Test Code : RE I / RE II (Short Answer type)
(Junior Research Fellowship in Economics)

The candidates for Junior Research Fellowship in Economics are required to take two short answer type tests - RE I (Mathematics) in the forenoon session and RE II (Economics) in the afternoon session.

Syllabus for RE I

1. Permutations and combinations.
2. Elementary set theory; Functions and relations; Matrices.
3. Convergence of sequences and series.
4. Functions of one and several variables: limits, continuity, differentiation, applications, integration of elementary functions, definite integrals, theory of quadratic equations.
5. Constrained and unconstrained optimization, convexity of sets and concavity and convexity of functions.
6. Elements of probability theory, discrete and continuous random variables, expectation and variance, joint conditional and marginal distributions, distributions of functions of random variable.

Syllabus for RE II

1. Theory of consumer behaviour; theory of production; market structure; general equilibrium and welfare economics; international trade and finance; public economics.
2. Macroeconomic theories of income determination, Rational Expectations, Phillips Curve, Neo-classical Growth Model, Inequality.
3. Game Theory: Normal and extensive forms, Nash and sub-game perfect equilibrium.
4. Statistical inference, regression analysis (including heteroscedasticity, autocorrelation and multicollinearity), least squares and maximum likelihood estimation, specification bias, endogeneity and exogeneity, instrumental variables, elementary time-series analysis.

Sample Questions RE I

1. (a) Find

$$\lim_{n\to\infty}\left(\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{n.(n+1)}\right).$$

   (b) Show that the value of e lies between 2 and 3.

(8+12)

2. (a) Find the complete set of solutions to each of the following optimization problems:

   (i) $\max_{x,y\in\Re} \left(x^2+y^2\right)$ subject to $|x|+|y|\leq 1$.

   (ii) $\min_{x,y\in\Re} \left(x^2+y^2\right)$ subject to $|x|+|y|=1$.

   (b) The set $P\triangle Q$ is defined as $P\triangle Q=(P-Q)\cup(Q-P)$. Three sets P, Q and R are given as $P=\{1,2,3,4,5,6\}$, $Q=\{1,3,5,9\}$ and $R=\{2,4,6\}$. Find the set $(P\triangle Q)\triangle R$.

(12+8)

3. (a) Find the signs separately on a, b and c such that the roots of the quadratic equation $ax^2+bx+c=0$ are of unequal magnitude and of opposite signs.

   (b) Find the rank of the following matrix

$$A=\begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix}.$$

(15+5)

4. (a) A function $f(x)$ is defined as

$$f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\leq x\leq 2 \\ x-\frac{1}{2}x^2 & \text{if } x>2. \end{cases}$$

   Show that the function $f(x)$ is continuous at $x=2$ but not differentiable at $x=1$.

(b) Find

$$\int_0^{\pi} \left| \frac{1}{2} + cos\ x \right| dx.$$

(10+10)

5. Consider the following definitions:

- If $x, y \in \Re^n$, the *line segment* joining points x and y is given by the set of points $\{z \in \Re^n : z = \theta x + (1 - \theta)\, y, \text{ for some } 0 \leq \theta \leq 1\}$.
- A set $S \subset \Re^n$ is a *convex set* if for every x, $y \in S$, the line segment joining x and y is contained in S.

Suppose A is a convex set in $\Re^n$ and $f : A \rightarrow \Re$. Prove that the following two statements are equivalent to each other.

(a) $f\left(x^2\right) \geq f\left(x^1\right)$ implies $f\left(\theta x^1 + (1 - \theta)\, x^2\right) \geq f\left(x^1\right)$ whenever $x^1, x^2 \in A$, and $0 \leq \theta \leq 1$;

(b) For every $\alpha \in \Re$, the set $S\,(\alpha) = \{x \in A : f\,(x) \geq \alpha\}$ is a convex set in $\Re^n$.

(20)

6. (a) Let $A(t)$ represent the area of the region enclosed by the curve $y = e^{-|x|}$ and the portion of x-axis between $-t$ and t. Show that $\lim\limits_{t \to \infty} A(t) = 2$.

(b) Suppose $f(x, y)$ is a differentiable function satisfying the following properties:

(i) $f(x + t, y) = f(x, y) + ty$ and $f(x, y + t) = f(x, y) + tx$ for all $x, y, t \in \Re$ and

(ii) $f(z, 0) = k$ for any $z \in \Re$ and k is an arbitrary constant.

