Test Code : CSB (Short Answer Type) 2013

Junior Research Fellowship (*JRF) in Computer Science*

The candidates for Junior Research Fellowship in Computer and Communication Sciences will have to take two tests - Test MMA (objective type) in the forenoon session and Test CSB (short answer type) in the afternoon session. The CSB test booklet will have two groups as follows:

GROUP A

A test for all candidates in the basics of computer programming and discrete mathematics.

GROUP B

A test, divided into five sections in the following areas at M.Sc./M.E./M.Tech. level:

- Mathematics,
- Statistics,
- Physics,
- Electrical and Electronics Engineering, and
- Computer Science.

A candidate has to answer questions from **only** one of these sections in GROUP B, according to his/her choice.

Group A carries **20 marks** *and Group B carries* **80 marks**.

The syllabi and sample questions of the CSB test are given overleaf.

Syllabi

GROUP A:

Logical reasoning; elementary combinatorics including inclusion-exclusion principle and pigeonhole principle; basics of programming (using pseudo-code); elementary data structures including array, stack and queue; Boolean algebra.

GROUP B:

Mathematics:

Graph theory and combinatorics: Graphs, paths and cycles, trees, Eulerian graphs, Hamiltonian graphs, chromatic numbers, planar graphs, digraphs and tournaments.

Linear algebra: Vector spaces, basis and dimension, linear transformations, matrices, rank, inverse, determinant, systems of linear equations, eigenvalues and eigenvectors, orthogonality.

Abstract algebra: Groups, subgroups, cosets, Lagrange's theorem, normal subgroups and quotient groups, permutation groups, rings, subrings, ideals, integral domains, fields, characteristic of a field, polynomial rings, unique factorization domains, field extensions, finite fields.

Elementary number theory: Elementary number theory, divisibility, congruences, primality.

Calculus and real analysis: Real numbers, convergence of sequences and series, limits, continuity, uniform continuity of functions, differentiability of functions of one or more variables, indefinite integral, fundamental theorem of integral calculus, Riemann integration, improper integrals, sequences and series of functions, uniform convergence.

Statistics:

Probability theory and distributions: Basic probability theory, discrete and continuous distributions, moments, characteristic functions, Markov chains.

Estimation and inference: Sufficient statistics, unbiased estimation, maximum likelihood estimation, consistency of estimates, most powerful and uniformly most powerful tests, unbiased tests and uniformly most powerful unbiased tests, confidence sets, Bayesian methods.

Linear models: Gauss-Markov set up and least squares theory, multiple linear regression, one and two way analysis of variance.

Multivariate analysis: Multivariate normal distribution, principal component analysis, multiple and canonical correlations, discriminant analysis.

Physics:

Classical mechanics: Lagrangian and Hamiltonian formulation of Newtonian mechanics, symmetries and conservation laws, motion in central field of force, small oscillations and normal modes, wave motion, special theory of relativity.

Electrodynamics: Electric and magnetic phenomena in dielectrics, Maxwell's equations, conservation laws, gauge invariance, electromagnetic waves.

Thermodynamics and statistical physics: Laws of thermodynamics, statistical basis of thermodynamics, thermodynamic potentials and Maxwell's relations, density matrix formulation, ensembles, partition function, classical and quantum statistics, blackbody radiation and Planck's distribution law.

Quantum physics: Basics of quantum mechanics, two-body problem, central potential, angular momentum algebra, quantum particles in electromagnetic fields, matrix mechanics.

Atomic and nuclear physics: Energy spectrum of an electron in hydrogen atom, electron spin, relativistic correction, selection rules, Zeeman effect, Stark effect, basic nuclear properties, nuclear force, nuclear models, radioactive decays.

Electronics: Basics of semiconductor physics, transistors, amplifiers including feedback, oscillators, operational amplifiers, RLC circuits, digital integrated circuits, A/D and D/A converters.

Electrical and Electronics Engineering:

Digital circuits and systems: Gates and logic circuits, combinational and sequential circuits, A/D and D/A converters.

Circuit theory: Kirchoff's laws, theorem of superposition, Thevenin's theorem, Norton's theorem, A.C. circuits.

Linear electronic devices and circuits: Transistors, amplifiers including feedback amplifiers, oscillators, operational amplifiers.

Digital communication: Information and coding theory, channel capacity, digital transmission.

Digital signal processing: Sampling, linear time invariant systems, Z-transform, discrete-time Fourier and discrete Fourier transforms.

Electrical machines: DC motors and generators, transformers, induction motors.

Computer Science:

Discrete mathematics: Functions and relations, recurrence relations, generating functions, graph theory - paths and cycles, trees, digraphs, planar graphs, Eulerian graphs, Hamiltonian paths.

Programming languages: Fundamental concepts - abstract data types, procedure call and parameter passing, C language.

Data structures and algorithms: Linked list, queue, binary tree, heap, AVL tree, Order notation, sorting, selection, searching, hashing, graph algorithms.

Computer organization and architecture: Number representation, computer arithmetic, memory organization, I/O organization, pipelining.

Operating systems: Process concept and management, scheduling, process synchronization, concurrency control, critical section problems, deadlocks, memory management, file systems.

Formal languages and automata theory: Finite automata and regular expressions, context-free grammars, Turing machines, elements of undecidability.

Database systems: Relational model, relational algebra, relational calculus, functional dependency, normalization (including multi-valued dependencies), query processing.

Computer networks: Layered network structures, network security, LAN technology - bus/tree, ring, star; data communications - data encoding, flow control, error detection/correction, TCP/IP networking.

Sample Questions

Note that all questions in the sample set are not of same marks and same level of difficulty.

GROUP A

A1. Show how two complex numbers $(a+ib)$ and $(c+id)$ may be multiplied using only three multiplications of real numbers, where $i = \sqrt{-1}$. You may use any number of additions and subtractions.

A2. Consider the pseudo-code given below.
Input: Integers b and c.

1. $a_0 \leftarrow \max(b,c)$, $a_1 \leftarrow \min(b,c)$.
2. $i \leftarrow 1$.
3. Divide a_{i-1} by a_i . Let q_i be the quotient and r_i be the remainder.
4. If $r_i = 0$ then go to Step 8.
5. $a_{i+1} \leftarrow a_{i-1} - q_i * a_i$.
6. $i \leftarrow i + 1$.
7. Go to Step 3.
8. Print a_i .

What is the output of the above algorithm when $b = 28$ and $c = 20$? What is the mathematical relation between the output a_i and the two inputs b and c?

A3. Consider the sequence $a_n = a_{n-1}\, a_{n-2} + n$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 1$. Is a_{2011} odd? Justify your answer.

A4. Given an array of n integers, write pseudo-code for reversing the contents of the array without using another array. For example, for the array
10 15 3 30 3
the output should be
3 30 3 15 10.
You may use one temporary variable.

A5. The integers 1, 2, 3, 4 and 5 are to be inserted into an empty stack using the following sequence of PUSH() operations:

PUSH(1) PUSH(2) PUSH(3) PUSH(4) PUSH(5)

Assume that POP() removes an element from the stack and outputs the same. Which of the following output sequences can be generated by inserting suitable POP() operations into the above sequence of PUSH() operations? Justify your answer.

(a) 5 4 3 2 1

(b) 1 2 3 4 5

(c) 3 2 1 4 5

(d) 5 4 1 2 3.

A6. Derive an expression for the maximum number of regions that can be formed within a circle by drawing n chords.

A7. Given $A = \{1, 2, 3, \ldots, 70\}$, show that for any six elements a_1, a_2, a_3, a_4, a_5 and a_6 belonging to A, there exists one pair a_i and a_j for which $|a_i - a_j| \leq 14$ $(i \neq j)$.

A8. Consider the multiplication of two 2-bit integers a_1a_0 and b_1b_0 to get a 4-bit output $c_3c_2c_1c_0$. Assuming that the rightmost bit is the least significant bit, derive Boolean functions for the output bits c_0 and c_3.

A9. Calculate how many integers in the set $\{1, 2, 3, \ldots, 1000\}$ are not divisible by 2, 5, or 11.

A10. Let M be a 4-digit positive integer. Let N be the 4-digit integer obtained by writing the digits of M in reverse order. If $N = 4M$, then find M. Justify your answer.

A11. Consider all the permutations of the digits $1, 2, \ldots, 9$. Find the number of permutations each of which satisfies *all* of the following:

- the sum of the digits lying between 1 and 2 (including 1 and 2) is 12,
- the sum of the digits lying between 2 and 3 (including 2 and 3) is 23,

- the sum of the digits lying between 3 and 4 (including 3 and 4) is 34, and
- the sum of the digits lying between 4 and 5 (including 4 and 5) is 45.

A12. A Boolean operation denoted by '$\implies$', is defined as follows:

A	B	$A \implies B$
T	T	T
F	T	T
T	F	F
F	F	T

Show that $A \vee B$ can be expressed in terms of '$\implies$' alone, without using any 0 or 1 input.

A13. A Boolean function f_1^D is said to be the dual of another Boolean function f_1 if f_1^D is obtained from f_1 by interchanging the operations '+' and '.', and the constants '0' and '1'. For example, if $f_1(a,b,c) = (a+b).(b+c)$ then $f_1^D(a,b,c) = a.b + b.c$.

A Boolean function f is self-dual if $f_1 = f_1^D$. Given $f_1(a,b,c) = a\bar{b} + \bar{b}c + x$, find the Boolean expression x such that f_1 is self-dual.

A14. (a) There are n students in a class. The students have formed k committees. Each committee consists of more than half of the students. Show that there is at least one student who is a member of more than half of the committees.

(b) Let $D = \{d_1, d_2, \ldots, d_k\}$ be the set of distinct divisors of a positive integer n (D includes 1 and n). Show that

$$\sum_{i=1}^{k} \sin^{-1} \sqrt{\log_n d_i} = \frac{\pi}{4} \times k.$$

HINT: $\sin^{-1} x + \sin^{-1} \sqrt{1-x^2} = \frac{\pi}{2}$

A15. (a) Give a strategy to sort four distinct integers a, b, c, d in increasing order that minimizes the number of pairwise comparisons needed to sort any permutation of a, b, c, d.

(b) An $n \times n$ matrix is said to be *tridiagonal* if its entries a_{ij} are zero except when $|i - j| \leq 1$ for $1 \leq i, j \leq n$. Note that only $3n - 2$ entries of a tridiagonal matrix are non-zero. Thus, an array L of size $3n - 2$ is sufficient to store a tridiagonal matrix. Given i, j, write pseudo-code to

i. store a_{ij} in L, and

ii. get the value of a_{ij} stored in L.

A16. (a) Consider an $m \times n$ integer grid. A *path* from the lower left corner at $(0, 0)$ to the grid point (m, n) can use three kinds of steps, namely (i) $(p, q) \rightarrow (p + 1, q)$ (horizontal), (ii) $(p, q) \rightarrow (p, q + 1)$ (vertical), or (iii) $(p, q) \rightarrow (p + 1, q + 1)$ (diagonal). Derive an expression for $D_{m,n}$, the number of such distinct paths.

(b) The numbers $1, 2, \ldots, 10$ are circularly arranged. Show that there are always three adjacent numbers whose sum is at least 17, irrespective of the arrangement.

A17. (a) Consider six distinct points in a plane. Let m and M denote respectively the minimum and the maximum distance between any pair of points. Show that $M/m \geq \sqrt{3}$.

(b) Consider the following intervals on the real line:

$$A_1 = (13.3, 18.3) \qquad A_3 = (8.3, 23.3) - A_1 \cup A_2$$
$$A_2 = (10.8, 20.8) - A_1 \quad A_4 = (5.8, 25.8) - A_1 \cup A_2 \cup A_3$$

where $(a, b) = \{x : a < x < b\}$.

Write pseudo-code that finds the interval to which a given input $x \in (5.8, 25.8)$ belongs, i.e., your pseudo-code should calculate $i \in \{1, 2, 3, 4\}$ such that $x \in A_i$. Your method should not use any comparison operation.

A18. (a) A group of 15 boys plucked a total of 100 apples. Prove that two of those boys plucked the same number of apples.

(b) How many 0's are there at the end of 50! ?

A19. Professor Hijibiji has defined the following Boolean algebra $\mathcal{B} = (B, +, *)$, where

- $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$, i.e., the set of all eight factors of 30;

- the two binary operators '+' and '$*$' respectively denote the LCM (least common multiple) and GCD (greatest common divisor) of two integer operands.

(a) Show that the two operations of $\mathcal{B}$ satisfy (i) associativity, (ii) commutativity, and (iii) distributivity.

(b) Which are the identity elements for $\mathcal{B}$?

(c) Define the complementation operation $\overline{a}$ for all $a \in B$ such that $\overline{\overline{a}} = a$.

