SYLLABUS AND SAMPLE QUESTIONS FOR MSQE
(Program Code: MQEK and MQED)
2014

Syllabus for PEA (Mathematics), 2014

Algebra: Binomial Theorem, AP, GP, HP, Exponential, Logarithmic Series, Sequence, Permutations and Combinations, Theory of Polynomial Equations; (up to third degree).
Matrix Algebra: Vectors and Matrices, Matrix Operations, Determinants.
Calculus: Functions, Limits, Continuity, Differentiation of functions of one or more variables. Unconstrained Optimization, Definite and Indefinite Integrals: Integration by parts and integration by substitution, Constrained optimization of functions of not more than two variables.
Elementary Statistics: Elementary probability theory, measures of central tendency, dispersion, correlation and regression, probability distributions, standard distributions - Binomial and Normal.

Sample questions for PEA (Mathematics), 2014

1. Let $f(x) = \frac{1-x}{1+x}$, $x \neq -1$. Then $f(f(\frac{1}{x}))$, $x \neq 0$ and $x \neq -1$, is

   (a) 1,

   (b) x,

   (c) x^2,

   (d) $\frac{1}{x}$.

2. What is the value of the following definite integral?

$$2\int_0^{\frac{\pi}{2}} e^x \cos(x)dx.$$

   (a) $e^{\frac{\pi}{2}}$.

   (b) $e^{\frac{\pi}{2}} - 1$.

   (c) 0.

(d) $e^{\frac{\pi}{2}} + 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows:

$$f(x) = |x - 1| + (x - 1).$$

Which of the following is not true for f?

(a) $f(x) = f(x')$ for all $x, x' < 1$.
(b) $f(x) = 2f(1)$ for all $x > 1$.
(c) f is not differentiable at 1.
(d) The derivative of f at $x = 2$ is 2.

4. Population of a city is 40 % male and 60 % female. Suppose also that 50 % of males and 30 % of females in the city smoke. The probability that a smoker in the city is male is closest to

(a) 0.5.
(b) 0.46.
(c) 0.53.
(d) 0.7.

5. A blue and a red die are thrown simultaneously. We define three events as follows:

- Event E: the sum of the numbers on the two dice is 7.
- Event F: the number on the blue die equals 4.
- Event G: the number on the red die equals 3.

Which of the following statements is true?

(a) E and F are disjoint events.
(b) E and F are independent events.

(c) E and F are not independent events.

(d) Probability of E is more than the probability of F.

6. Let $p > 2$ be a prime number. Consider the following set containing 2×2 matrices of integers:

$$T_p = \left\{ A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} : a, b \in \{0, 1, \ldots, p-1\} \right\}.$$

A matrix $A \in T_p$ is p-special if determinant of A is not divisible by p. How many matrices in T_p are p-special?

(a) $(p-1)^2$.

(b) $2p - 1$.

(c) p^2.

(d) $p^2 - p + 1$.

7. A "good" word is any seven letter word consisting of letters from $\{A, B, C\}$ (some letters may be absent and some letter can be present more than once), with the restriction that A cannot be followed by B, B cannot be followed by C, and C cannot be followed by A. How many good words are there?

(a) 192.

(b) 128.

(c) 96.

(d) 64.

8. Let n be a positive integer and $0 < a < b < \infty$. The total number of real roots of the equation $(x-a)^{2n+1} + (x-b)^{2n+1} = 0$ is

(a) 1.

(b) 3.

(c) $2n - 1$.

(d) $2n + 1$.

9. Consider the optimization problem below:

$$\begin{aligned} \max_{x,y} \; & x + y \\ \text{subject to } & 2x + y \leq 14 \\ & -x + 2y \leq 8 \\ & 2x - y \leq 10 \\ & x, y \geq 0. \end{aligned}$$

The value of the objective function at optimal solution of this optimization problem:

(a) does not exist.

(b) is 8.

(c) is 10.

(d) is unbounded.

