JRF IN MATHEMATICS 2013

TEST CODE MTA, MTB

There will be two tests MTA and MTB of 2 hours duration each in the forenoon and in the afternoon. Topics to be covered in these tests along with an outline of the syllabus and sample questions are given below:

1) Topics for MTA (Forenoon examination) : Real Analysis, Measure and Integration, Complex Analysis, Ordinary Differential Equations and General Topology.

2) Topics for MTB (Afternoon examination) : Algebra, Linear Algebra, Functional Analysis, Elementary Number Theory and Combinatorics.

Candidates will be judged based on their performance in **both** the tests.

OUTLINE OF THE SYLLABUS

1. **General Topology :** Topological spaces, Continuous functions, Connectedness, Compactness, Separation Axioms. Product spaces. Complete metric spaces. Uniform continuity. Baire category theorem.

2. **Functional Analysis :** Normed linear spaces, Banach spaces, Hilbert spaces, Compact operators. Knowledge of some standard examples like $C[0,1], L^p[0,1]$. Continuous linear maps (linear operators). Hahn-Banach Theorem, Open mapping theorem, Closed graph theorem and the uniform boundedness principle.

3. **Real analysis :** Sequences and series, Continuity and differentiability of real valued functions of one variable and applications, uniform convergence, Riemann integration, continuity and differentiability of real valued functions of several variables, partial derivatives and mixed partial derivatives, total derivative.

4. **Linear algebra :** Vector spaces, linear transformations, characteristic roots and characteristic vectors, systems of linear equations, inner product spaces, diagonalization of symmetric and Hermitian matrices, quadratic forms.

5. **Elementary number theory and Combinatorics:** Divisibility, congruences, standard arithmetic functions, permutations and combinations, and combinatorial probability.

6. **Lebesgue integration :** Lebesgue measure on the line, measurable functions, Lebesgue integral, convergence almost everywhere, monotone and dominated convergence theorems.

7. **Complex analysis :** Analytic functions, Cauchy's theorem and Cauchy integral formula, maximum modulus principle, Laurent series, Singularities, Theory of residues, contour integration.

8. **Abstract algebra :** Groups, homomorphisms, normal subgroups and quotients, isomorphism theorems, finite groups, symmetric and alternating groups, direct product, structure of finite Abelian groups, Sylow theorems. Rings and ideals, quotients, homomorphism and isomorphism theorems, maximal ideals, prime ideals, integral domains, field of fractions; Euclidean rings, principal ideal domains, unique factoristation domains, polynomial rings. Fields, characteristic of a field, algebraic extensions, roots of polynomials, separable and normal extensions, finite fields.

9. **Ordinary differential equations :** First order ODE and their solutions, singular solutions, initial value problems for first order ODE, general theory of homogeneous and nonhomogeneous linear differential equations, and Second order ODE and their solutions.

SAMPLE QUESTIONS

Topology

(1) Let (X, d) be a compact metric space. Suppose that $f : X \to X$ is a function such that

$$d(f(x), f(y)) < d(x, y) \text{ for } x \neq y, x, y \in X.$$

Then show that there exists $x_0 \in X$ such that $f(x_0) = x_0$.

(2) Let X be a Hausdorff space. Let $f : X \to \mathbb{R}$ be such that $\{(x, f(x)) : x \in X\}$ is a compact subset of $X \times \mathbb{R}$. Show that f is continuous.

(3) Let X be a compact Hausdorff space. Assume that the vector space of real-valued continuous functions on X is finite dimensional. Show that X is finite.

(4) Let $n > 1$ and let $X = \{(p_1, p_2, \ldots, p_n) | p_i \text{ is rational}\}$. Show that X is disconnected.

(5) Let $A = \{(x, y) \in \mathbb{R}^2 | \max\{|x|, |y|\} \leq 1\}$ and $B = \{(0, y) \in \mathbb{R}^2 | y \in \mathbb{R}\}$. Show that the set $A + B = \{a + b | a \in A, b \in B\}$ is a closed subset of $\mathbb{R}^2$.

Functional analysis and Linear algebra

(6) Let $y_1, y_2, \ldots$ be a sequence in a Hilbert space. Let V_n be the linear span of $\{y_1, y_2, \ldots, y_n\}$. Assume that $\|y_{n+1}\| \leq \|y - y_{n+1}\|$ for each $y \in V_n$ for $n = 1, 2, 3, \ldots$. Show that $\langle y_i, y_j \rangle = 0$ for $i \neq j$.