A20. Given an array $A = \{a_1, a_2, \ldots, a_n\}$ of unsorted distinct integers, write a program in *pseudo-code* for the following problem: given an integer u, arrange the elements of the array A such that all the elements in A which are less than or equal to u are at the beginning of the array, and the elements which are greater than u are at the end of the array. You may use at most 5 extra variables apart from the array A.

GROUP B

(i) MATHEMATICS

M1. (a) Show that the sequence given by $x_n = \int_0^n \frac{\sin x}{x} dx, n \geq 1$ is Cauchy.

(b) Define the function on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \frac{\sin x \sin y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Is f continuous at $(0,0)$? Justify your answer.

M2. (a) Let f be a real-valued continuous function on $[0,1]$ such that $\int_0^1 f^4(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [0,1]$.

(b) Let f be a real-valued continuously differentiable function with $f(0) = 1$. Find $\lim_{x \to 0} \frac{1}{x^2} \int_{-x^2}^{x^2} f(t)dt$.

(c) Let $D \subset [0,1] \times [0,1]$ be defined as $D = \{(x,y) : xy \geq 3/4\}$. Compute $\int\int_D \theta^2 (xy)^{\theta-1} dxdy$, where θ is a constant.

M3. (a) Let $f : [0,1] \to [0,1]$ be such that

$$f(x) = nx - [nx]; \quad \frac{1}{n} < x \leq \frac{1}{n-1}; \quad n = 2,3,4,\ldots; \ x \neq 0$$

and $f(0) = 0$. Show that $\int_0^1 f(x)dx$ exists and find its value.
Note: $[y]$ = Largest integer $\leq y$; $y \in \mathbb{R}$.

(b) Let $f : [0,1] \to (0,\infty)$ be continuous. Let

$$a_n = \left(\int_0^1 (f(x))^n dx \right)^{\frac{1}{n}}; \quad n = 1,2,3,\ldots.$$

Find $\lim_{n\to\infty} a_n$.

M4. (a) Suppose f is a continuous real valued function on $[0,1]$. Show that

$$\int_0^1 x f(x)\, dx = \frac{1}{2} f(\xi)$$

for some $\xi \in [0,1]$.

(b) For every $x \geq 0$, prove that

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x + \theta(x)}},$$

for a unique $\theta(x), 0 < \theta(x) < 1$. Also prove that,
(i) $\frac{1}{4} \leq \theta(x) \leq \frac{1}{2}$, and
(ii) $\lim_{x \to 0} \theta(x) = \frac{1}{4}$ and $\lim_{x \to \infty} \theta(x) = \frac{1}{2}$.

M5. (a) If p is a prime ($p \neq 2$) and $p \mid (m^p + n^p)$, (i.e., p divides $(m^p + n^p)$) then show that $p^2 \mid (m^p + n^p)$.

(b) Let f be a continuous function on $[0, 1]$. Suppose for each integer $n \geq 1$,

$$\int_0^1 f^3(x) x^n dx = 0, \quad \text{where} \quad f^3(x) = (f(x))^3.$$

Show that $\int_0^1 f^4(x) dx = 0$. Hence, or otherwise, show that $f \equiv 0$.

M6. Let G be a finite group and f be an automorphism of G such that $f(f(x)) = x$ for all $x \in G$. Also, the only fixed point of f is e, i.e., if $g \in G$ and $f(g) = g$, then $g = e$.

(a) Show that the map $g \mapsto g^{-1} f(g)$ is a bijection.

(b) Hence or otherwise, show that $f(g) = g^{-1}$ for all $g \in G$.

(c) Hence or otherwise, show that G is abelian.

M7. Let S_n denote the group of permutations of $\{1, 2, \cdots n\}$.

(a) Give an example of an element $\alpha \in S_7$ of order 6.

(b) Find the number of elements in S_7 of order 5. Do such elements belong to the alternating group A_7?

(c) Suppose $(G, \cdot)$ is a finite group of order n with identity e.

i. Show that if n is odd then there is no element $x \neq e$ such that $x^2 = e$. Hence, or otherwise, show that every element $x \in G$ can be expressed as $x = y^2$ for some $y \in G$.

ii. Conversely, if every element $x \in G$ can be expressed as $x = y^2$ for some $y \in G$, then show that n is odd.

M8. Let $R = (S, +, \cdot, 0, 1)$ be a commutative ring and n be a positive integer such that $n = 2^k$ for some positive integer k.

(a) Show that for every $a \in S$

$$\sum_{i=0}^{n-1} a^i = \prod_{i=0}^{k-1}\left(1 + a^{2^i}\right).$$

(b) Let $m = w^{\frac{n}{2}} + 1$ where $w \in S$, $w \neq 0$. Show that for $1 \leq p < n$,

$$\sum_{i=0}^{n-1} w^{ip} \equiv 0 \mod m.$$

M9. (a) Show that there is a basis consisting of only symmetric and skew-symmetric matrices for the vector space of all $n \times n$ real matrices over $\mathbb{R}$.

(b) Does there exist a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that (0,1,1), (1,1,0), (1,2,1) are mapped to (1,0,0), (0,1,0), (0,1,1) respectively? Justify your answer.

M10. (a) Let A be a square matrix such that $A^2 = A$. Show that all eigen values of A are 0 or 1.

(b) Let A be a symmetric matrix whose eigenvalues are 0 or 1. Show that $A^2 = A$.

(c) Suppose A is an $n \times n$ matrix such that $A^2 = A$. If $rank(A) = r$ and $rank(I - A) = s$, show that $r + s = n$.

(d) Show that given any idempotent matrix A, every vector $x \in \mathbb{R}^n$ can be uniquely written as $x = y + z$, where $Ay = y$ and $Az = 0$.

(e) Suppose A and B are idempotent matrices satisfying $A+B = I_n$, where I_n is the identity matrix of order n. Show that $AB = BA = 0$.

M11. (a) Show that in a connected graph, any two longest paths have at least one vertex in common.

(b) Construct a cubic graph with $2n$ vertices having no triangles. (A graph is cubic if every vertex has degree three.)

(c) Let G be a graph on 9 vertices. Show that either G has a triangle or $\overline{G}$ contains K_4. (Here $\overline{G}$ is the complement of the graph G, and K_4 is the complete graph with 4 vertices). [Hint: any vertex has degree at least 4 in either G or $\overline{G}$.]

(d) Let $S = \{0, 1, 2, 3\}$ and let a_k be the number of strings of length k over S having an even number of zeros. Find a recurrence relation for a_k and then solve for a_k.

M12. Let T be a tree with n vertices. For vertices u, v of T, define $d(u, v)$ to be the number of edges in a path from u to v. Let $W(T)$ be the sum of $d(u, v)$ over all $\binom{n}{2}$ pairs of vertices $\{u, v\}$.

(a) Suppose the tree T is a path on n vertices. Show that

$$W(T) = \frac{1}{2}\sum_{k=1}^{n-1}(k^2 + k).$$

(b) Now, suppose T is a star graph on n vertices. Show that $W(T) = (n-1)^2$. (N.B. The edge set of the star graph is equal to $\{(u_1, u_i) : 2 \leq i \leq n\}$.)

(c) Hence or otherwise show that for any tree T,

$$(n-1)^2 \leq W(T) \leq \frac{1}{2}\sum_{k=1}^{n-1}(k^2 + k).$$

M13. (a) Let T be a tree with n vertices ($n \geq 3$). For any positive integer i, let p_i denote the number of vertices of degree i. Prove that

$$p_1 - p_3 - 2p_4 - \cdots - (n-3)p_{n-1} = 2.$$

(b) Show that 2^n does not divide $n!$ for any $n \geq 1$.

M14. (a) Show that, given $2^n + 1$ points with integer coordinates in $\mathbb{R}^n$, there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.

(b) Find the number of functions from the set $\{1, 2, 3, 4, 5\}$ onto the set $\{1, 2, 3\}$.

M15. (a) A set S contains integers 1 and 2, and all integers of the form $3x + y$ where x and y are distinct elements of S. What is S ? Justify your answer.

(b) Let $\phi(n)$ denote the number of positive integers m relatively prime to $n; m < n$. Let $n = pq$ where p and q are prime numbers. Then show that $\phi(n) = (p-1)(q-1) = pq(1 - \frac{1}{q})(1 - \frac{1}{p})$.

M16. Consider the $n \times n$ matrix $A = ((a_{ij}))$ with $a_{ij} = 1$ for $i < j$ and $a_{ij} = 0$ for $i \geq j$. Let

$$V = \{f(A) : f \text{ is a polynomial with real coefficients}\}.$$

Note that V is a vector space with usual operations. Find the dimension of V, when (a) $n = 3$, and (b) $n = 4$. Justify your answer.

M17. Consider the following system of equations over a field $\mathbf{F}$.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{F}$. State the conditions for which the above system of equations has (a) no solution, (b) a unique solution and (c) more than one solution.

Next consider the following equations over the field $\mathbf{Z}_p$, the field of integers modulo p, where p is a prime.

$$\begin{aligned} 7x - y &= 1 \\ 11x + 7y &= 3 \end{aligned}$$

For what value of p does this system of equations have (a) no solution, (b) a unique solution and (c) more than one solution? Find the solution in the case of (b).

M18. In each of the following four problems, two sets S and T are given. You need to define a continuous and onto function $f : S \to T$ in each of the cases. You need to provide mathematical reasons if you cannot define such an f.

(a) $S = (0, 1)$, $T = (0, 1]$.

(b) $S = (0, 1]$, $T = (0, 1)$.

(c) $S = (0, 1) \cup (2, 3)$, $T = (0, 1)$.

(d) $S = (0, 1)$, $T = (0, 1) \cup (2, 3)$.

M19. You are given 49 balls of colour red, black and white. It is known that, for any 5 balls of the same colour, there exist at least two among them possessing the same weight. The 49 balls are distributed in two boxes. Prove that there are at least 3 balls which lie in the same box possessing the same colour and having the same weight.

M20. Consider a sequence $\{a_n\}$ such that $0 < a_1 < a_2$ and $a_i = \sqrt{a_{i-1}a_{i-2}}$ for $i \geq 3$.

(a) Show that $\{a_{2n-1} : n = 1, 2, \ldots\}$ is an increasing sequence and $\{a_{2n} : n = 1, 2, \ldots\}$ is a decreasing sequence.

(b) Show that $\lim_{n\to\infty} a_n$ exists.

(c) Find $\lim_{n\to\infty} a_n$.

M21. (a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} z^{n^4}$.

(b) Test whether the series $\sum_{n=1}^{\infty} \frac{e^{inx}}{n^2}$ is uniformly convergent on $\mathbb{R}$.

(c) Consider all triangles on the plane with fixed perimeter $c \in \mathbb{R}$. Find the triangle whose area is maximum.

M22. If every vertex of a graph $G = (V, E)$ has degree at least $\frac{|V|}{2}$, then prove that there is a simple cycle containing all the vertices. Note that, a simple cycle with a vertex set $U \subseteq V$ is a cycle where each vertex of U appears only once if one traverses along the cycle.

M23. (a) Consider three non-null matrices (not necessarily square) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ that satisfy $\mathbf{ACC^t} = \mathbf{BCC^t}$, where $\mathbf{C^t}$ denotes the transpose of $\mathbf{C}$. Is it necessarily true that

i. $\mathbf{A} = \mathbf{B}$?

ii. $\mathbf{AC} = \mathbf{BC}$?

Justify your answers.

(b) Consider a real matrix $\mathbf{C}_{n\times p}$, whose columns are linearly independent and all elements in the first column are 1. Define $\mathbf{D} = \mathbf{I} - \mathbf{C}(\mathbf{C^t}\mathbf{C})^{-1}\mathbf{C^t}$, where $\mathbf{I}$ is the identity matrix of order n and $\mathbf{C^t}$ denotes the transpose of $\mathbf{C}$.

i. Show that $\mathbf{D1} = \mathbf{0}$, where $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{t}}$.

ii. Find the rank of $\mathbf{D}$.

M24. (a) Consider a graph with 8 vertices. If the degrees of seven of the vertices are $1, 2, 3, 4, 5, 6$ and 7, find the degree of the eighth vertex. Argue whether the graph is planar. Also find its chromatic number.

(b) Let $\mathbf{S}$ be a subset of $\{10, 11, 12 \ldots, 98, 99\}$ containing 10 elements. Show that there will always exist two disjoint subsets $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{S}$ such that the sum of the elements of $\mathbf{A}$ is the same as that of $\mathbf{B}$.

M25. (a) Determine the product of all distinct positive integer divisors of 630^4.

(b) Let $p_1 < p_2 < \ldots < p_{31}$ be prime numbers such that 30 divides $p_1^4 + p_2^4 + \cdots + p_{31}^4$. Prove that $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$.

(c) Find all primes p and q such that $p + q = (p - q)^3$. Justify your answer.

M26. (a) Show that each of the equations $\sin(\cos x) = x$ and $\cos(\sin y) = y$ has exactly one root in $[0, \pi/2]$. If x_1 and x_2 are the roots of these two equations respectively, then show that $x_1 < x_2$.

