Test Code : PHB : (Short Answer type) 2013

Junior Research Fellowship in Theoretical Physics and Applied Mathematics

The candidates for Junior Research Fellowships in Applied Mathematics and Theoretical Physics will have to write two papers −Test MMA (objective type) in the forenoon session and Test PHB (short answer type) in the afternoon session.

The PHB test booklet will consist of two parts. The candidates are required to answer Part I and only one of the remaining parts II & III.

The syllabi and sample questions for the test are as follows.

PART-I

Mathematical and logical reasoning

Syllabus

B.Sc. Pass Mathematics syllabus of Indian Universities.

Sample Questions

1. If

$$S = \left\{\frac{p}{q}\right\} + \left\{\frac{2p}{q}\right\} + \left\{\frac{(q-1)p}{q}\right\}$$

where p and q are relatively prime positive integers, then show that $2S$ is divisible by $q-1$. [$\{x\}$= fractional part of x]

2. Let f be a real valued function defined on the interval $[-2, 2]$ as:

$$f(x) = \begin{cases} (x+1)2^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

   i) Find the range of the function.

   ii) Is f continuous at every point in $(-2, 2)$? Justify your answer.

3. Let $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \omega_1 \\ \omega_2 & 0 \end{pmatrix}$, where ω_1 and ω_2 are roots of the equation $x^2 - 2x + 2 = 0$. If $B = A^{99} - A^{97} + A^{95}$, then show that

$$Tr\ (A) = Tr\ (B).$$

4. The position of a particle moving in a plane is given by $x = \sin \omega t$, $y = \cos \alpha\omega t$. Show that the trajectory repeats itself periodically, only if α is a rational number.

5. It is given that $\phi(1) = 2$ and $f(x) = \int_{x^2}^{x} \phi(t)dt$. Find $f'(1)$.

6. X is a uniformly distributed random variable with probability density function

$$f(x) \;=\; \begin{cases} \dfrac{5}{a} & \text{for} \quad -\frac{a}{10} \leq x \leq \frac{a}{10} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

where a is a non-negative constant. If $P(|x| < 2) = 2P(|x| > 2)$, then find a.

7. Find the roots of the equation $z^5 = -i$, and indicate their locations in the complex plane.

8. A ball falling with zero initial velocity on a smooth inclined plane forming an angle α with the horizontal, traverses a distance h before it strikes the plane. The ball rebounds elastically off the inclined plane. At what distance from the impact point the ball will rebound for the second time?

9. Displacement of a particle executing periodic motion is given by $y = 4\cos^2(t)\sin(5t)$. How many harmonic waves need to be superposed to get the above displacement?

10. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + (tan\ x)^{100}}$.

11. Let p, q be two prime numbers each greater than or equal to 5 and $p > q$. Show that $p^2 - q^2$ is divisible by 24.

12. If $f(x, f(y)) = x^p y^q$ for all x, y, then show that $p^2 = q$ and find $f(x)$.

13. Show that the area of the triangle formed by z, iz, and $z + iz$ is $\frac{r^2}{2}$, where $r = |z|$ and $z = a + ib$, with a, b being real nonzero numbers.

14. A particle sliding on a smooth inclined plane requires 4 sec to reach the bottom starting from rest at the top. How much time does it take to cover $\frac{1}{4}th$ distance starting from rest at the top?

15. Given any polynomial $A(x)$ with coefficients in R, show that there exists a polynomial $B(x)$ such that $A(x).B(x) = C(x^2)$, where $C(y)$ is some polynomial in y with coefficients in R.

16. Find the maximum possible value of xy^2z^3 subject to the conditions x, y, $z \geq 0$ and $x + y + z = 3$.

17. A point moves in the (x, y)-plane such that at time $t(> 0)$ it has the coordinates $(1/t, (1 + t)/\sqrt{2})$. Find the time when it comes closest to the origin.