(7) Let E and F be real or complex normed linear spaces. Let $T_n : E \to F$ be a sequence of continuous linear transformations such that $\sup_n \|T_n\| < \infty$. Let

$$M = \{x \in E | \text{ The sequence } \{T_n(x)\} \text{is Cauchy}\}.$$

Show that M is a closed set.

(8) Suppose that X is a normed linear space over $\mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ is a linear functional. Show that the kernel of f is either closed or dense.

(9) Let X be an infinite dimensional Banach space. Prove that every basis of X is uncountable.

(10) Let X and Y be complex, normed linear spaces which are not necessarily complete. Let $T : X \to Y$ be a linear map such that $\{Tx_n\}$ is a Cauchy sequence in Y whenever $\{x_n\}$ is a Cauchy sequence in X. Show that T is continuous.

(11) Let H be a Hilbert space and $S \subseteq H$ be a finite subset. Show that $(S^{\perp})^{\perp}$ is a finite dimensional vector space.

(12) Find an $n \times n$ matrix with real entries whose minimal polynomial is x^{n-1}.

Real Analysis and Measure Theory

(13) Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. Define

$$s_n = \frac{(a_1 + a_2 + \ldots + a_n)}{n}, n = 1, 2, 3, \ldots$$

Show that $\liminf\limits_{n\to\infty} a_n \leq \liminf\limits_{n\to\infty} s_n$.

(14) Let $p(x)$ be an odd degree polynomial in one variable with coefficients from the set $\mathbb{R}$ of real numbers. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Prove that there exists an $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

(15) Suppose that U is a connected open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is such that $\frac{\partial f}{\partial x} \equiv 0$ and $\frac{\partial f}{\partial y} \equiv 0$ on U. Show that f is a constant function.

(16) Let $f_1, f_2, f_3, \ldots$ and f be nonnegative Lebesgue integrable functions on $\mathbb{R}$ such that

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{y} f_n(x)dx &= \int_{-\infty}^{y} f(x)dx \text{ for each } y \in \mathbb{R} \\ \text{and } \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx &= \int_{-\infty}^{\infty} f(x)dx. \end{aligned}$$

Show that $\liminf\limits_{n\to\infty} \int_U f_n(x)dx \geq \int_U f(x)dx$ for any open subset U of $\mathbb{R}$.

(17) Let f be a uniformly continuous real valued function on the real line $\mathbb{R}$. Assume that f is integrable with respect to the Lebesgue measure on $\mathbb{R}$. Show that $f(x) \to 0$ as $|x| \to \infty$.

(18) Let $f : \mathbb{R} \longrightarrow [0, \infty)$ be a measurable function. If $\int_{-\infty}^{\infty} f(x)dx = 1$ then prove that $\int_{-\infty}^{\infty} \frac{1}{1+f(x)}dx = \infty$. (Hint: First show that $m\{x : f(x) < 1\} = \infty$ where m is the Lebesgue measure.)

Elementary Number Theory and Combinatorics

(19) Let p be a prime and r an integer, $0 < r < p$. Show that $\frac{(p-1)!}{r!(p-r)!}$ is an integer.

(20) If a and b are integers such that 9 divides $a^2 + ab + b^2$ then show that 3 divides both a and b.

(21) Let c be a 3^n digit number whose digits are all equal. Show that 3^n divides c.

(22) Prove that $x^4 - 10x^2 + 1$ is reducible modulo p for every prime p.

(23) Does there exist an integer x satisfying the following congruences?

$$\begin{aligned} 10x &= 1(\text{mod } 21) \\ 5x &= 2(\text{mod } 6) \\ 4x &= 1(\text{mod } 7) \end{aligned}$$

Justify your answer.

(24) Suppose that there are n boxes labelled $1, 2, \ldots n$ and there are n balls also labelled similarly. The balls are thrown into boxes completely randomly so that each box receives one ball.

(a) How many possible arrangements of balls in boxes is possible?

(b) Find the probability that the ball labelled 1 goes into the box labelled 1.

(c) Find the probability that at least one ball is in the box with the same label.

Complex Analysis

(25) Suppose for an analytic function f its zero set Z_f is uncountable. Show that $f \equiv 0$.