10. A random variable X is distributed in $[0, 1]$. Mr. Fox believes that X follows a distribution with cumulative density function (cdf) $F : [0, 1] \to [0, 1]$ and Mr. Goat believes that X follows a distribution with cdf $G : [0, 1] \to [0, 1]$. Assume F and G are differentiable, $F \neq G$ and $F(x) \leq G(x)$ for all $x \in [0, 1]$. Let $\mathbb{E}_F[X]$ and $\mathbb{E}_G[X]$ be the expected values of X for Mr. Fox and Mr. Goat respectively. Which of the following is true?

(a) $\mathbb{E}_F[X] \leq \mathbb{E}_G[X]$.

(b) $\mathbb{E}_F[X] \geq \mathbb{E}_G[X]$.

(c) $\mathbb{E}_F[X] = \mathbb{E}_G[X]$.

(d) None of the above.

11. Let $f : [0,2] \to [0,1]$ be a function defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \le \alpha \\ \frac{1}{2} & \text{if } x \in (\alpha, 2]. \end{cases}$$

where $\alpha \in (0,2)$. Suppose X is a random variable distributed in $[0,2]$ with probability density function f. What is the probability that the realized value of X is greater than 1?

(a) 1.

(b) 0.

(c) $\frac{1}{2}$.

(d) $\frac{3}{4}$.

12. The value of the expression

$$\sum_{k=1}^{100} \int_0^1 \frac{x^k}{k} dx$$

is

(a) $\frac{100}{101}$.

(b) $\frac{1}{99}$.

(c) 1.

(d) $\frac{99}{100}$.

13. Consider the following system of inequalities.

$$\begin{aligned} x_1 - x_2 &\le 3 \\ x_2 - x_3 &\le -2 \\ x_3 - x_4 &\le 10 \\ x_4 - x_2 &\le \alpha \\ x_4 - x_3 &\le -4, \end{aligned}$$

where α is a real number. A value of α for which this system has a solution is

(a) -16.
(b) -12.
(c) -10.
(d) None of the above.

14. A fair coin is tossed infinite number of times. The probability that a head turns up for the first time after even number of tosses is

(a) $\frac{1}{3}$.
(b) $\frac{1}{2}$.
(c) $\frac{2}{3}$.
(d) $\frac{3}{4}$.

15. An entrance examination has 10 "true-false" questions. A student answers all the questions randomly and his probability of choosing the correct answer is 0.5. Each correct answer fetches a score of 1 to the student, while each incorrect answer fetches a score of zero. What is the probability that the student gets the mean score?

(a) $\frac{1}{4}$.
(b) $\frac{63}{256}$.
(c) $\frac{1}{2}$.
(d) $\frac{1}{8}$.

16. For any positive integer k, let S_k denote the sum of the infinite geometric progression whose first term is $\frac{(k-1)}{k!}$ and common ratio is $\frac{1}{k}$. The value of the expression $\sum_{k=1}^{\infty} S_k$ is

(a) e.
(b) $1+e$.
(c) $2+e$.
(d) e^2.

17. Let $G(x) = \int_0^x te^t dt$ for all non-negative real number x. What is the value of $\lim_{x\to 0} \frac{1}{x} G'(x)$, where $G'(x)$ is the derivative of G at x.

    (a) 0.
    (b) 1.
    (c) e.
    (d) None of the above.

18. Let $\alpha \in (0,1)$ and $f(x) = x^\alpha + (1-x)^\alpha$ for all $x \in [0,1]$. Then the maximum value of f is

    (a) 1.
    (b) greater than 2.
    (c) in $(1,2)$.
    (d) 2.

19. Let n be a positive integer. What is the value of the expression

$$\sum_{k=1}^{n} kC(n,k),$$

    where $C(n,k)$ denotes the number of ways to choose k out of n objects?

    (a) $n2^{n-1}$.
    (b) $n2^{n-2}$.
    (c) 2^n.
    (d) $n2^n$.