18. A particle of mass $4kg$ is moving under the action of the force $\mathbf{F} = (4\hat{\imath} + 12\hat{\jmath})$ N, where t is the time in seconds. The initial velocity of the particle is $(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$m/s. Calculate the work done by $\mathbf{F}$ during the time interval $0 \leq t \leq 1$.

PART-II

Applied Mathematics

Syllabus

1. *Abstract algebra* : Groups, rings, fields.
2. *Real analysis* : Functions of single and several variables, metric space, normed linear space, Riemann Integral, Fourier series, Integral Transform.
3. *Differential equations* : ODE − Existence of solution, fundamental system of integrals, elementary notions, special functions. PDE upto second order, equations of parabolic, hyperbolic and elliptic type.
4. *Dynamics of particles and rigid bodies* : Motion of a particle in a plane and on a smooth curve under different laws of resistance, kinematics of a rigid body, motion of a solid body on an inclined smooth or rough plane.
5. *Functions of complex variables* : Analytic function, Cauchy's theorem Taylor and Laurent series, singularities, branch-point, contour integration, analytic continuation.
6. *Fluid Mechanics* : Kinematics of fluid, equation of continuity, irrotational motion, velocity potential, dynamics of ideal fluid, Eulerian and Lagrangian equations of motion, stream function, sources, sinks and doublets, vortex, surface waves, group velocity, viscous flow − Navier Stokes equation, boundary layer theory, simple problems.
7. *Probability and statistics* : Probability axioms, conditional probability, probability distribution, mathematical expectations, characteristic functions, covariance, correlation coefficient. Law of large numbers, central limit theorem. Random samples, sample characteristics, estimation, statistical hypothesis, Neyman Pearson theorem, likelihood ratio testing.

Sample Questions

1. Let m and n be positive integers and F is a field. Let f_i $(i = 1, 2, \ldots, m)$ be linear functionals on F^n. For α in F^n, let $T\alpha = (f_1(\alpha), f_2(\alpha), \ldots \ldots, f_m(\alpha))$. Show that T is a linear transformation from F^n into F^m. Also show that every linear transformation from F^n into F^m is of the above form for some f_i $(i = 1, 2, \ldots, m)$.
2. a) Let $X_1 = [1, 2]$ and $X_2 = [0, 1]$. Let d_1 denote the Euclidean metric in X_1 and let $d_2(x, y) = 2|x - y|$ in X_2. Show that (X_1, d_1) and (X_2, d_2) are equivalent metric spaces.

b) Two different metrics on the space $X = x \in R : 0 < x \leq 1$ are defined by $d_1(x,y) = |x-y|$ and $d_2(x,y) = |\frac{1}{x} - \frac{1}{y}|$. Are the spaces (X, d_1) and (X, d_2) equivalent? Give reasons for your answer.

3. a) If G is a group of even order, then prove that it has an element $a \neq e$, satisfying $a^2 = e$.

   b) Let $A = \{a \ : \ a = 9x + 15y, \ \ x\ , y$ are integers and $|a| \leq \ 1000\ \}$. Find the cardinality of A?

4. a) Does there exist a hexagon with sides of lengths $2, 2, 3, 3, 4, 4$ (with certain order) and with each angle equal? Justify your answer.

   b) Let a, b be positive integers with a odd. Define the sequence $\{u_n\}$,

$$\begin{aligned} u_{n+1} &= \frac{1}{2}u_n, \qquad \text{if } u_n \text{ is even} \\ &= u_n + a \qquad \text{otherwise} \end{aligned}$$

   Show that $u_n \leq a$ for some $n \in N$.

5. a) A body of mass M is suspended from a fixed point O by a light inextensible string of length l and mass m.

   (i) Find the tension in the rope at a distance z below O

   (ii) If the point of support now begins to rise with velocity $2g$, what is the tension in the string?

   b) A stone of mass m is thrown vertically upwards with initial speed V. If the air resistance at speed v is mkv^2, where k is a positive constant, show that the stone returns to its starting point with speed $V\sqrt{1 + kV^2/g}$.