Find $f(x, y)$. Give explanations for your answer.

(8+12)

7. Show that the number of ways in which three numbers can be selected from the set $\{1, 2, \ldots, 3n\}$ so that their sum is divisible by 3 is $3\binom{n}{3} + n^3$.

(20)

8. (a) Prove that if A and B are independent events, then so are their complementary events.

   (b) A randomly chosen group is tested for a disease. Within this group, each individual has a probability of 0.1 of having the disease. A test is performed to identify the individuals with the disease. The test has two outcomes: positive or negative. If the individual does not have the disease, the test outcome is "negative" 90 percent of the time. If the individual has the disease, the test outcome is "negative" 20 percent of the time. Individuals who test "positive" are sent to a hospital for further treatment. What is the probability that an individual sent to the hospital indeed has the disease?

(5+15)

9. Consider the following set $\mathcal{F} = \{F_0, F_1, F_2, \ldots\}$. This set consists of positive integers which satisfy the following properties: (i) $F_0 = F_1 = 1$ and (ii) $F_n = F_{n-1} + F_{n-2}$ for all positive integers $n \geq 2$. Prove that, for all positive integers n, the elements of the set $\mathcal{F}$ satisfy the following identity :

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{n+1} = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix}$$

where $|\ \ |$ denotes the determinant.

(20)

10. (a) Show that $(1^r + 2^r + \ldots + n^r)^n > n^n(n!)^r$, r being a real number.

   (b) Consider a rational number $\frac{m}{n}$ and the equation $\frac{1}{x} + \frac{1}{y} = \frac{m}{n}$. Show that this equation cannot have infinitely many positive integer solutions (x, y).

(5+15)

Sample Questions RE II

1. Consider an IS-LM model for a closed economy with proportional consumption function, autonomous money supply and government final expenditure, lump-sum direct taxes and investment inversely related to rate of interest.

   (a) Suppose the government wants to raise investment keeping output constant. What mix of monetary and fiscal policies will achieve the goal? Give explanations for your answer.

   (b) The central bank is considering two alternative policies:
   (i) holding money supply constant,
   (ii) adjusting money supply in order to hold interest rate constant.
   Which one of these two policies will stabilize output better, if all the shocks to the economy arise from exogenous changes in demand for goods and services?

(10+15)

2. A community of n individuals produces two goods. Individual i has resources ω_i which must be divided between the production of a private good and contributions to a public good. Let c_i and g_i be the resources devoted to the production of the private and the public goods respectively. The payoff for individual i is given by $f(c_i) + v(g)$ where $f(.)$, $v(.)$ are production functions of the private and the public goods respectively, $f' > 0$, $v' > 0$, $f'' < 0$, $v'' < 0$ and $\sum_{i=1}^{n} g_i = g$, that is the public good is produced by the joint contribution of all the individuals. Note that the payoff functions of all individuals are identical.

   (a) Prove that in the Nash equilibrium every individual who makes a positive contribution to the production of the public good must get the same payoff regardless of wealth.

   (b) Suppose $n = 3$. Also assume that $f'(0)$ and $v'(0)$ are arbitrarily large. Show that in the Nash equilibrium while the rich individual always contributes to the production of the public good, the poor individual may or may not contribute. **(12+13)**

3. A monopolist faces a linear inverse market demand curve with an unknown intercept. The equation is $p = a - q$. The firm knows that a is either a_2 with probability β, or it is a_1 with probability $1 - \beta$. Assume $a_2 > a_1 > 0$ and $\beta \in (0, 1)$. If it produces an amount q, it incurs a production cost of $\frac{q^2}{2}$. Before deciding on production, the firm has the option of discovering, at cost $k > 0$, the true value of a. It can always decide to produce without acquiring such information. The monopolist's aim is to make information acquisition and production decisions to maximize its ex ante expected profit.

   (a) Derive conditions on a_1, a_2, β and k such that it is profitable for the monopolist to acquire information.

   (b) Given a_1 and a_2, for what value(s) of β does the firm have the strongest incentive to acquire information?