20. The first term of an arithmetic progression is a and common difference is $d \in (0,1)$. Suppose the k-th term of this arithmetic progression equals the sum of the infinite geometric progression whose first term is a and common ratio is d. If $a > 2$ is a prime number, then which of the following is a possible value of d?

(a) $\frac{1}{2}$.
(b) $\frac{1}{3}$.
(c) $\frac{1}{5}$.
(d) $\frac{1}{9}$.

21. In period 1, a chicken gives birth to 2 chickens (so, there are three chickens after period 1). In period 2, each chicken born in period 1 either gives birth to 2 chickens or does not give birth to any chicken. If a chicken does not give birth to any chicken in a period, it does not give birth in any other subsequent periods. Continuing in this manner, in period $(k+1)$, a chicken born in period k either gives birth to 2 chickens or does not give birth to any chicken. This process is repeated for T periods - assume no chicken dies. After T periods, there are in total 31 chickens. The maximum and the minimum possible values of T are respectively

(a) 12 and 4.
(b) 15 and 4.
(c) 15 and 5.
(d) 12 and 5.

22. Let a and p be positive integers. Consider the following matrix

$$A = \begin{bmatrix} p & 1 & 1 \\ 0 & p & a \\ 0 & a & 2 \end{bmatrix}$$

If determinant of A is 0, then a possible value of p is

(a) 1.
(b) 2.
(c) 4.
(d) None of the above.

23. For what value of α does the equation $(x-1)(x^2-7x+\alpha)=0$ have exactly two unique roots?

   (a) 6.
   (b) 10.
   (c) 12.
   (d) None of the above.

24. What is the value of the following infinite series?

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \log_e 3^k.$$

   (a) $\log_e 2$.
   (b) $\log_e 2 \log_e 3$.
   (c) $\log_e 6$.
   (d) $\log_e 5$.

25. There are 20 persons at a party. Each person shakes hands with some of the persons at the party. Let K be the number of persons who shook hands with odd number of persons. What is a possible value of K?

   (a) 19.
   (b) 1.
   (c) 20.
   (d) All of the above.

26. Two independent random variables X and Y are uniformly distributed in the interval $[0,1]$. For a $z \in [0,1]$, we are told that probability that $\max(X,Y) \leq z$ is equal to the probability that $\min(X,Y) \leq (1-z)$. What is the value of z?

   (a) $\frac{1}{2}$.

(b) $\frac{1}{\sqrt{2}}$.

(c) any value in $[\frac{1}{2}, 1]$.

(d) None of the above.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies for all $x, y \in \mathbb{R}$

$$f(x+y)f(x-y) = \big(f(x) + f(y)\big)^2 - 4x^2 f(y).$$

Which of the following is not possible for f?

(a) $f(0) = 0$.

(b) $f(3) = 9$.

(c) $f(5) = 0$.

(d) $f(2) = 2$.

28. Consider the following function $f : \mathbb{R} \to \mathbb{Z}$, where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}$ is the set of all integers.

$$f(x) = \lceil x \rceil,$$

where $\lceil x \rceil$ is the smallest integer that is larger than x. Now, define a new function g as follows. For any $x \in \mathbb{R}$, $g(x) = |f(x)| - f(|x|)$, where $|x|$ gives the absolute value of x. What is the range of g?

(a) $\{0, 1\}$.

(b) $[-1, 1]$.

(c) $\{-1, 0, 1\}$.

(d) $\{-1, 0\}$.

29. The value of $\lim_{x \to -1} \frac{x+1}{|x+1|}$ is.

(a) 1.

(b) -1.

(c) 0.

(d) None of the above.

30. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 2$ if $x \leq 2$ and $f(x) = a^2 - 3a$ if $x > 2$, where a is a positive integer. Which of the following is true?

(a) f is continuous everywhere for some value of a.

(b) f is not continuous.

(c) f is differentiable at $x = 2$.

(d) None of the above.