6. a) Use the generating function for Bessel functions to show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2x}{\pi}} \sin x.$$

   b) The Hermite polynomials $H_n(x)$ may be defined by the relation

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

   Using the above definition, find a relation connecting $H_{n-1}(x)$, $H_n(x)$ and $H_{n+1}(x)$.

7. Show that if the solution of the ODE

$$2xy'' + (3 - 2x)y' + 2y \ = \ 0$$

is expressed in the form $y = \sum_{n=0}^{\infty} a_n\, x^{n+\sigma}$, σ can take two possible values. Find the relation between a_n and a_{n+1}, and show that one solution reduces to a polynomial.

8. a) Show that $x_n^3 + y_n^3 \rightarrow 0$ implies $x_n + y_n \rightarrow 0$. Is the reverse implication true?

b) A function is defined as follows:

$$\begin{aligned} f(x) &= 0, \qquad \text{where } x \text{ is irrational} \\ &= \frac{1}{q}, \qquad \text{where } x = \frac{p}{q}, \end{aligned}$$

where p, q are two positive integers prime to each other. Show that $f(x)$ is continuous at $x = a$, if a is irrational and $f(x)$ is discontinuous at $x = a$, if a is rational.

9. a) Using Laplace transformation, solve the differential equation

$$\frac{d^2y(t)}{dt^2} + at\frac{dy(t)}{dt} - 2ay(t) = 1$$

subject to the conditions $y(0) = y'(0) = 0, \quad a > 0$ being a constant.

b) Using $f(x) = x^2, -\pi < x < \pi$, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

10. Show that

$$\frac{\cos 2nx}{\sqrt{\pi/2}}, \frac{\sin 2nx}{\sqrt{\pi/2}}, \quad (n = 1, 2, ...)$$

are orthogonal functions on $[0, \pi]$. Hence or otherwise show that the Fourier series expansion of $\cos x$, when $0 < x < \pi$, is given by

$$\cos x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin 2nx}{4n^2 - 1}.$$

11. Show that

$$\frac{\sin x}{x} = \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2\pi^2}\right)$$

(Hint: if n = even integer and $n = 2m$, $x^n - 1 = 0$ has two real roots and $m - 1$ complex conjugate pairs)

12. a) Find the set of all possible z in C when it is given that the group (with respect to multiplication) generated by the complex number $z = re^{i\theta}$ is finite.

b) Let $A : k \times k$ be real symmetric matrix and x_n be a sequence in R^k. Show that if A is positive definite then $x_n' A x_n \rightarrow 0 \Rightarrow x_n \rightarrow 0$.

13. Calculate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Show that the eigenvectors are orthogonal and form a basis in C^2. Find also a unitary matrix U.

14. Find the integral surface of the equation

$$(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$$

passing through the curve $xz=a^3$, $y=0$, where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$.

15. a) Use Contour integration to find the value of $\int_0^\infty \frac{dx}{x^4+1}dx$

b) Consider a function $f(z)=\dfrac{sinz}{z-\pi}$. Identify the nature of singularity of this function and obtain a Laurent series expansion about the singularity.

16. a) Evaluate $\int\int \sqrt{4x^2-y}\ dxdy$ over the triangle formed by the straight lines $y=0, x=1, y=x$.

b) Use Laplace transform to solve the following differential equation

$$Y''(t)-Y(t)=1\ +e^{3t},$$

given $Y(0)=-\frac{7}{8}$, $Y'(0)=0$. [Here, $Y'(t)=\frac{dY}{dt}, Y''(t)=\frac{d^2Y}{dt^2}$]

17. a) The velocity along the centre line of a nozzle of length L is given by

$$u=2t\left(1-\frac{0.5x}{L}\right)^2$$

where u is the velocity in m/s, t is the time in seconds from the commencement of flow and x is the distance of the inlet from the nozzle. Find the convective acceleration and the local acceleration when $t=3,\ x=L/2,\ L=0.8$m.

b) Identify the type of the partial differential equation

$$\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}=x-y$$

and find its generalised solution including the particular integral.