(b) Let f be a real-valued continuous function on $\mathbb{R}$ satisfying the inequality

$$f(x) \leq \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy, \quad \forall x \in \mathbb{R},\ \forall h > 0.$$

Prove that for any bounded closed interval, the maximum of f on that interval is attained at one of its end points.

(c) Define $f(x) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt$ for $x > 0$. Show that $0 < f(x) < 1/x$ and $f(x)$ is monotonically decreasing for $x > 0$.

M27. (a) For $n \in \mathbb{N}$, let the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by

$$0 < b_1 < a_1, \quad a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}, \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

Show that both the sequences are monotone and they have the same limit.

(b) Consider a polynomial $f_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ with integer coefficients. If $f_n(b_1) = f_n(b_2) = f_n(b_3) = f_n(b_4) = f_n(b_5) = 19$ for five distinct integers b_1, b_2, b_3, b_4 and b_5, then how many different integer solutions exist for $f_n(x) = 23$? Justify your answer.

M28. (a) Show that $f(x) = e^{|x|} - x^5 - x - 2$ has at least two real roots, where e is the base of natural logarithms.

(b) Let $\sum a_n$ be a convergent series such that $a_n \geq 0$ for all n. Show that $\sum \sqrt{a_n}/n^p$ converges for every $p > \frac{1}{2}$.

(ii) STATISTICS

S1. (a) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables satisfying $X_{n+1} = X_n + Z_n$ (addition is modulo 5), where $\{Z_n\}_{n\geq 1}$ is a sequence of independent and identically distributed random variables with common distribution
$P(Z_n = 0) = 1/2$, $P(Z_n = -1) = P(Z_n = +1) = 1/4$.
Assume that X_1 is a constant belonging to $\{0, 1, 2, 3, 4\}$. What happens to the distribution of X_n as $n \to \infty$?

(b) Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with a common uniform distribution on $\{1, 2, \ldots, m\}$. Define a sequence of random variables $\{X_n\}_{n\geq 1}$ as $X_{n+1} = MAX\{X_n, Y_n\}$ where X_1 is a constant belonging to $\{1, 2, \ldots, m\}$. Show that $\{X_n\}_{n\geq 1}$ is a Markov chain and classify its states.

S2. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the gamma distribution with density function

$$f(x, \theta) = \frac{\theta^k}{\Gamma(k)} e^{-\theta x} x^{k-1}, \quad 0 < x < \infty,$$

where $\theta > 0$ is unknown and $k > 0$ is known. Find a minimum variance unbiased estimator for $\frac{1}{\theta}$.

S3. Let $0 < p < 1$ and $b > 0$. Toss a coin once where the probability of occurrence of head is p. If head appears, then n independent and identically distributed observations are generated from Uniform $(0, b)$ distribution. If the outcome is tail, then n independent and identically distributed observations are generated from Uniform $(2b, 3b)$ distribution. Suppose you are given these n observations $X_1, \ldots, X_n$, but not the outcome of the toss. Find the maximum likelihood estimator of b based on $X_1, \ldots, X_n$. What happens to the estimator as n goes to ∞?

S4. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common density function f. Define the random variable N as

$$N = n, \text{ if } X_1 \geq X_2 \geq \cdots \geq X_{n-1} < X_n; \text{ for } n = 2, 3, 4, \ldots.$$

Find $Prob(N = n)$. Find the mean and variance of N.

S5. (a) Suppose $X_1, \dots, X_n$ $(n > 2)$ are independent and identically distributed observations from a normal population with mean θ and variance 1, $\theta \in \mathbb{R}$. Let $S_n = X_1 + \cdots + X_n$. Consider the conditional expectation of $X_1 X_2$ given S_n. Decide, with adequate reasons, if this conditional expectation depends on θ. Find an expression for the conditional expectation.

(b) Suppose $X_1, \dots, X_n$ $(n > 2)$ are independent observations from a Poisson population with mean $\theta, \theta > 0$. Suppose we are interested in estimating $\psi(\theta) = P_\theta(X_1 = 0) = e^{-\theta}$. Let $S_n = X_1 + \cdots + X_n$. As $\mathrm{E}(S_n/n) = \theta$, an estimator of $\psi(\theta)$, obtained by the method of moments, is given by $T_n = \exp(-S_n/n)$. Find the mean squared error of T_n. Also, find the limit of the mean squared error as n tends to infinity.

S6. (a) Let X and Y be two random variables such that

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N(\mu, \Sigma).$$

Find a formula for $\phi(t, r) = E(X^t Y^r)$, where t and r are real numbers, and E denotes the expectation.

(b) Consider the linear model $y_{n\times1} = A_{n\times p}\beta_{p\times1} + \epsilon_{n\times1}$ and the usual Gauss-Markov set up where $E(\epsilon) = 0$ and $D(\epsilon) = \sigma^2 I_{n\times n}$, E denotes the *expectation* and D denotes the covariance matrix. Assume that A has full rank. Show that $Var\left(\beta_1^{LS}\right) = (\alpha - \Gamma^T B^{-1}\Gamma)^{-1}\sigma^2$ where

$$A^T A = \begin{bmatrix} \alpha_{1\times1} & \Gamma^T \\ \Gamma & B_{\overline{p-1}\times\overline{p-1}} \end{bmatrix}$$

Γ is a $\overline{p-1} \times 1$ matrix and β_1^{LS} = the least square estimate of β_1, the first component of the vector β, Var denotes the variance, and A^T denotes the transpose of the matrix A.

S7. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two independent and identically distributed multivariate random vectors with mean $\mathbf{0}$ and covariance matrix $\sigma^2 \mathbf{I}_n$, where $\sigma^2 > 0$ and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

(a) Show that $\mathbf{X}^T\mathbf{Y}/(\|\mathbf{X}\| \cdot \|\mathbf{Y}\|)$ and $V = \sum(X_i^2 + Y_i^2)$ are independent.
(Here, $\|(a_1, \dots, a_n)\| = \sqrt{a_1^2 + \cdots + a_n^2}$).

(b) Obtain the probability density function of $\left(\sum_{i=1}^{n} X_i^2 / \sum_{i=1}^{n} Y_i^2\right)$.

S8. Let $X_1, X_2, \ldots, X_n$ be independent random variables. Let $E(X_j) = j\theta$ and $V(X_j) = j^3\sigma^2$, $j = 1, 2, \ldots, n$, $-\infty < \theta < \infty$ and $\sigma^2 > 0$. Here $E(X)$ denotes the expectation and $V(X)$ denotes the variance of the random variable X. It is assumed that θ and σ^2 are unknown.

(a) Find the best linear unbiased estimate for θ.

(b) Find the uniformly minimum variance unbiased estimate for θ under the assumption that X_i's are normally distributed; $1 \leq i \leq n$.

S9. Let (X, Y) follow a bivariate normal distribution. Let *mean* of $X =$ *mean* of $Y = 0$, variance of $X =$ variance of $Y = 1$, and the correlation coefficient between X and Y be ρ. Find the correlation coefficient between X^3 and Y^3.

S10. Let the probability density function of a random variable X be $f(x)$, $-\infty < x < \infty$, and we have two hypotheses

$H_0 : f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ and

$H_1 : f(x) = (1/2) \exp(-|x|)$.

In connection with testing H_0 against H_1, derive the most powerful test at level $\alpha = 0.05$.

S11. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed observations with a common exponential distribution with mean μ. Show that there is no uniformly most powerful test for testing $H_0 : \mu = 1$ against $H_A : \mu \neq 1$ at a given level $0 < \alpha < 1$, but there exists a uniformly most powerful unbiased test. Derive that test.

S12. (a) An unbiased die is rolled once. Let the score be $N \in \{1, 2, \ldots, 6\}$. The die is then rolled n times. Let X be the maximum of these n scores. Find the probability of the event $(X = 6)$.

(b) The unit interval (0,1) is divided into two sub-intervals by picking a point at random from the interval. Let Y and Z be the lengths of the longer and shorter sub-intervals, respectively. Find the distribution of Z and show that $\frac{Y}{Z}$ does not have a finite expectation.

S13. Let X_1, X_2, X_3 be independent and identically distributed observations with a common double exponential distribution with density

$$f(x, \theta) = \frac{1}{2} \exp(-|x - \theta|), \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Suppose that the observations are all distinct.

(a) Find the maximum likelihood estimator of θ. Give a complete argument for your answer.

(b) Suppose it is known that $-10 \leq \theta \leq 10$. Find the maximum likelihood estimator of θ. Justify your answer.

S14. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ are two independent multivariate normal random vectors with $E(\mathbf{X}) = E(\mathbf{Y}) = \mathbf{0}$, $D(\mathbf{X}) = A = ((a_{ij}))$ and $D(\mathbf{Y}) = B = ((b_{ij}))$, where $D(.)$ denotes the dispersion matrix.

(a) Using $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, construct another set of random variables $\mathbf{Z} = (Z_1, \ldots, Z_n)$ such that $E(\mathbf{Z}) = \mathbf{0}$ and $Cov(Z_i, Z_j) = c_{ij} = a_{ij}b_{ij}, 1 \leq i, j \leq n$, so that $D(\mathbf{Z}) = C = ((c_{ij}))$.

(b) Using (a) or otherwise show that if $a_{ii}{=}b_{ii}{=}1$ for $1 \leq i \leq n$, then $max(\underline{\lambda}(A), \underline{\lambda}(B)) \leq \underline{\lambda}(C) \leq \overline{\lambda}(C) \leq min(\overline{\lambda}(A), \overline{\lambda}(B))$, where $\underline{\lambda}$ and $\overline{\lambda}$ denote respectively the smallest and largest eigen values of a matrix.

S15. (a) Let $(i, y_i), 1 \leq i \leq 5$, be 5 data points as shown in the table below,

i	1	2	3	4	5
y_i	2	3	3	4	4

where $y_1, \ldots, y_5$ are independently normally distributed with $E(y_i) = a + bi$ and $Var(y_i) = \sigma^2 > 0$, $1 \leq i \leq 5$, where a, b are two constants. Find the equation of the least squares line. Also, find an unbiased estimate of σ^2.

(b) Consider the following balanced two-way ANOVA model without interaction:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} \text{ for } i = 1, 2 \text{ and } j = 1, 2,$$
$$\epsilon_{ij} \overset{i.i.d.}{\sim} \mathrm{N}(0, \sigma^2) \text{ for } i = 1, 2 \text{ and } j = 1, 2.$$

Consider a linear parametric function

$$\psi := c_1\alpha_1 + c_2\alpha_2 + d_1\beta_1 + d_2\beta_2.$$

Find a necessary and sufficient condition on the c_i's and the d_j's such that ψ is estimable.

S16. Let $A = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ be the space obtained by tossing a coin three times. Let $f : A \to (0, \infty)$ and $x_1 \in A$. For any $x_i \in A$, x_{i+1} is found in the following way.
Toss a fair coin three times and let the outcome be z.
If $f(z) \geq f(x_i)$ then $x_{i+1} = z$, otherwise $x_{i+1} = x_i$.

What can you say about $\lim_{i\to\infty} f(x_i)$? Justify your answer.

S17. Let there be two classes C_1 and C_2. Let the density function for class C_i be $p_i(x)$ for $i = 1, 2$ where $p_i(x) = ie^{-ix}$; $x > 0$, $i = 1, 2$. Let the prior probability for C_1 be 0.4 and the prior probability for C_2 be 0.6. Find the decision rule for classification of an observation, which provides the minimum probability of misclassification and find its value for that decision rule.

S18. (a) Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_{10}$ be a random sample of 10 observations from a 100 dimensional multivariate normal distribution with dispersion matrix $\sum = ((\sigma_{ij}))_{100\times 100}$. Let $\hat{\sigma}_{ij}$ be maximum likelihood estimate of σ_{ij} based on $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3, \ldots, \underset{\sim}{x}_{10}$. Let $\hat{\sum} = ((\hat{\sigma}_{ij}))_{100\times 100}$. Show that $rank\ (\hat{\sum}) \leq 10$.

(b) Show that $X_1 + 2X_2$ is not sufficient for μ where X_1 and X_2 are two independent and identically distributed observations from $N(\mu, 1)$ distribution.

S19. (a) Suppose $X \sim \mathrm{N}(\theta, 1)$, where $\theta \geq 0$. For any fixed $x > 0$, define

$$g(\theta) = P_\theta(X^2 > x), \quad \theta \geq 0.$$

Show that $g(\theta)$ is a strictly increasing function of θ on $[0, \infty)$.