(26) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic. Suppose that $|f(\frac{1}{n})| \leq \frac{1}{n^{3/2}}$ for each $n \in \mathbb{N}$. Prove that $\{n^2 f(\frac{1}{n})\}$ is bounded.

(27) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous. If f^2 and f^3 are analytic prove that f is analytic at every point of $\mathbb{C}$.

Abstract Algebra

(28) Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$ and let k be an integer between 1 and n. Find the number of elements x in S_n such that the cycle containing 1 in the cycle decomposition of x has length k.

(29) Let $\mathbb{C}$ be the field of complex numbers and $\varphi : \mathbb{C}[X, Y, Z] \to \mathbb{C}[t]$ be the ring homomorphism such that

$$\begin{aligned} \varphi(a) &= a \text{ for all } a \text{ in } \mathbb{C}, \\ \varphi(X) &= t, \\ \varphi(Y) &= t^2, \text{ and} \\ \varphi(Z) &= t^3. \end{aligned}$$

Determine the kernel of φ.

(30) Show that there is no field isomorphism between $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$. Are they isomorphic as vector spaces over $\mathbb{Q}$?

(31) Let K be a subfield of $\mathbb{C}$ not contained in $\mathbb{R}$. Show that K is dense in $\mathbb{C}$.

(32) Determine the additive group of the field of four elements.

(33) Let $\mathbb{Z}[X]$ denote the ring of polynomials in X with integer coefficients. Find an ideal I in $\mathbb{Z}[X]$ such that $\mathbb{Z}[X]/I$ is a field of order 4.

Differential Equations

(34) Let $y : [a, b] \to \mathbb{R}$ be a solution of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y(x) = 0,$$

where $P(x)$ and $Q(x)$ are continuous functions on $[a, b]$. If the graph of the function $y(x)$ is tangent to X-axis at any point of this interval, then prove that y is identically zero.

(35) Consider the ordinary differential equation

$$y''(t) + py'(t) + qy(t) = 0, \;\; t > 0$$

where p and q are real constants such that $p^2 - 4q > 0$. Show that $|y(t)| \to 0$ as $t \to \infty$.

(36) Let $p \in \mathbb{C}$. Consider the differential equation

$$u''(x) - p^2 u(x) = 0.$$

If every solution of this equation satisfies

$$\sup_{T>0} \frac{1}{2T}\int_{-T}^{T} |u(t)|dt < \infty,$$

prove that $Re(p) = 0$.

JRF IN MATHEMATICS 2012

TEST CODE MTA

(1) Let $a_1, a_2, \ldots$ be a sequence of real numbers with $a_i \geq 0$. If $\Sigma_1^\infty \frac{1}{1+a_n} < \infty$, then show that $\Sigma_1^\infty \frac{1}{1+x_n a_n} < \infty$ for each real sequence $x_1, x_2, \ldots$, with $x_i \geq 0$ and $\liminf_{n\to\infty} x_n > 0$.

(2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with the intermediate value property: that is, f maps intervals to intervals. Let $x \in \mathbb{R}$. Suppose to each sequence (x_n) converging to x there exists a constant M such that

$$|f(x) - f(x_n)| \leq M \cdot \sup_{n,m} |f(x_n) - f(x_m)|.$$

Then show that f is continuous at x.

(3) Define $f_1, f_2, \ldots : [0,1) \longrightarrow \mathbb{R}$ as follows: For $n = 2^k + p$, $0 \leq p < 2^k$

$$f_n(t) = \begin{cases} 1, & \text{if } t \in [\frac{p}{2^k}, \frac{p+1}{2^k}) \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $\int_0^1 |f_n(t)| dt \longrightarrow 0$ as $n \longrightarrow \infty$.
(b) Find $\limsup f_n(t)$ and $\liminf f_n(t)$.

(4) Let m be the Lebesgue measure on $\mathbb{R}$ and $f : \mathbb{R} \longrightarrow [0, \infty)$ be a Lebesgue integrable function. Show that there exists a Lebesgue measurable set $E \subseteq [0, \infty)$ such that $m(E) \neq m(f^{-1}(E))$.

(5) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded Lebesgue measurable function such that, for all $a, b \in \mathbb{R}$ with $-\infty < a < b < +\infty$, $\int_a^b f(x)\, dx = 0$.
(a) Show that $\int_E f = 0$ for each subset E of $\mathbb{R}$ of finite Lebesgue measure.
(b) Deduce that $f = 0$ a.e.