18. Water flows through a circular pipe. At one section, diameter of the pipe is $0.3m$, static pressure is 260 KPa gauge, velocity is $3m/sec$ and the elevation is $10m$. The pipe diameter at the other section is 0.15 m with zero elevation. Find the pressure at the downstream section neglecting the frictional effect. Density of water may be assumed as 999 Kg/m^3.

19. For steady state axisymmetric flow of incompressible fluid through a constricted tube, sketch the pressure distribution along the axis (the dash-dot line) of the tube for (a) low Reynolds number (Re=1) and (b) high Reynolds number (Re=1500).

20. a) Let a number be drawn at random from $\{1, 2, ...n\}$. Call it X. A number is drawn at random from $\{1, 2, ...x\}$. Call it Z. Find $E(z)$ and Var (z).

    b) Let $X \sim Exponential(\lambda)$ with $\lambda > 0$. Show that for all $t > 0$, the value of $E(X/X > t) - t$ does not depend on t.

PART-III

Theoretical Physics

Syllabus

1. **Classical Mechanics**

   Mechanics of a particle and system of particles−conservation laws− scattering in a central field, Lagrange's equation and their applications, Hamilton's equation, Canonical transformation, Special theory of relativity, Small oscillation, Vibration & acoustics.

2. **Electromagnetic theory**

   Electrostatics, Magnetostatics, Classical electrodynamics, Maxwell's equations, Gauge transformation, Poynting's theorem, Wave equation and plane waves, Radiating system and scattering.

3. **Statistical Physics & Condensed Matter Physics**

   Thermodynamic equilibrium, Partition functions, Density matrix, Phase transitions, Spin systems, Statistical fluctuations, Band theory of electrons, Semiconductor Physics.

4. **Quantum Mechanics and Quantum Field Theory**

   Inadequacy of classical physics, Schroedinger wave equation, General formalism of wave mechanics, Exactly soluble eigenvalue problems, Approximation methods, Scattering theory, Time dependent perturbation theory, Symmetries and Conservation Laws, Relativistic equations, Klein-Gordon/Dirac equations, Lagrangian field theory, Examples of quantum field theory $-\ \phi^4$, Quantum electrodynamics.

5. **Elementary Particles**

   Elementary particles, Weak and strong interactions, Selection rules, CPT theorem, Symmetry Principles in Particle Physics.

Sample questions for Theoretical Physics

1. Consider a particle of mass m constrained to move on a frictionless circular loop of radius R. The loop is rotated with angular frequency ω about a vertical axis passing through its center. [Assume that at any instant of time the mass is at a position $\theta(t)$].

   i) Set up the Lagrangian for this system (upto a constant).

   ii) Write down the equation of motion for the particle.

iii) Find out the equilibrium positions where the mass would settle down when ω changes.

2. a) Let an observer on Earth see two particles born from a source and moving in opposite directions. The particle moving in the left has a velocity $\frac{1}{3}c$ km/sec with respect to the observer, where c is the velocity of light in vacuum . The observer sees the rate of increase of the distance between the particles as c km/sec. Find the relative velocity of one particle with respect to the other.

   b) Show that the transformation

$$\begin{aligned} q &= \sqrt{2P}\sin\theta \\ p &= \sqrt{2P}\cos\theta \end{aligned}$$

   is a canonical transformation.

3. Two pendulums of mass m and length l are coupled by a massless spring of spring constant k. The unstretched length of the spring is equal to the distance between the supports of the two pendulums. Set up the Lagrangian in terms of generalized coordinates and velocities and derive the equations of motion .

4. A uniform flat disc of mass M and radius r rotates about a horizontal axis through its center with angular speed ω_0. A chip of mass m breaks off the edge of the disc at an instant such that the chip rises vertically above the point at which it broke off. How high does the chip rise above the point before it starts to fall off? What is the final angular momentum of the disc?