(b) Suppose $\boldsymbol{X} \sim \mathrm{N}_p(\boldsymbol{\theta}, \boldsymbol{I}_p)$, where $\boldsymbol{\theta} \in I\!R^p$, and $\boldsymbol{I}_p$ is the identity matrix of order p. For any fixed $x > 0$, define

$$h(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} > x), \quad \boldsymbol{\theta} \in I\!R^p.$$

Assuming $h(\boldsymbol{\theta})$ depends on $\boldsymbol{\theta}$ only through $\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}$, show that $h(\boldsymbol{\theta})$ is a strictly increasing function of $\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}$.

S20. (a) Suppose independent observations are generated from the linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \ldots, n$, where $x_1, \ldots, x_n$ are known positive numbers, and $\mathrm{E}(\epsilon_i) = 0\ \forall i, \mathrm{Var}(\epsilon_i) = \sigma^2\ \forall i$. However, we fit the model $y_i = \beta_1 x_i + \epsilon_i$. Show that the bias in the least squares estimate of β_1 is at most $\beta_0/\overline{x}$.

(b) Consider the linear model $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2 \boldsymbol{I}_n)$, where $\boldsymbol{X}$ is an $n \times p$ matrix with entries 0, 1 and -1. Assume that $\boldsymbol{X}$ is of full column rank. Show that if $\hat{\beta}_j$ denotes the best linear unbiased estimate (BLUE) of β_j, then

$$\mathrm{Var}(\hat{\beta}_j) \geq \sigma^2/n.$$

S21. Suppose that $\boldsymbol{X}$ has uniform distribution over the square defined by the four vertices $(0, \theta), (\theta, 0), (0, -\theta)$ and $(-\theta, 0)$ in the plane, where $\theta > 0$. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be independent and identically distributed (i.i.d.) observations from this distribution. Find the maximum likelihood estimate of θ based on these observations.

S22. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli(p) distribution. We want to test $\mathrm{H}_0 : p = 0.4$ against $\mathrm{H}_1 : p = 0.6$.

(a) Find the most powerful test for testing H_0 against H_1.

(b) Use the Central Limit Theorem to determine the sample size needed so that the two probabilities of error of the test obtained in (a) are both approximately equal to α. Find the result in terms of z_α, the $(1-\alpha)$-th quantile of $N(0, 1)$.

S23. (a) Let $\{X_n\}_{n=1}^{\infty}$ be independent and identically distributed as $U(0, \theta)$, where $\theta > 0$. Let $X_{(k)}^{(n)}$ denote the k-th order statistic based on $X_1, \ldots, X_n$. Show that for any fixed k, $nX_{(k)}^{(n)}$ converges in distribution to a non-degenerate random variable as $n \to \infty$.

(b) Let X_1 and X_2 be independent and identically distributed as $U(0, 1)$. Find the distribution of $Z = \sqrt{X_1 X_2}$.

S24. Let $\mathcal{X} = \{1, 2, 3, \ldots\}$ be the state space of a Markov chain with the transition probability matrix $P = ((p_{i,j}))$ where $p_{i,1} = 1/i^2$ and $p_{i,i+1} = 1 - 1/i^2$ for all $i \geq 1$.

(a) Derive the stationary distribution of the Markov chain.

(b) Is the Markov chain irreducible? Justify your answer.

(c) Does the Markov chain converge to the stationary distribution for all initial states? Justify your answer.

S25. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed as $U(\theta, \theta + |\theta|)$, where $\theta \neq 0$. Find the MLE of θ based on $X_1, X_2, \ldots, X_n$.

(b) Let X be a random variable with probability density function f_θ. If ϕ is the test function of a most powerful test at level α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, then prove that either $\beta_\phi(\theta_1) = 1$ or $\beta_\phi(\theta_0) = \alpha$, where $\beta_\phi(\theta) = E_\theta \phi(X)$.

S26. Consider a linear regression model containing an intercept. If c is the number of times the i-th row of the design matrix is replicated, then show that the i-th leverage value h_{ii} satisfies $n^{-1} \leq h_{ii} \leq c^{-1}$, where n denotes the number of observations.

S27. (a) Let $p_1(x)$ and $p_2(x)$ denote the probability density functions representing two populations, namely, Class-1 and Class-2 respectively. Let P and $(1 - P)$ be the prior probabilities of the Class-1 and Class-2 respectively. Let

$$p_1(x) = \begin{cases} x - 1 & \text{for } x \in [1, 2] \\ 3 - x & \text{for } x \in (2, 3] \\ 0, & \text{otherwise;} \end{cases}$$

$$p_2(x) = \begin{cases} x - 2 & \text{for } x \in [2, 3] \\ 4 - x & \text{for } x \in (3, 4] \\ 0, & \text{otherwise.} \end{cases}$$

Find the Bayes risk of the optimal Bayes rule for this classification problem.

(b) Suppose $\{X_1, X_2, \ldots, X_n, \ldots\}$ is a time-homogeneous Markov chain with the finite state space $\mathcal{S} = \{1, 2, \ldots, p\}$ where the transition probabilities for $n \geq 1$ are given by

- $P(X_{n+1} = i | X_n = i) = \frac{1}{2}$ for all $i \in \mathcal{S}$
- $P(X_{n+1} = i + 1 | X_n = i) = P(X_{n+1} = i - 1 | X_n = i) = \frac{1}{4}$ for all $i \in \{2, 3, \ldots, p - 1\}$
- $P(X_{n+1} = 2 | X_n = 1) = P(X_{n+1} = p | X_n = 1) = P(X_{n+1} = p - 1 | X_n = p) = P(X_{n+1} = 1 | X_n = p) = \frac{1}{4}$

Find the limiting distribution of X_n as $n \to \infty$

S28. (a) Let $X_1, \ldots, X_5$ be independent and identically distributed as $N(0,1)$. Calculate the probability $P(X_1 > X_2X_3X_4X_5)$.

(b) Let X be a random variable taking values $1, 2, \ldots, k$, with probabilities $p_1, p_2, \ldots, p_k$, respectively. Prove that

$$\sum_{i=1}^{k} p_i^2 \leq \max\{p_1, p_2, \ldots, p_k\}.$$

(c) Let $(i_1, i_2, \ldots, i_{2n})$ and $(j_1, j_2, \ldots, j_{2n})$ be two random permutations of $(1, 2, \ldots, 2n)$.

i. Let S denote the number of indices k such that $i_k = 2j_k$. For example, if $n = 3$, then $S = 2$ for the permutations $(1,4,5,3,2,6)$ and $(3,2,6,4,1,5)$. Determine $E(S)$, the expected value of S.

ii. Let Z be a discrete uniform random variable taking values in $\{1, 2, \ldots, 2n\}$. Show that $E(i_Z/j_Z) \geq 1$.

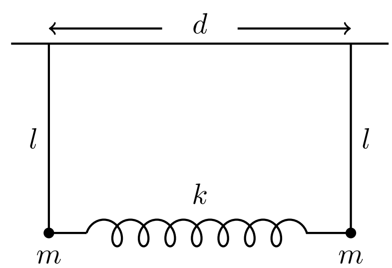
(iii) PHYSICS

P1. A particle of mass m is constrained to move on a smooth sphere of radius a and is acted on by gravity. Choosing θ and ϕ (polar and azimuthal angles, with polar axis vertically up) as the generalized coordinates, write down the Hamiltonian for the particle. Obtain Hamilton's equations of motion and show that

$$a\ddot{\theta} = \frac{k^2 \cos\theta}{m^2 a^3 \sin^3\theta} + g\sin\theta,$$

where k is the constant value of p_ϕ, the generalized momentum corresponding to ϕ.

P2. Two pendulums of mass m and length l are coupled by a massless spring of spring constant k, and are moving in a plane (see figure). The unstretched length of the spring is equal to the distance d between the supports of the two pendulums.



(a) Set up the Lagrangian in terms of generalized coordinates and velocities.

(b) Derive the equations of motion.

(c) Consider small vibrations and simplify the equations of motion.

(d) Find the frequencies of the two normal modes.

P3. (a) The molar internal energy of a monatomic gas obeying van der Waals equation is given by $\frac{3}{2}RT - \frac{a}{V}$, where a is a constant, and R, T and V carry their usual meanings.

(i) If the gas is allowed to expand adiabatically into vacuum (i.e., free expansion) from volume V_1 to V_2, and T_1 is its initial temperature, what is the final temperature of the gas?

(ii) What would be the final temperature for an ideal gas?

(b) A single classical particle of mass m with energy $\epsilon \leq E$, where $E = P^2/2m$, is enclosed in a volume V; P being the momentum corresponding to the energy E.

(i) Determine the (asymptotic) number of accessible microstates in the energy range ϵ to $\epsilon + d\epsilon$.

(ii) Using this result, obtain the partition function of the aforesaid system if the energy varies between 0 and ∞.

(iii) What would be the free energy of the particle?

P4. A nuclear fission explosion produces a fire ball that can be approximated at some instant to be a black body of 10 cm radius having a temperature $10^8 K$. Calculate

(i) the total electromagnetic power radiated,

(ii) the radiation flux at a distance of 1 km, and

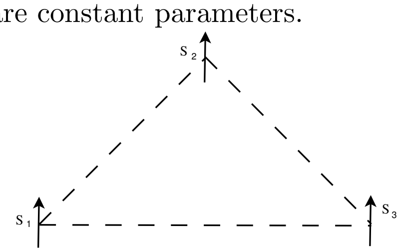
(iii) the wavelength corresponding to maximum energy radiated.

[Stefan's constant (σ)= 0.57×10^{-7} Watt $m^{-2}K^{-4}$ and
Wien's (displacement law) constant = 2.9×10^{-3} meter Kelvin.]

P5. Assume that three spins S_1, S_2, S_3 are arranged in the form of an equilateral triangle with each spin interacting with its two neighbors (see figure below). Each spin can take values +1 or −1. The energy of this system in a magnetic field perpendicular to the plane of the triangle, is

$$H = -J(S_1S_2 + S_2S_3 + S_3S_1) - F(S_1 + S_2 + S_3).$$

Here J and F are constant parameters.



(i) Find the partition function of the system.

(ii) Find the average spin.

P6. Consider a simple harmonic oscillator in one dimension with the Hamiltonian $H = \hbar\omega\left(a^\dagger a + \dfrac{1}{2}\right)$, where a and $a^\dagger$ are annihilation and creation operators respectively. The other symbols have their usual meanings.

(a) Show that $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, $|n\rangle$ being the eigen function of H corresponding to the eigen value $E_n = \hbar\omega(n + \dfrac{1}{2})$.

(b) Consider the wave function of a harmonic oscillator at $t = 0$

$$\psi(0) = N\left(|0\rangle + 2|1\rangle + 3|2\rangle\right),$$

where N is the normalization constant.

(i) Find the value of N.

(ii) Calculate the probability of observing the energy of the oscillator to be $\frac{3}{2}\hbar\omega$ upon measurement.

(iii) Find the wave function $\psi(t)$ at time t and calculate the expectation value of the energy for this wave function.

P7. (a) An electron of mass m has the energy of a photon of wavelength λ. Find the velocity of the electron.

(b) Free neutrons have a decay constant k. Consider a non-relativistic situation where the de Broglie wavelength of neutrons in a parallel beam is λ. Determine the distance from the source where the intensity of the beam drops to half of its value at source.

P8. (a) Consider a non-relativistic particle of mass m moving in the potential

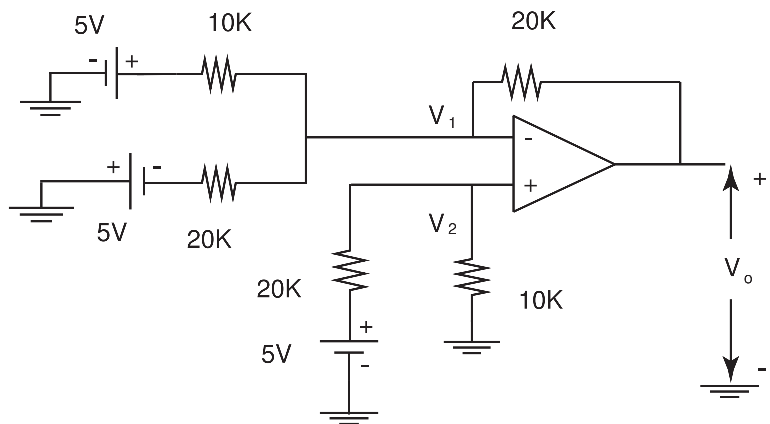
$$V(x) = \begin{cases} V_0 & 0 < x < a \\ \infty & \text{elsewhere} \end{cases}$$

where $V_0 > 0$. Find the energy levels and the corresponding wave functions for all x.

