Let V_n be the linear span of $\{y_1, y_2, \ldots, y_n\}$. Assume that $\|y_{n+1}\| \leq \|y - y_{n+1}\|$ for each $y \in V_n$ for $n = 1, 2, 3, \ldots$. Show that $\langle y_i, y_j \rangle = 0$ for $i \neq j$.

(7) Let E and F be real or complex normed linear spaces. Let $T_n : E \to F$ be a sequence of continuous linear transformations such that $\sup_n \|T_n\| < \infty$. Let

$$M = \{x \in E | \text{ The sequence } \{T_n\,(x)\} \text{is Cauchy}\}.$$

Show that M is a closed set.

(8) Suppose that X is a normed linear space over $\mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ is a linear functional. Show that the kernel of f is either closed or dense.

(9) Let X be an infinite dimensional Banach space. Prove that every basis of X is uncountable.

(10) Let X and Y be complex, normed linear spaces which are not necessarily complete. Let $T : X \to Y$ be a linear map such that $\{Tx_n\}$ is a Cauchy sequence in Y whenever $\{x_n\}$ is a Cauchy sequence in X. Show that T is continuous.

(11) Let L and T be two linear functionals on a real vector space V such that $L(v) = 0$ implies $T(v) = 0$. Show that $T = cL$ for some real number c.

(12) Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Show that, for each nonzero scalar $\lambda, (\lambda I - B)^{-1} = P_\lambda(B)$ for some polynomial $P_\lambda(X)$ of degree 3.

(13) Let A be an $n \times n$ square matrix such that A^2 is the identity. Show that any $n \times 1$ vector can be expressed as a sum of at most two eigenvectors of A.

Real Analysis and Measure Theory

(14) Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. Define

$$s_n = \frac{(a_1 + a_2 + \ldots + a_n)}{n}, n = 1, 2, 3, \ldots$$

Show that $\liminf\limits_{n\to\infty} a_n \leq \liminf\limits_{n\to\infty} s_n$.

(15) Let $p(x)$ be an odd degree polynomial in one variable with coefficients from the set $\mathbb{R}$ of real numbers. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Prove that there exists an $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(16) Suppose that U is a connected open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is such that $\frac{\partial f}{\partial x} \equiv 0$ and $\frac{\partial f}{\partial y} \equiv 0$ on U. Show that f is a constant function.

(17) Let $f_1, f_2, f_3, \ldots$ and f be nonnegative Lebesgue integrable functions on $\mathbb{R}$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{y} f_n(x)dx = \int_{-\infty}^{y} f(x)dx \text{ for each } y \in \mathbb{R}$$

$$\text{and } \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

Show that $\liminf\limits_{n\to\infty} \int_U f_n(x)dx \geq \int_U f(x)dx$ for any open subset U of $\mathbb{R}$.

(18) Let f be a uniformly continuous real valued function on the real line $\mathbb{R}$. Assume that f is integrable with respect to the Lebesgue measure on $\mathbb{R}$. Show that $f(x) \to 0$ as $|x| \to \infty$.

(19) Let (Ω, A, μ) be a probability space, i.e. $\mu(\Omega) = 1$. Let $f \geq 0$ be a measurable function on Ω. Show that

$$\int_\Omega \frac{1}{f}\, d\mu \geq \frac{1}{\int_\Omega f\, d\mu}.$$

(20) Let (Ω, A, μ) be a probability space. Let $f \geq 0$ be measurable. Show that $\int f(w)d\mu(w) = \int_0^\infty \mu\{w \mid f(w) \geq x\}dx$.

Elementary Number Theory and Combinatorics

(21) Let p be a prime and r an integer, $0 < r < p$. Show that $\frac{(p-1)!}{r!(p-r)!}$ is an integer.

(22) If a and b are integers such that 9 divides $a^2 + ab + b^2$ then show that 3 divides both a and b.

(23) Let c be a 3^n digit number whose digits are all equal. Show that 3^n divides c.

(24) Prove that $x^4 - 10x^2 + 1$ is reducible modulo p for every prime p.

(25) Does there exist an integer x satisfying the following congruences?

$$\begin{aligned} 10x &= 1 (\text{mod } 21) \\ 5x &= 2 (\text{mod } 6) \\ 4x &= 1 (\text{mod } 7) \end{aligned}$$

Justify your answer.

(26) Suppose that there are n boxes labelled $1, 2, \ldots n$ and there are n balls also labelled similarly. The balls are thrown into boxes completely randomly so that each box receives one ball.

(a) How many possible arrangements of balls in boxes is possible?

(b) Find the probability that the ball labelled 1 goes into the box labelled 1.

(c) Find the probability that at least one ball is in the box with the same label.

Complex Analysis

(27) Suppose for an analytic function f its zero set Z_f is uncountable. Show that $f \equiv 0$.

(28) Let $\mathbb{C}$ be the set of complex numbers and let f be an analytic function on the open disc $\{z \in \mathbb{C} \mid |z| < 1\}$. Assume that $\left\{\frac{d^n f}{dz^n}(0)\right\}$ is a bounded sequence. Show that f has an analytic extension to $\mathbb{C}$.

(29) Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire functions with $g(z) \neq 0$ for all $z \in \mathbb{C}$. Assume that $\lim_{|z|\to\infty} \frac{f(z)}{g(z)}$ exists. Show that either $f(z) = 0$ for all $z \in \mathbb{C}$ or $f(z) \neq 0$ for all $z \in \mathbb{C}$.

Abstract Algebra

(30) Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$ and let k be an integer between 1 and n. Find the number of elements x in S_n such that the cycle containing 1 in the cycle decomposition of x has length k.

(31) Let $\mathbb{C}$ be the field of complex numbers and $\varphi : \mathbb{C}[X, Y, Z] \to \mathbb{C}[t]$ be the ring homomorphism such that

$$\begin{aligned} \varphi(a) &= a \text{ for all } a \text{ in } \mathbb{C}, \\ \varphi(X) &= t, \\ \varphi(Y) &= t^2, \text{ and} \\ \varphi(Z) &= t^3. \end{aligned}$$

Determine the kernel of φ.

(32) Show that there is no field isomorphism between $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$. Are they isomorphic as vector spaces over $\mathbb{Q}$?

(33) Determine finite subgroups of the multiplicative group of non-zero complex numbers.

(34) Let $\mathbb{Z}[X]$ denote the ring of polynomials in X with integer coefficients. Find an ideal I in $\mathbb{Z}[X]$ such that $\mathbb{Z}[X]/I$ is a field of order 4.

Differential Equations

(35) Let $y : [a, b] \to \mathbb{R}$ be a solution of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y(x) = 0,$$

where $P(x)$ and $Q(x)$ are continuous functions on $[a, b]$. If the graph of the function $y(x)$ is tangent to X-axis at any point of this interval, then prove that y is identically zero.

(36) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the differential equation

$$y'(t) + y(t) = f(t) \ (*)$$

on $\mathbb{R}$.

a) Show that $(*)$ can have at most one bounded solution.

b) If f is bounded, show that $(*)$ has a bounded solution.

(37) Let $q(X)$ be a polynomial in X of degree n with real coefficients and let k be a non-zero real number. Show that the differential equation

$$\frac{dy}{dx} + ky(x) = q(x)$$

has exactly one polynomial solution of degree n.