   (15+10)

4. There are two coffee shops on a street. In each shop the cost of producing a cup of coffee is c. There are N consumers and N is large. A fraction λ of the consumers is informed about the price prevailing in each shop and chooses the one which charges the lower price; and if the shops charge the same price an informed consumer chooses a shop with equal probability. The remaining $(1 - \lambda)$ fraction, who is not informed about the prices, randomly goes to a shop with probability half-half. Finally, each consumer, whether informed or uninformed, is willing to pay a maximum price R for a cup of coffee where $R > c$.

   (a) Write down the profit (payoff) of each shop as a function of the prices chosen by the two shops.

   (b) Determine the Nash equilibrium of the game for $\lambda = 0$ and for $\lambda = 1$.

(c) Prove that for $0 < \lambda < 1$, a pure strategy Nash equilibrium does not exist.

(4+10+11)

5. Suppose the production function of the economy is of the form, $Y = (B_t K_t)^{\alpha}(A_t L_t)^{1-\alpha}$, where Y denotes output, K denotes the capital stock, and L denotes the population size. Population grows at the constant rate n that is $\frac{\dot{L}}{L} = n$. Both capital and labor augmenting technical progress rates are given by $\frac{\dot{B}}{B} = g_B$, and $\frac{\dot{A}}{A} = g_A$. Derive the law of motion for the capital labor ratio in effective units $Z = \frac{BK}{AL}$ under the assumptions of the Solow Model (that is $\dot{K} = sY - \delta K$, where $0 < s < 1$ is the savings rate, and $0 < \delta < 1$ is the depreciation rate on capital). Show that the system has a balanced growth path, i.e., a constant z solution, if and only if $g_B = 0$ or in other words, technical progress is labor augmenting.

(25)

6. Consider an economy producing two goods X and Y with two factors of production, Capital (K) and labour (L). The production function in the two sectors are given by $X = A_x L_x^{\alpha} K_x^{1-\alpha}$ and $Y = A_y L_y^{\alpha} K_y^{1-\alpha}$ where L_i, K_i are labour and capital employed in sector i, $i = X, Y$, α is a constant, $0 < \alpha < 1$ and A_x, A_y are also given constants. The economy is endowed with $\bar{L}$ units of labour and $\bar{K}$ units of capital. Perfect competition prevails in each sector and both factors are fully mobile across sectors.

   (a) Show that the production possibility frontier for this economy is linear.

   (b) Assume that $\alpha = \frac{1}{2}$, $\bar{L} = 400$, $\bar{K} = 100$, $A_x = 10$, $A_y = 15$. Also assume that all consumers have an identical utility function given by $U = \min\{X, Y\}$. Determine the equilibrium commodity prices, factor prices and output levels. (You may assume that one of the goods is the numeraire).

(c) Suppose there is another country with identical technology and preferences but a different factor endowment ratio. Would you expect the two countries to trade with each other? Give reasons for your answer. **(10+10+5)**

7. (a) Consider an exchange economy $\mathcal{E} \equiv (N, u_i, \omega_i), i \in N$. Here $N = \{1, .., n\}$ is the set of agents. There are L commodities and $u_i : \mathcal{R}^L \to \mathcal{R}$ and $\omega_i \in \mathcal{R}_{++}$ is agent i's utility function and endowment vector respectively. Assume that u_i is a twice continuously differentiable concave function. An allocation is a vector $x \equiv (x_1, x_2, .., x_n)$ where $x_i \in \mathcal{R}^L, i \in N$ such that $\sum_{i \in N} x_i = \sum_{i \in N} \omega_i$. Provide formal definitions of the following.

   (i) A Pareto-optimal allocation.

   (ii) A perfectly competitive allocation.

   (2+2)

   (b) Consider two exchange economies $\mathcal{E} \equiv (N, u_i, \omega_i), i \in N$ and $\mathcal{E}' \equiv (N', u'_j, \omega'_j), j \in N'$ defined over the same set of L commodities but such that the set of agents is disjoint, i.e. $N \cap N' = \emptyset$. Let $\mathcal{E} \cup \mathcal{E}'$ be the concatenated economy $((N \cup N'), (u_i, u'_j), \omega_i, \omega'_j), i \in N, j \in N'))$. Let $\bar{x}$ and $\bar{x}'$ be perfectly competitive allocations for $\mathcal{E}$ and $\mathcal{E}'$ respectively. Note that the concatenated vector $(\bar{x}, \bar{x}')$ is an allocation in $\mathcal{E} \cup \mathcal{E}'$.