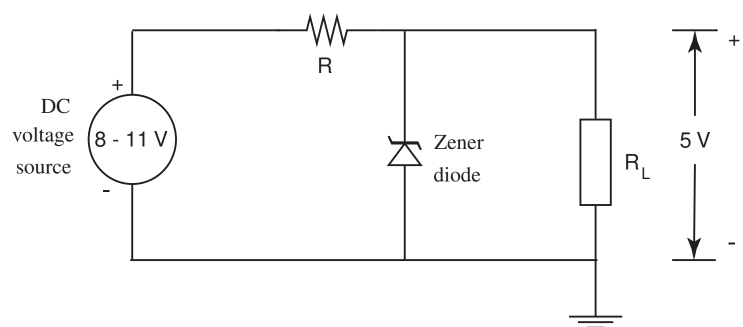
(b) A light beam is propagating through a block of glass with index of refraction μ. If the block of glass is moving at a constant velocity v in the same direction as the beam, what is the velocity of light in the block as measured by an observer in the laboratory?

P9. (a) A negative feedback amplifier has a voltage gain of 100. Variations of the voltage gain up to $\pm 2\%$ can be tolerated for some specific application. If the open-loop gain variations of $\pm 10\%$ are expected owing to spread in device characteristics because of variation in manufacturing conditions, determine the minimum value of the feedback ratio β and also the open loop gain to satisfy the above requirements.

(b) Calculate the output voltage V_0 for the following network:

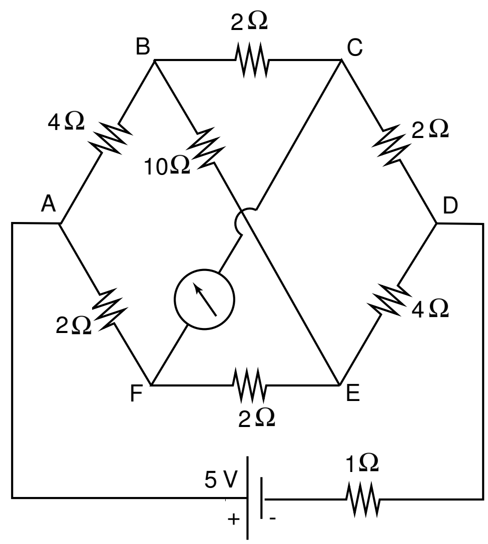


P10. (a) Consider the following circuit for deriving a +5 volt power supply to a resistive load R_L from an input d-c voltage source whose voltage may vary from 8V to 11V. The load R_L may draw a maximum power of 250 mW. The Zener diode has a breakdown voltage of 5 volts. Compute the maximum value of the resistance



R and also the power dissipation requirements for R and the Zener diode. Assume that the minimum breakdown current of the Zener diode is negligible compared to the load current.

(b) Consider the following circuit. Calculate the potential difference between the points F and C, as shown by the ideal voltmeter.



P11. (a) Write the properly normalized Maxwell-Boltzmann distribution $f(u)$ for finding particles of mass m with magnitude of velocity in the interval $[u, u + du]$ at a temperature T.

i. What is the most probable speed at temperature T?

ii. What is the average speed?

iii. What is the average squared speed?

(b) A container is divided into two compartments I and II by a partition having a small hole of diameter d. The two compartments are filled with Helium gas at temperatures $T_1 = 150K$ and $T_2 = 300K$, respectively.

i. How does the diameter d determine the physical process by which the gas comes to steady state?

ii. If λ_1 and λ_2 are the mean free paths in the compartments I and II, respectively, find $\lambda_1 : \lambda_2$ when $d \ll \lambda_1$ and $d \ll \lambda_2$.

iii. Find $\lambda_1 : \lambda_2$ when $d \gg \lambda_1$ and $d \gg \lambda_2$.

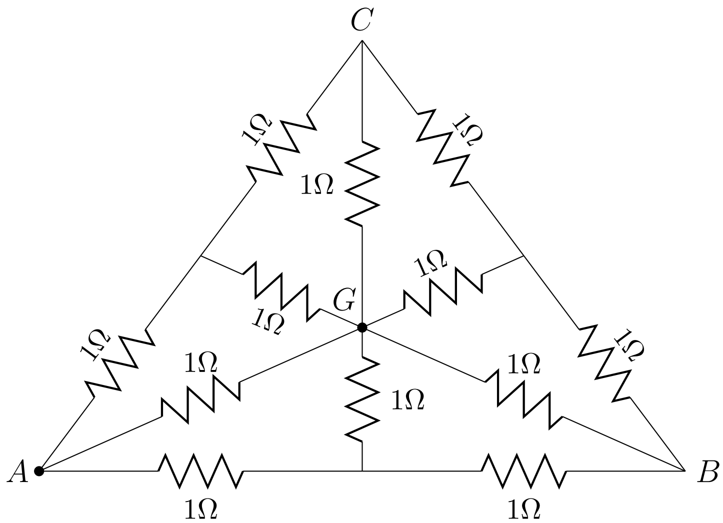
P12. (a) A gas obeys the equation of state $\frac{pV}{RT} = 1 + pB$, where B is a function of temperature T only. Show that

$$C_p - C_0 = -RTp\frac{d^2}{dT^2}(TB),$$

where C_0 is the value of C_p when $p = 0$.

(b) Show that for two dimensional electron gas, the number of electrons per unit area is $n = \frac{4\pi m k_B T}{h^2} ln(1 + \exp(E_F/k_B T))$, where k_B is the Boltzmann constant, h is the Planck's constant, E_F is the Fermi energy and n is the electron mass.

P13. (a) Determine the equivalent resistance between A and G of the following circuit. Each resistance has the value 1Ω.



(b) Consider a four-input hypothetical logic gate G with the following Karnaugh map.

AB \ CD	00	01	11	10
00	1			
01		1		
11			1	
10				1

i. Prove or disprove that G can be treated as a universal logic gate. A large number of 0 and 1 lines are available to you.

ii. Implement $A + B$ using minimum number of G gates.

P14. (a) Consider an iron sphere of radius R that carries a charge Q and a uniform magnetization $\vec{M} = M\hat{z}$. Compute the angular momentum stored in the electromagnetic fields.

(b) Find the charge and current distributions that would give rise to the potentials

$$\phi = 0, \ \vec{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2 \hat{z} & \text{for} \ \ |x| < ct \\ \\ 0 & \text{for} \ \ |x| \geq ct \end{cases}$$

where k is a constant and $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$. What would be the surface current?

P15. (a) A substance shows a Raman line at 4567 Å when the exciting line 4358 Å is used. Deduce the positions of Stokes and anti-Stokes lines for the same substance when the exciting line 4047 Å is used.

(b) The ground state of chlorine is $^2P_{\frac{3}{2}}$. Find its magnetic moment. How many substates will the ground state split into in a weak magnetic field?

P16. The Λ-particle is an unstable sub-atomic particle that can decay spontaneously into a proton and a negatively charged pion

$$\Lambda \rightarrow p + \pi^-.$$

In a certain experiment, the outgoing proton and pion were observed, both travelling in the same direction along the positive x-axis with momenta P_p and P_{π^-} respectively. Find the rest mass of Λ-particle given the rest masses of proton and pion to be m_p and m_{π^-} respectively.

P17. (a) Consider the multiplication of two 2-bit integers a_1a_0 and b_1b_0 to get a 4-bit output $c_3c_2c_1c_0$. Design a circuit for deriving the bit c_2 using only 2-input NAND gates.

(b) Suppose in a voltage sensitive Wheatstone bridge, each arm has a resistance R. Now the resistance of one arm is changed to $R+r$, where $r \ll R$. The Wheatstone bridge is supplied with an input voltage of e_i. Show that on account of imbalance, the output voltage is

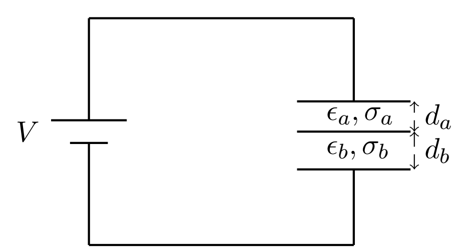
$$\left[\frac{(r/R)}{4+2(r/R)}\right] e_i.$$

P18. (a) Consider a possible solution of Maxwell's equations in vacuum given by

$$\vec{A}(\vec{x},t) = \vec{A}_0 e^{i(\vec{K}.\vec{x}-\omega t)} \quad , \quad \phi(\vec{x},t) = 0.$$

Here $\vec{A}(\vec{x},t)$ is the vector potential and $\phi(\vec{x},t)$ is the scalar potential. $\vec{A}_0$, $\vec{K}$, ω are constants. Show that Maxwell's equations impose the relation, $\mid \vec{K} \mid = \frac{\omega}{c}$, where c is the velocity of the electromagnetic wave in vaccum.

(b) A parallel plate capacitor is filled with two layers of dielectric material a and b (see figure) and is connected to a battery with potential V. The dielectric constant and conductivity of materials a and b are ϵ_a, σ_a and ϵ_b, σ_b respectively. The thicknesses of the materials a and b are d_a and d_b respectively.



i. Calculate the electric fields in the materials a and b.

ii. Find the current flowing through the capacitor.

(iv) ELECTRICAL AND ELECTRONICS ENGINEERING

E1. (a) Consider the multiplication of two 2-bit integers a_1a_0 and b_1b_0 to get a 4-bit output $c_3c_2c_1c_0$. Design a circuit for deriving the bit c_2 using only 2-input NAND gates.

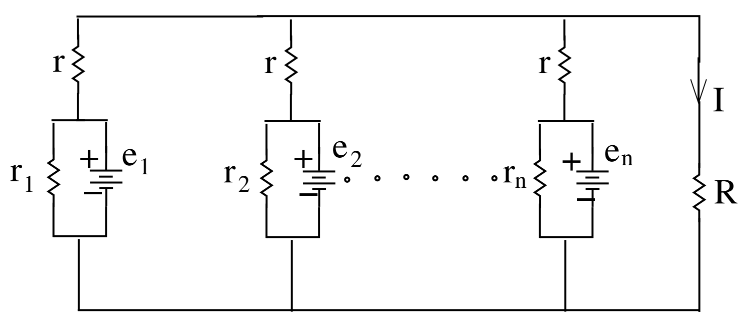
(b) Consider a bit sequence a_i, $i \geq 0$ which has the property $a_{i+4} = a_{i+1} \oplus a_i$.

i. To generate this sequence, which bits of the sequence need to be initialized?

ii. Design a logic circuit using flip-flops and NAND gates to generate the above bit sequence.

iii. For any given value of i, identify the points in the circuit at which the values of $a_i, a_{i+1}, \ldots, a_{i+4}$ may be obtained.

E2. A resistor **R** is getting supply from n e.m.f. sources $e_1, e_2, \ldots, e_n$ where $e_1 < e_2 < \cdots < e_n$, connected with corresponding resistors as shown in the figure.



Calculate the current **I** flowing through the resistor **R**.

E3. (a) A 44 KW, 220 V d.c. machine has 110 Ω shunt resistance and 0.05 Ω armature resistance. Calculate total armature power developed when the machine is working as (i) a generator and (ii) a motor.

(b) Test data of a 200/400 V, 10 KVA, 50 Hz single phase transformer is as follows.

Short circuit test on secondary side: 20 V, 10 A, 80 W

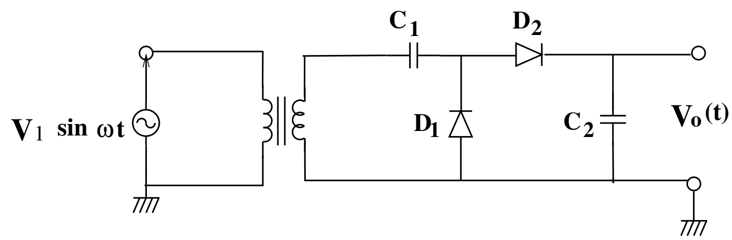
Open circuit test on primary side: (i) 200 V, 50 Hz, 2500 W

(ii) 100 V, 25 Hz, 1000 W

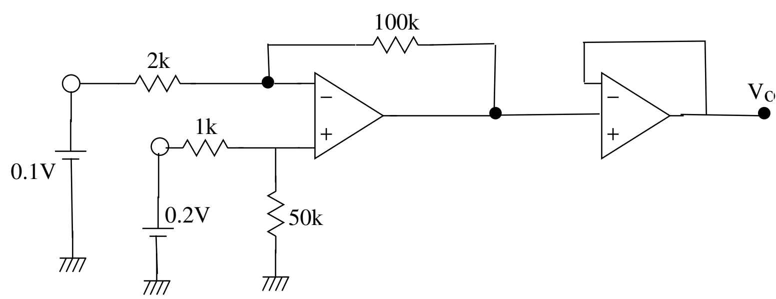
At full-load and unity power factor, calculate:

i. copper loss,
ii. hysteresis loss, and
iii. eddy current loss.

E4. (a) In the following circuit, the diodes D_1 and D_2 and the capacitors C_1 and C_2 are assumed to be ideal. At the input, a sinusoidal voltage $V_1 \sin(\omega t)$ is applied. Sketch the output waveform $V_o(t)$ as a function of time t.