(6) Let $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ be an analytic function with a simple pole of order 1 at 0 with residue a_{-1}. Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic

with $g(0) \neq 0$. Calculate for $r > 0$,

$$\frac{1}{2\pi i} \int_{|z|=r} f(z)g(z)\, dz.$$

(7) Let $f : \{z \in \mathbb{C} \mid 0 < |z| < 1\} \longrightarrow \mathbb{C}$ be analytic such that $n \leq |f(1/n)| \leq n^{(3/2)}$ for $n = 2, 3, \ldots$. Assume that $z^2 f(z)$ is bounded in $|z| < 1$. Show that f has a pole of order 1 at 0.

(8) Consider the ordinary differential equation

$$y''(t) + py'(t) + \frac{p^2}{4} y(t) = 0$$

on $\mathbb{R}$ with $p \in \mathbb{C}$ purely imaginary and $p \neq 0$. Note that solutions will be complex valued, not necessarily real valued.

(a) Show that the equation has at least one unbounded solution.

(b) If f is any unbounded solution then show that the limit

$$\lim_{|t| \to \infty} \left| \frac{f(t)}{t} \right|$$

exists and is nonzero.

(9) The cofinite topology on $\mathbb{R}$ is the topology in which a subset $F \subseteq \mathbb{R}$ is closed if and only if F is either finite or $F = \mathbb{R}$. Let $X = \mathbb{R}$ with the cofinite topology and $Y = \mathbb{R}$ with the usual topology. Show that any continuous map $f : X \longrightarrow Y$ is a constant.

(10) Let $X = [-1, 1] \times [-1, 1]$ and $Y = \{0\} \times [-1/2, 1/2]$. Give an example of a continuous map $r : X \longrightarrow Y$ such that $r(x) = x$ for each $x \in Y$.

JRF IN MATHEMATICS 2012

TEST CODE MTB

(1) Let A and B be $n \times n$ matrices with real entries. Show that the matrix $\begin{pmatrix} A & I \\ I & B \end{pmatrix}$ has rank n if and only if A is nonsingular and $B = A^{-1}$.

(2) Let n be a positive odd integer and let A be a symmetric $n \times n$ matrix of integer entries such that $a_{ii} = 0$, $i = 1, 2, \ldots, n$. Show that the determinant of A is even.

(3) Let A be a $n \times n$ real matrix with $A^2 = A^t$. Show that every real eigen-value of A is either 0 or 1.

(4) Let $\mathbb{Q}$, $\mathbb{R}$ denote the fields of rational numbers and real numbers respectively. Which of the following rings are *not* isomorphic.
 (a) $\mathbb{Q}[x]/ < x^2 + 1 >$ and $\mathbb{Q}[x]/ < x^2 + x + 1 >$.
 (b) $\mathbb{R}[x]/ < x^2 + 1 >$ and $\mathbb{R}[x]/ < x^2 + x + 1 >$. Justify your answer.

(5) Let S_4 denote the group of permutations of $\{1, 2, 3, 4\}$ and let H be a subgroup of S_4 of order 6. Show that there exists an element $i \in \{1, 2, 3, 4\}$ which is fixed by each element of H.

(6) Let R be a commutative integral domain and K be a subring of R. If K is a field and the dimension of R as a vector space over K is finite, then show that R is a field.

(7) Let X be a Banach space. Let $T : X \longrightarrow X$ be a invertible linear operator and $M > 0$ be such that $||T^{-k}|| \leq M$ for all $k \geq 1$. Prove that $\inf_{n \geq 1} ||T^n(x)|| > 0$ for all $x \neq 0$ in X.

(8) Let X be a Hilbert space and (x_n) be a sequence in X. If $||x_n|| \leq 1$ and $x_n \to x$ weakly for some x in X with $||x|| = 1$, then show that $||x_n - x||^2 \to 0$.

(9) Show that the number of bijections f of $\{1, 2, \ldots, n\}$ such that $f(i) \neq i$ for any i is equal to

$$\Sigma_{j=0}^{n}(-1)^j \frac{n!}{j!}.$$

(10) Find the smallest integer $n > 0$ such that 2012 divides $9^n - 1$. (Hint: 251 and 503 are prime numbers).