   (i) Show by means of an example (you may choose L, N, N', $u_i, u'_j, \omega_i, \omega'_j$ suitably) that $(\bar{x}, \bar{x}')$ is not a perfectly competitive allocation in $\mathcal{E} \cup \mathcal{E}'$.

   (ii) Show that if $(\bar{x}, \bar{x}')$ is Pareto-optimal in $\mathcal{E} \cup \mathcal{E}'$, then it is a competitive allocation in $\mathcal{E} \cup \mathcal{E}'$.

   (10+11)

8. Let $\{X_t\}$ be a special moving average process of order 2 given by $X_t = \epsilon_t + \theta \epsilon_{t-2}$, where $\{\epsilon_t\}$ is a white noise process with mean zero and variance 1.

(a) State the stationarity and invertibility conditions, if any, for $\{X_t\}$. Give explanations for your answer.

(b) Find the autocorrelation function of this process when $\theta = 0.8$. Suggest another (admittedly similar) time series with the same autocorrelation function.

(c) Compute the variance of the sample mean $\frac{1}{4}\sum_{t=1}^{4} X_t$ when $\theta = 0.8$.

(8+11+6)

9. (a) Let the distribution function of an income distribution be given by $F(x) = 1 - \left(\frac{x_0}{x}\right)^{\alpha}$, $x \geq x_0$ and $\alpha > 1$. Then show that the share of income of persons earning income $\leq x$ is given by $1 - (1 - F(x))^{\frac{\alpha - 1}{\alpha}}$.

(b) Let $x = (x_1, x_2, \ldots, x_n)$ be an income profile. The Atkinson-Kolm-Sen (AKS) measure of income inequality is given by

$$I_{AKS} = 1 - \frac{x_e}{\mu},$$

where μ is the mean income and x_e is the 'equally distributed equivalent income'. The latter, that is, the 'equally distributed equivalent income' is defined as that level of income which, if given to each individual, will yield the same social welfare as given by the existing profile.

Show that if the AKS measure is a relative measure of income inequality, then the Social Welfare Function is homothetic.

(10+15)

10. Consider the standard classical linear regression model (CLRM)

$$Y = \beta X + \epsilon$$

where Y is an $n \times 1$ vector of observations on the dependent variable, X is an $n \times k$ matrix of observations on k independent variables, β is a $k \times 1$ vector of associated regression coefficients and ϵ is an $n \times 1$ vector of errors.

(a) Clearly state the assumptions of the CLRM.

(b) Derive the maximum likelihood (ML) estimator of β.

(c) Suppose that there is heteroscedasticity in the error and this is given by $V(\epsilon_i) = \sigma^2 z_i^2$, $i = 1, 2, \ldots, n$ for some known variable Z. Find the true standard error of the ML estimator of β obtained in (b) above. **(5+10+10)**

11. (a) A researcher estimates the following model based on 100 observations, for stock market returns, but thinks that there may be an econometric problem with it.

$$\underset{}{y_t} = \underset{(0.436)}{0.638} + \underset{(0.291)}{0.402x_{1t}} + \underset{(0.763)}{0.891x_{2t}}, \quad R^2 = 0.97$$

where figures in parentheses are the standard errors of the corresponding coefficient estimates.

Suggest, along with justifications, what the econometric problem might be. Also indicate how you might go about solving the perceived problem. (*The critical values of the standard normal distribution at 5 percent and 1 percent levels of significance are 1.96 and 2.57 respectively.*)

(b) (i) Consider the following regression model

$$y_i = \alpha + \beta x_i + \epsilon_i.$$

Suppose that the regressor is correlated with the error term. Would ordinary least squares (OLS) be a valid method of estimation in this case? Explain.

(ii) From a sample of 100 observations, the following data are obtained. Calculate the instrumental variable estimates of α and β.

$\sum y_i^2 = 350$, $\sum x_i y_i = 150$, $\sum z_i y_i = 100$, $\sum y_i = 100$, $\sum x_i^2 = 400$, $\sum z_i x_i = 200$, $\sum x_i = 100$, $\sum z_i^2 = 400$ and $\sum z_i = 50$.

(12+13)