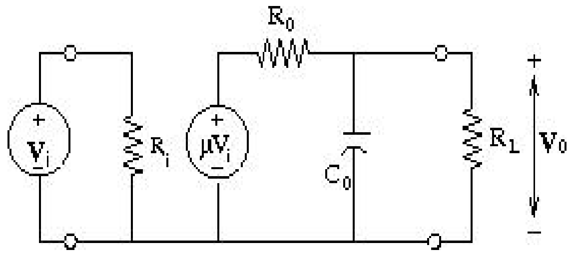
(b) Consider the following circuit with two ideal OP-Amps. Calculate the output voltage, V_o.



E5. Consider a voltage amplifier circuit shown in the figure below, where R_i and R_0 represent the input and output impedances respectively, C_0 is the total parasitic capacitance across the output port, μ is the amplifier gain and the output is terminated by a load resistance R_L.

(a) Calculate the current, voltage and power gain in decibels (dB) of the amplifier, when
$R_i = 1M\Omega$, $R_L = 600\Omega$, $R_0 = 100M\Omega$, $C_0 = 10pf$, $\mu = 10$.

(b) Calculate the 3-dB cutoff frequency of the amplifier when
$R_i = 5K\Omega$, $R_L = 1K\Omega$, $R_0 = 100\Omega$, $C_0 = 10pf$, $\mu = 2$.



E6. Open-circuit and short-circuit tests are conducted on a 220/440 V, 4.4 KVA, single phase transformer. The following readings are obtained.

Open Circuit test with voltage applied on low-voltage side:

Voltage = 110 V,
Current = 1.1A, and
Power = 150 W.

Short Circuit test with voltage applied on high-voltage side:

Voltage = 24 V,
Current = 8A, and
Power = 64 W.

At 0.8 p.f. lagging, calculate

(i) the efficiency of the transformer at full-load, and

(ii) the output voltage at the secondary terminal when supplying full-load secondary current.

E7. Assume that an analog voice signal which occupies a band from 300 Hz to 3400 Hz, is to be transmitted over a Pulse Code Modulation (PCM) system. The signal is sampled at a rate of 8000 samples/sec. Each sample value is represented by 7 information bits plus 1 parity bit. Finally, the digital signal is passed through a raised cosine roll-off filter with the roll-off factor of 0.25. Determine

(a) whether the analog signal can be exactly recovered from the digital signal;

(b) the bit duration and the bit rate of the PCM signal before filtering;

(c) the bandwidth of the digital signal before and after filtering;

(d) the signal to noise ratio at the receiver end (assume that the probability of bit error in the recovered PCM signal is zero).

E8. (a) Given a library of 2-input AND, NOT and 2-input XOR gates, synthesize the function $f(A, B, C, D)$ as shown in the Karnaugh map using minimum number of gates of the library.

$AB \backslash CD$	00	01	11	10
00	1		1	
01		1	1	
11		1		1
10		1	1	

(b) A sequential lock circuit has two push-buttons **A** and **B** which cannot be pressed simultaneously. It has one output **z** which becomes 1 and opens the lock, only when the buttons are pressed in the sequence **ABBA**. Find a reduced state table for the lock circuit.

E9. Consider the following waveform of a signal $f(t) = 3\cos 500t + 4\cos 1500t$ volts, which is coded using Delta Modulation (DM).

(a) Compute the minimum integer sampling rate with justification for exact reconstruction of the signal from the sampled data.

(b) Assuming a quantizer step size of $\frac{1}{2}$ volt, determine the mean-square quantization noise power.

(c) Determine the cut-off frequency of the low-pass filter in the DM receiver and calculate the corresponding signal-to-noise ratio.

E10. (a) Consider the discrete-time sequence

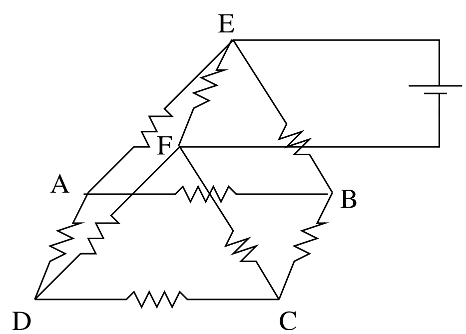
$$x[n] = \begin{cases} (-0.5)^n & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

i. Determine the inverse Z-transform of $X(z^3)$ without computing $X(z)$.

ii. Let $y[n] = e^{j(\pi/3)n}x[n]$. Sketch the pole-zero plot and indicate the Region of Convergence (ROC) of $Y(z)$.

(b) Consider the complex sequence $v[n] = Re\{v[n]\} + jIm\{v[n]\}$. Compute the Z-transform of $Im\{v[n]\}$ in terms of $V(z)$ and indicate the ROC of $Im\{v[n]\}$ in terms of the ROC of $V(z)$.

E11. A prism is made of wire mesh with each side having equal resistance R. (See the figure given below.) A battery of 6 V and zero internal resistance is connected across E and F. If R is 0.5 Ω, find the current in the branch AD.



E12. (a) Consider a Discrete Memory-less System (DMS) with eight symbols $x_0, x_1, \ldots, x_7$ occurring with probabilities 0.005, 0.005, 0.04, 0.10, 0.10, 0.15, 0.25 and 0.35 respectively.

i. Devise an optimum variable length code for this system.

ii. What is the entropy of the DMS in bits?

iii. What is the average number of bits required by this coding system?

[$\log_2 3 = 1.58496$, $\log_2 5 = 2.32193$, and $\log_2 7 = 2.80736$.]

(b) Define Mutual Information (MI) between two random variables X and Y when

i. both X and Y are discrete,

ii. X is discrete and Y is continuous.

iii. In case (ii) above, consider that X has two outcomes $x_1 = A$ and $x_2 = -A$. If the conditional probability density function

$p(y/x_i)$ is Gaussian with mean x_i and variance σ^2 for $i = 1, 2$, then find the MI of the system.

E13. (a) A sequence $x[n]$ of length 8 has a Discrete Fourier Transform (DFT) $X[k]$ as follows:

$$X[k] = \begin{cases} 1; & k = 0, 1, 2, 3 \\ k - 2; & k = 4, 5, 6, 7 \\ 0; & \text{otherwise.} \end{cases}$$

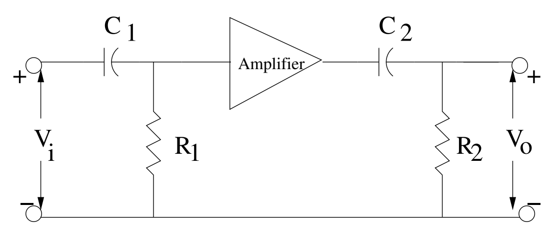
$x[n]$ is now up-sampled by a factor of 2 by inserting a zero between every pair of samples of $x[n]$, and appending a zero at the end to produce a sequence $y[n]$ of length 16. Sketch $Y[k]$, the 16-point DFT of $y[n]$.

(b) A zero-mean, wide-sense stationary sequence with variance σ_x^2 passes through a stable and causal all-pass filter with z-transform

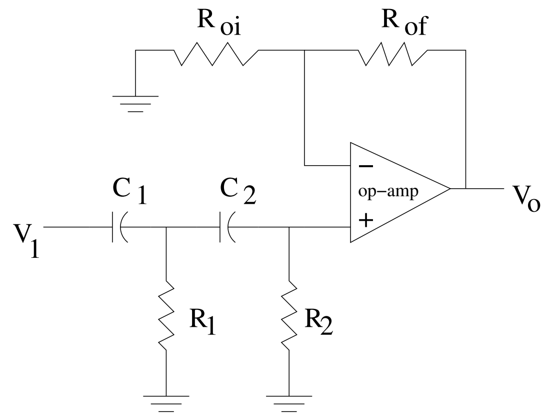
$$H(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}}.$$

Determine the range of values that α can take. Also calculate the power spectrum of the output, i.e., the Discrete Time Fourier Transform (DTFT) of the auto-correlation sequence of the output.

E14. (a) Consider the following circuit where the amplifier has a voltage gain of A with zero phase shift. Let a ramp voltage $V_i = \alpha t$, α (> 0) being a constant, be applied to the input. Derive an expression for the output voltage V_o. Also, find the initial slope of V_o as time $t \to 0$.



(b) Calculate the gain and cutoff frequency of the following circuit when $R_1 = R_2 = 2$ kΩ, $C_1 = C_2 = 0.05$ μF, $R_{oi} = 10$ kΩ and $R_{of} = 50$ kΩ.



E15. (a) Design a digital circuit to compare two three bit numbers $A = a_2a_1a_0$ and $B = b_2b_1b_0$; the circuit should have three outputs indicating $A = B$, $A < B$ and $A > B$. Give a gate-level diagram of this comparator circuit.

(b) Using two such comparator circuits and as few basic gates as possible, design a digital circuit to compare two six bit numbers $A = a_5a_4a_3a_2a_1a_0$ and $B = b_5b_4b_3b_2b_1b_0$. This circuit should also have only three outputs indicating $A = B$, $A < B$ and $A > B$.

E16. A 22 KVA, 2200/220V two-winding transformer is converted to an auto-transformer with additive polarity.

(a) Calculate the percent increase in KVA of the auto-transformer with respect to the original two-winding transformer.

(b) The auto-transformer has a full-load efficiency of 90% at unity power factor. Calculate the efficiency of the auto-transformer when the load is reduced to half at the same power factor. Assume that iron loss is 100 W.

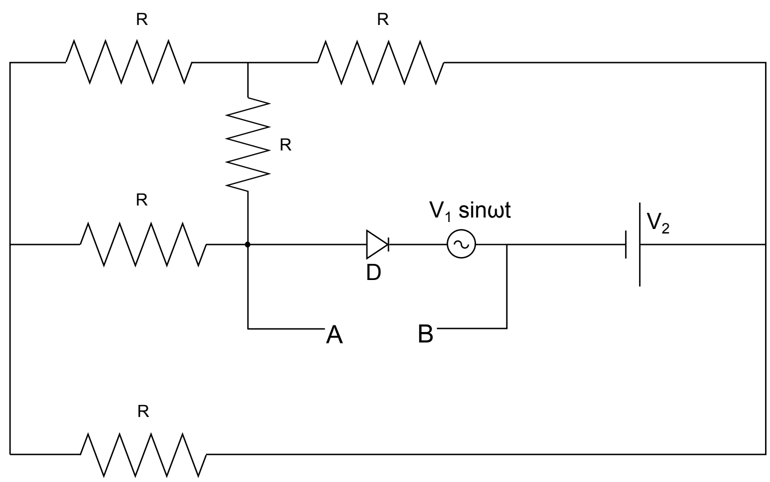
E17. A DC shunt motor of rated power 20 KW is connected to a supply voltage S = 220 V. The armature and field copper losses are 0.8 KW and 0.22 KW, respectively. The rotor speed is 1360 r.p.m. Assume that all other losses are negligible.

(a) Calculate the efficiency, field and armature current at the rated operation.

(b) Suppose the above machine has to perform as a DC generator providing an output voltage of 120 V and a power of 12 KW.

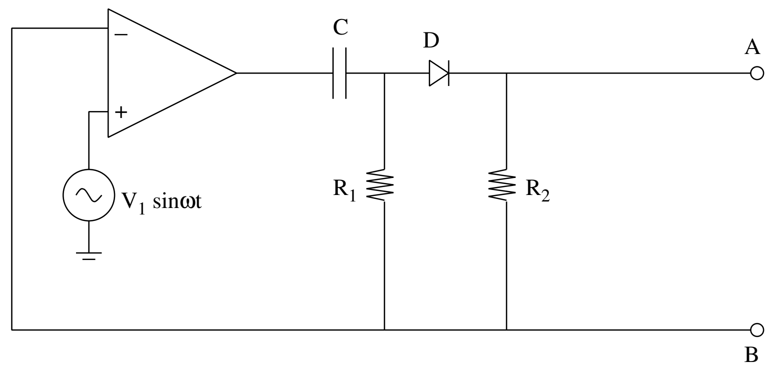
Calculate the necessary rotor speed and the required torque at normal operation.

E18. (a) Consider the following circuit, where a sinusoidal source $V_1 \sin \omega t$ and a DC source V_2 are connected as shown. Assume $V_1 > V_2$. Let V_{AB} be the voltage between A and B. The value of each resistance is R.



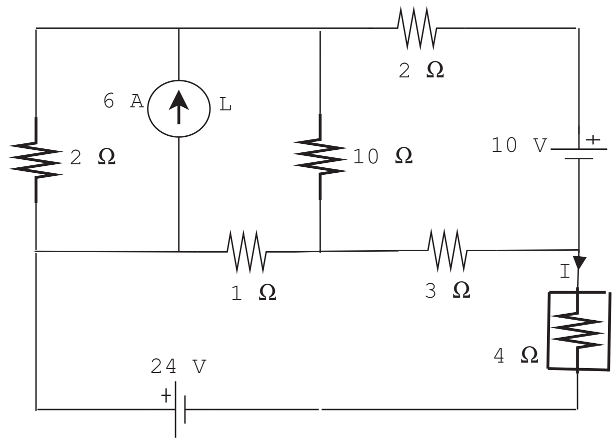
The voltage sources and the diode D are assumed as ideal. Draw the waveform of V_{AB} and justify your answer.

(b) Consider the following circuit. An a.c. source $V_1 \sin \omega t$ is connected to the non-inverting input of the OP-AMP. Draw the voltage waveform V_{AB} and justify your answer.

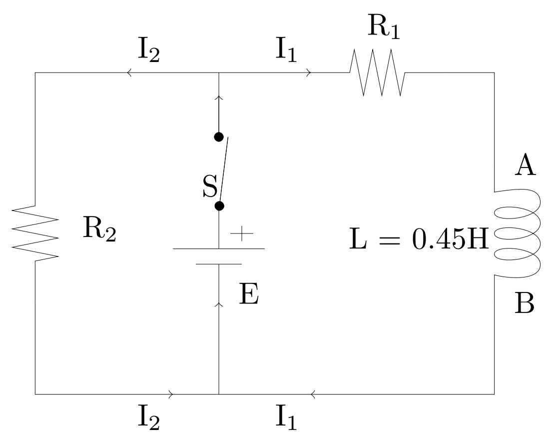


E19. In the following circuit, determine the current through the 4Ω resistor.

It is assumed that the two batteries are ideal and L is an ideal current source.



E20. (a) Consider the following circuit. The emf E = 15V, $R_1 = 1.5K\Omega$, and $R_2 = 3.0K\Omega$. The switch is closed for $t < 0$, and steady-state conditions are established. The switch is now thrown open at $t = 0$.

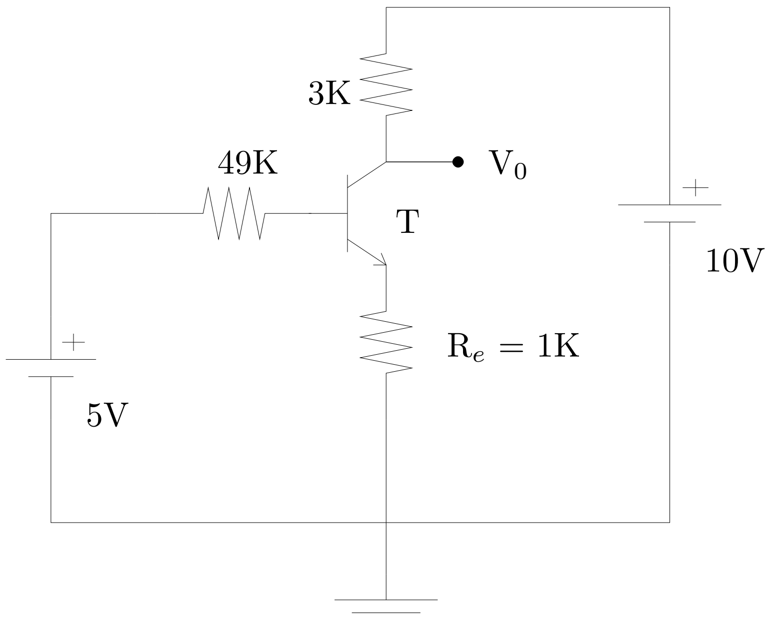


i. Find the initial voltage emf_0 across L just after $t = 0$. Which end of the coil is at higher potential: A or B?

ii. Draw freehand graphs of the currents in R_1 and R_2 as a function of time, indicating the values before and after $t = 0$.

(b) For the circuit shown below, assume $\beta = h_{FE} = 100$, $V_{CE,\ sat} = 0.2V$, $V_{BE,\ sat} = 0.8V$, $V_{BE,\ active} = 0.7V$, and

$V_{BE,\ cutoff} = 0.0$. [All symbols follow the standard notations.]



i. Determine whether the transistor T is in the *cutoff*, *saturation* or *active* region.

ii. Find the minimum value of R_e for which the transistor is in the active region. [Assume $I_{CO} \ll I_B$]

(v) COMPUTER SCIENCE

C1. (a) Write the smallest real number greater than 6.25 that can be represented in the IEEE-754 single precision format (32-bit word with 1 sign bit and 8-bit exponent).

(b) Convert the sign-magnitude number **10011011** into a 16-bit 2's complement binary number.

(c) The CPU of a machine is requesting the following sequence of memory accesses given as word addresses: 1, 4, 8, 5, 20, 17, 19, 56. Assuming a direct-mapped cache with 8 one-word blocks, that is initially empty, trace the content of the cache for the above sequence.

C2. (a) A machine $\mathcal{M}$ has the following *five* pipeline stages; their respective time requirements in nanoseconds (ns) are given within parentheses:
F-stage — instruction fetch (9 ns),
D-stage — instruction decode and register fetch (3 ns),
X-stage — execute/address calculation (7 ns),
M-stage — memory access (9 ns),
W-stage — write back to a register (2 ns).
Assume that for each stage, the pipeline overhead is 1 ns. A program P having 100 machine instructions runs on $\mathcal{M}$, where every 3^{rd} instruction needs a 1-cycle stall before the X-stage. Calculate the CPU time in seconds for completing P.

(b) The CPU of a computer has a ripple-carry implementation of a 2's complement adder that takes two 8-bit integers $A = a_7a_6 \ldots a_0$ and $B = b_7b_6 \ldots b_0$ as inputs, and produces a sum $S = s_7s_6 \ldots s_0$, where $a_i,\ b_i,\ c_i \in \{0, 1\}$ for $(0 \leq i \leq 7)$.
Let $A = 1001\ 1001$ and $B = 1000\ 0110$. What will be the output S of the adder? How will the value of S be interpreted by the machine?

(c) Add the following two floating point numbers A and B given in IEEE 754 single precision format and show the sum S in the same format.
A: 0000011000100 0000 000000000000001
B: 1000011000100 0000 000000000000001

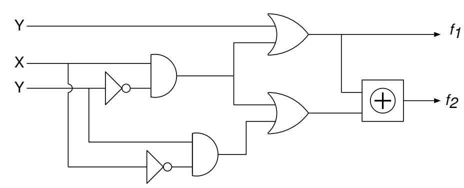
C3. Consider a bit sequence a_i, $i \geq 0$ which has the property $a_{i+4} = a_{i+1} \oplus a_i$.

(a) To generate this sequence, which bits of the sequence need to be initialized?

(b) Design a logic circuit using flip-flops and NAND gates to generate the above bit sequence.

(c) For any given value of i, identify the points in the circuit at which the values of $a_i, a_{i+1}, \ldots, a_{i+4}$ may be obtained.

C4. (a) A Boolean function f is said to be positive unate if f can be expressed in a form where all the variables appear in uncomplimented form, and only the AND and OR Boolean operators are used. For example, the function $g_1 = X_1X_2 + X_2X_3$ is positive unate but $g_2 = X_1X_2 + \bar{X}_2X_3$ is not. Consider the following circuit and determine which of the two functions f_1 and f_2 are positive unate.



(b) Consider a machine with four registers (one of which is the accumulator `A`) and the following instruction set.

- `LOAD R` and `STORE R` are indirect memory operations that load and store using the address stored in the register `R`. Thus, `LOAD R` loads the contents of memory[R] into `A` and `STORE R` stores the contents of `A` in memory[R].
- `MOV R1 R2` copies the contents of register `R1` into register `R2`.
- `ADD R` and `SUB R` operate on the accumulator and one other register `R`, such that `A = A op R`.
- `LDC n` stores the 7-bit constant `n` in the accumulator.
- `BRA`, `BZ`, and `BNE` are branch instructions, each taking a 5-bit offset.

Design an instruction encoding scheme that allows each of the above instructions (along with operands) to be encoded in 8 bits.

C5. (a) In a Buddy memory allocation system, a process is allocated an amount of memory whose size is the smallest power of 2 that is greater than or equal to the amount requested by the process.

A system using buddy memory allocation has 1MB memory. For a given sequence of nine processes, their respective memory requirements in KB are:

50, 150, 90, 130, 70, 80, 120, 180, 68.

(i) Illustrate with an allocation diagram to justify whether all the requests, in the given order, can be complied with. Assume that memory once allocated to a process is no longer available during the entire span of the above sequence.

(ii) Calculate the total memory wasted due to fragmentation in your memory allocation by the above scheme.

(b) Two processes P_1 and P_2 have a common shared variable *count*. While P_1 increments it, P_2 decrements it. Given that R_0 is a register, the corresponding assembly language codes are:

P_1:	$count++$		P_2:	$count--$	
MOV	*count*	R_0	MOV	*count*	R_0
ADD	#1	R_0	SUB	#1	R_0
MOV	R_0	*count*	MOV	R_0	*count*

Give an example to justify whether a race condition may occur if P_1 and P_2 are executed simultaneously.

C6. (a) Five batch jobs $P_1, \ldots, P_5$ arrive almost at the same time. They have estimated run times of 10, 6, 2, 4 and 8 ms, respectively. Their priorities are 3, 5, 2, 1 and 4 respectively, where 1 indicates the highest priority and 5 indicates the lowest. Determine the average turnaround and waiting time for the following scheduling algorithms:

(i) Round robin with time quantum of 5 ms,

(ii) Priority scheduling.

(b) The access time of a cache memory is 100 ns and that of main memory is 1000 ns. It is estimated that 80% of the memory requests are for read and the remaining 20% are for write. The hit ratio for read access is 0:9. A write through procedure is used.

(i) What is the average access time of the system considering only memory read cycles?

(ii) What is the average access time of the system considering both read and write requests?

C7. (a) What are the conditions which must be satisfied by a solution to the critical section problem?

(b) Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1, share the following variables:

```
char flag[2] = {0,0};
char turn = 0;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
do {
 flag[i] = 1;
 while (flag[j])
       if (turn == j) {
          flag[i] = 0;
          while (turn == j) {};
       }
 ...
 CRITICAL SECTION
 ...
 turn = j;
 flag[i] = 0;
 ...
 REMAINDER SECTION
 ...
} while (1);
```

Does this solution satisfy the required conditions?

C8. An operating system allocates memory in units of 1 KB pages. The address space of a process can be up to 64 MB in size; however, at any point of time, a process can be allocated at most 16 MB of physical memory. In addition the kernel uses 65 KB of physical memory to store page table entries of the current process. The OS also uses a translation-lookaside buffer (TLB) to cache *page table entries*. You are also given the following information:

- size of a page table entry is 4 bytes,
- TLB hit ratio is 90%,
- time for a TLB lookup is negligible,
- time for a memory read is 100 nanoseconds,

- time to read a page from the swapping device into physical memory is 10 milliseconds.

Calculate the effective memory access time for a process whose address space is 20 MB? Assume that memory accesses are random and distributed uniformly over the entire address space.

C9. (a) The *C* function `divby3` given below is intended to check whether a given number is divisible by 3. It assumes that the argument `number` is a string containing the decimal representation of a positive integer, and returns 1 or 0 depending on whether the integer is divisible by 3 or not.

```
int divby3(char *number)
{
    int sum = 0;
    while (*number != '\0') {
        sum += *number - '0';
        number++;
    }
    return (sum % 3) ? 0 : 1;
}
```

Assume that a variable of type `int` is stored using 4 bytes and the decimal representations of arbitrarily large positive integers can be passed as arguments to `divby3`.

i. Show that the given function does not work correctly for some integers larger than 10^{10^9}.

ii. Modify the above function so that it works as intended for all positive integers.
NOTE: The smaller the number of ALU operations used by your function, the more marks you will get.

(b) There are n students standing in a line. The students have to rearrange themselves in ascending order of their roll numbers. This rearrangement must be accomplished only by successive swapping of adjacent students.

i. Design an algorithm for this purpose that minimises the number of swaps required.

ii. Derive an expression for the number of swaps needed by your algorithm in the worst case.

C10. Consider the C program given below.

```
#include <stdio.h>
#define n 5
#define m n*n
int **a;

main(){
        int i, j, k;
        void printarray();
        a = (int **)calloc(n, sizeof(int *));
        for(i = 0; i < n; i++)
            *(a+i) = (int *)calloc(n, sizeof(int));

        for(k=1, i=0, j=(n-1)/2; k<=m; i--, j--, k++){
            i = (i + m) % n;
            j = (j + m) % n;
            if (a[i][j] != 0){
               i = (i + 2) % n;
               j = (j + 1) % n;
            }
               a[i][j] = k;
        }
        printarray();
        return 0;
}

void printarray(){
        int i, j;

        for(i = 0; i < n; i++){
            for (j = 0; j < n; j++)
                printf("%d", a[i][j]);
            printf("\n");
        }
}
```

(a) Write the output of the above program.

(b) Modify the subroutine "printarray" to print all the elements of the array in decreasing order.

C11. (a) Let $R = (A, B, C, D, E, F)$ be a schema with the set $F = \{A \to BC,\ CD \to E, B \to D, E \to A\}$ of functional dependencies. Suppose R is decomposed into two schemata $R_1 = (A, B, C)$ and $R_2 = (A, D, E, F)$

(i) Is this decomposition loss-less? Justify.

(ii) Is this decomposition dependency preserving? Justify.

(iii) Identify all the candidate keys for R.

(iv) Decompose R into normalized sets of relations using 3NF.

(v) If a new dependency $A \twoheadrightarrow F$ (multi-valued dependency) is introduced, what would be the new set of normalized relations?

(b) Consider the relations $r1(A, B, C)$, $r2(C, D, E)$ and $r3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r1)$ be 500, $V(C, r2)$ be 1000, $V(E, r2)$ be 50, and $V(E, r3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear in relation r for attribute X . If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.

C12. (a) Consider a LIBRARY database consisting of the following entity sets:

Book (bookid, title, publishername)
Book_authors (bookid, authorname)
Publisher (publishername, address, phonenumber)
Bookcopies (bookid, accessionnumber)
Book_loans (bookid, cardnumber, issuedate, duedate)
Borrower (cardnumber, name, address, phonenumber)

Write a relational algebra expression for retrieving the names of the borrowers who do not have any book issued. Hence write an equivalent SQL statement for the above query.

(b) A network has 125 stations attached by a dedicated pair of lines to a hub in a star topology. The distance from each station to the hub is 25 meters, the speed of the transmission lines is 10 Mbps, all frames are of length 12500 bytes, and the signal propagates on the line at a speed of 2.5×10^8 meters/second. Assume that token-ring protocol is used for medium access control. Assume single-frame operation, eight-bit latency at each station, and a free token is of three bytes long.

(i) Find the effective frame transmission time.

(ii) Assume that each station can transmit up to a maximum of $k = 2$ frames/token. Find the maximum throughput of the network.

C13. (a) A program P consisting of 1000 instructions is run on a machine at 1 *GHz* clock frequency. The fraction of floating point (FP) instructions is 25%. The average number of clock-cycles per instruction (CPI) for FP operations is 4.0 and that for all other instructions is 1.0.

(i) Calculate the average CPI for the overall program P.

(ii) Compute the execution time needed by P in seconds.

(b) Consider a 100mbps token ring network with 10 stations having a ring latency of 50 μs (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is 200 μs.

(i) Express the maximum efficiency of this network when only a single station is active in the network.

(ii) Find an upper bound on the token rotation time when all stations are active.

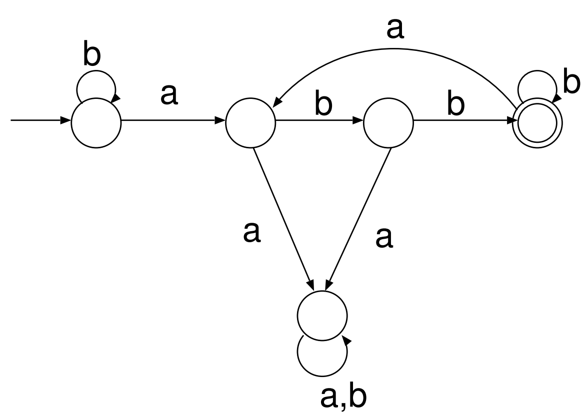
(iii) Calculate the maximum throughput rate that one host can achieve in the network.

C14. (a) Station A is sending data to station B over a full duplex error free channel. A sliding window protocol is being used for flow control. The send and receive window sizes are 6 frames each. Each frame is 1200 bytes long and the transmission time for such a frame is 70 μS. Acknowledgment frames sent by B to A are very small and require negligible transmission time. The propagation delay over the link is 300 μS. What is the maximum achievable throughput in this communication?

(b) Consider a large number of hosts connected through a shared communication channel. The hosts transmit whenever they have any data. However, if two data packets try to occupy the channel at the same time, there will be a collision and both will be garbled. The hosts retransmit these packets that suffered collisions. Assume that the generation of new packets by a host is a Poisson process and is independent of other hosts. The total number of transmissions (old and new packets combined) per packet time follows Poisson distribution with mean 2.0 packets per packet time. Compute the throughput of the channel. (Packet time is the amount of time needed to transmit a packet.)

C15. (a) Construct a finite state machine that accepts all the binary strings in which the number of 1's and number of 0's are divisible by 3 and 2, respectively.

(b) Describe the language recognized by the following machine.



(c) Consider the grammar $E \to E + n|E \times n|n$. For a sentence $n + n \times n$, find the handles in the right sentential form of the reductions.

C16. Design a Turing machine that recognizes the unary language consisting of all strings of 0's whose length is a power of 2, i.e., $L = \{0^{2^n} | n \geq 0\}$.

C17. (a) Write a context-free grammar for the language consisting of all strings over $\{a, b\}$ in which the number of a's is not the same as that of b's.

(b) Let the set $D = \{p \mid p$ is a polynomial over the single variable x, and p has a root which is an integer (positive or negative)$\}$.

i. Design a Turing machine (TM) to accept D.

ii. Can D be decided by a TM? Justify your answer.

C18. Consider the context-free grammar $G = (\{S, A\}, \{a, b\}, S, P)$, where

$$\begin{aligned} P \;=\; \{ & S \rightarrow AS, \\ & S \rightarrow b, \\ & A \rightarrow SA, \\ & A \rightarrow a\}. \end{aligned}$$

Show that G is left-recursive. Write an equivalent grammar G' free of left-recursion.

C19. Let $S = \{x_1, x_2, \ldots x_n\}$ be a set of n integers. A pair (x_i, x_j) (where $i \neq j$) is said to be the closest pair if $|x_i - x_j| \leq |x_{i'} - x_{j'}|$, for all possible pairs $(x_{i'}, x_{j'})$, $i', j' = 1, 2, \ldots, n, i' \neq j'$.

(a) Describe a method for finding the closest pair among the set of integers in S using $O(n \log_2 n)$ comparisons.

(b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S, the current closest pair can be reported in $O(\log_2 n)$ time.

(c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.

C20. Let A be an $n \times n$ matrix such that for every 2×2 sub-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A, if $a < b$ then $c \leq d$. Moreover, for any pair of rows i and j, if a_{ik} and a_{jl} are the largest elements in i-th and j -th rows of A, respectively, then $k \leq l$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

(a) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n \log n)$.

(b) Establish its correctness, and justify the time complexity of the proposed algorithm.

C21. Consider a graph G (called an interval graph) whose nodes correspond to a set of intervals on the real line. The i-th interval is denoted by $[\alpha_i, \beta_i]$, where $0 \leq \alpha_i < \beta_i$. An edge between two nodes (i, j) implies that the corresponding intervals $[\alpha_i, \beta_i]$ and $[\alpha_j, \beta_j]$ overlap.

(a) Consider the set of intervals $[3, 7]$, $[2, 4]$, $[2, 3]$, $[1, 5]$, $[1, 2]$, $[6, 7]$, $[10, 16]$, $[11, 12]$. Draw the corresponding interval graph and identify the largest subgraph where all the nodes are connected to each other.

(b) Write an algorithm which takes the interval graph G as input and finds the largest subgraph of G in which all the nodes are connected to each other. What is the time complexity of your algorithm?

(c) Given a list of intervals, write an algorithm to list all the connected components in the corresponding interval graph. What is the time complexity of your algorithm?

C22. For a graph $G = (V, E)$, a vertex coloring with x colors is defined as a function $C : V \to \{1, 2, \ldots, x\}$, such that $C(u) \neq C(v)$ for every edge $(u, v) \in E$. The chromatic polynomial $P(G, x)$ denotes the number of distinct vertex colorings for G with at most x colors. For example, $P(K_2, x) = x(x - 1)$.

(a) Derive the chromatic polynomial for the graphs

i. $K_{1,5}$

ii. C_4

where $K_{m,n}$ denotes the complete bipartite graph of m and n vertices in the two partitions, and C_n is the simple cycle with n vertices.

(b) Consider K_n to be a complete graph with n vertices. Express $P(K_n, x)$ as a recurrence relation and solve it.

C23. The diameter of a tree $T = (V, E)$ is given by $\max_{u,v \in V} \{\delta(u, v)\}$, where $\delta(u, v)$ is the shortest path distance (i.e., the length of the shortest path) between vertices u and v. So, the diameter is the largest of all shortest path distances in the tree.

(a) Write pseudo-code for an efficient algorithm to compute the diameter of a given tree T.

(b) Analyze the time complexity of your algorithm.

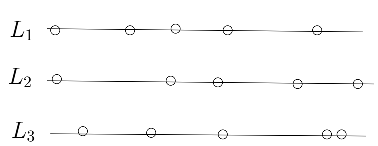
(c) What is its space complexity?

(d) Clearly mention the data structure(s) used by your algorithm.

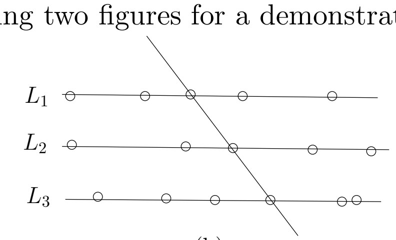
(e) A vertex c is called a center of a tree T if the distance from c to its most distant vertex is the minimum among all vertices in V. Write an algorithm to determine and report a center of the given tree T.

C24. Consider three parallel lines L_1, L_2 and L_3. On each line L_i, a set of n points $\{p_{i1}, p_{i2}, \ldots, p_{in}\}$ is placed.

The objective is to identify a triplet of indices (k, ℓ, m) (if exists) such that a straight line can pass through p_{1k}, $p_{2\ell}$ and p_{3m} on L_1, L_2 and L_3 respectively. [See the following two figures for a demonstration.]



(a)



(b)

In Figure (a), there does not exist any triplet (k, ℓ, m) such that a straight line can pass through $p_{1k}, p_{2\ell}$ and p_{3m}. In Figure (b), the triplet $(3, 3, 4)$ is a solution since a straight line passes through p_{13}, p_{23} and p_{34}.

Present an efficient algorithm for solving this problem. Justify its correctness and worst case time complexity.

[Full credit will be given if your algorithm is correct, and of worst case time complexity $O(n^2)$.]

C25. Let $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 2$.

(a) Express 63 as a sum of distinct a_i's.

(b) Write an algorithm to express any positive integer k as a sum of at most $\lceil \log_2 k \rceil$ many distinct a_i's.

(c) Prove the correctness of your algorithm.

C26. (a) How many distinct labeled spanning trees does a complete graph of n vertices have? Give a formal argument to establish your answer.

(b) A least expensive connection route among n houses needs to be designed for a cable-TV network. Consider the following algorithm $\mathcal{A}$ for finding a spanning tree.

```
Algorithm 𝒜
Input: G = (V, E)
Output: Set of edges M ⊆ E
Sort E in decreasing order of cost of edges;
i ← 0;
while i < |E| do
begin
    Let temp = (u1, u2) be the i-th edge E[i] in E;
    Delete E[i], i.e., replace E[i] by φ;
    if u1 is disconnected from u2 then
        restore temp in list E as E[i];
    i ← i + 1;
end
return the edges in E which are not φ;
```

i. Prove that the algorithm $\mathcal{A}$ can be used to correctly find a least cost connection route, given a set of n houses and information about the cost of connecting any pair of houses.

ii. What is the worst case time complexity of $\mathcal{A}$?

iii. If all the edges have distinct cost, how many solutions can there be?

C27. (a) Let $T = (V, E)$ be a tree, and let $v \in V$ be any vertex of T.

- The *eccentricity* of v is the maximum distance from v to any other vertex in T.
- The *centre* C of T is the set of vertices which have the minimum eccentricity among all vertices in T.
- The *weight* of v is the number of vertices in the largest subtree of v.
- The *centroid* G of T is the set of vertices with the minimum weight among all vertices in T.

Construct a tree T that has disjoint centre and centroid, each having two vertices, i.e., $C \cap G = \varnothing$ and $|C| = |G| = 2$.

(b) A vertex cover of a graph $G = (V, E)$ is a set of vertices $V' \subseteq V$ such that for any edge $(u, v) \in E$, either u or v (or both) is in V'. Write an efficient algorithm to find the minimum vertex cover of a given tree T. Establish its correctness. Analyse its time complexity.

C28. You are given k sorted lists, each containing m integers in ascending order. Assume that (i) the lists are stored as singly-linked lists with one integer in each node, and (ii) the head pointers of these lists are stored in an array.

(a) Write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(k)$ additional storage.

(b) Next, write an efficient algorithm that merges these k sorted lists into a single sorted list using $\Theta(1)$ additional storage.

(c) Analyse the time complexity of your algorithm for each of the above two cases.