5. a) It is known from special theory of relativity that the Doppler shift is given $\lambda' = \lambda\sqrt{(1+V/c)\,(1-V/c)}$ where λ is the emitted wavelength as seen in a reference frame at rest with respect to the source, and λ' is the wavelength measured in a frame moving with velocity V away from the source along the line of sight. Show that the Doppler shift in wavelength is $z \approx V/c$ for $V < c$ with c being the velocity of light.

   b) A rod of proper length L_0 is at rest in a reference frame S'. It lies in the (x', y') plane and makes an angle of $sin^{-1}\frac{3}{5}$ with the x' axis. If S' moves with constant velocity v parallel to the x axis of another frame S:
   (a) What is the value of v if, as measured in S, the rod is at 45^0 to the x axis? (b) What is the length of the rod as measured in S under these conditions?

6. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction $\vec{B}$, the vector $\vec{B}$ being perpendicular to the plane (x, y). At time $t = 0$, a connector starts sliding translation wise from the vertex of the

parabola with a constant acceleration f. Find the e.m.f. of electromagnetic induction in the loop thus formed as a function of y.

7. a) Consider a possible solution to Maxwell's equation given by

$$\vec{A}(\vec{x},t) \;=\; A_0\; e^{i(\vec{k}.\vec{x}-\omega t)},\;\; \phi(\vec{x},t)=0$$

where $\vec{A}$ is the vector potential and ϕ is the scalar potential. Further, suppose $\vec{A}_0, \vec{k}$ and ω are constants in space-time. Give, and interpret the constraints on $\vec{A}_0, \vec{k}$ and ω imposed by each of the Maxwell's equations given below

$$\text{a) } \vec{\nabla}.\vec{B} \;=\; 0, \qquad \text{b) } \vec{\nabla}\times\vec{\mathrm{E}}+\frac{1}{\mathrm{c}}\frac{\partial\vec{\mathrm{B}}}{\partial \mathrm{t}} \;=\; 0,$$

$$\text{c) } \vec{\nabla}.\vec{E} \;=\; 0, \qquad \text{d) } \vec{\nabla}\times\vec{\mathrm{B}}-\frac{1}{\mathrm{c}}\frac{\partial\vec{\mathrm{E}}}{\partial \mathrm{t}} \;=\; 0,$$

b) A parallel plate capacitor with plate separation d is filled with two layers of dielectric material a and b . The dielectric constant and conductivity of materials a and b are ϵ_a, σ_a and ϵ_b, σ_b respectively. The thicknesses of the materials a and b are d_a and d_b respectively.

(i) Calculate the electric fields in the materials a and b.

(ii) Find the current flowing through the capacitor.

8. a) Consider a gas in a container obeying Van der Waals gas equation

$$(P+\frac{a}{v^2})(V-b)=nRT$$

where a and b are constants. The initial volume is V and then isothermally it is compressed to one half of its volume. Find the work done by the gas.

b) If the effective density of states in valence band is eight times that in conduction band in a pure semiconductor at 27^oC, find the shift of Fermi level from the middle of the energy gap assuming low concentration of electrons and holes in the semiconductor. (Boltzmann's constant $K_B = 1.33\times 10^{-23} J/\text{ mole}^o\ K,\ \ln 2 = .693$)

9. a) Consider a line of $2N$ ions of alternating charges $\pm q$ with a repulsive potential A/R^n between nearest neighbours in addition to the usual Coulomb potential. Neglecting surface effects find the equilibrium separation R_0 for such a system. Let the crystal be compressed so that R_0 becomes $R_0(1-\delta)$. Calculate work done in compressing a unit length of the crystal to order δ^2.

b) An assembly of N particles with spin $\frac{1}{2}$ and magnetic moment μ_0 is in a uniform applied magnetic field. The spins interact with the applied field (otherwise free).

i) Express the energy of the system as a function of its total magnetic moment and the applied field.

ii) Find the total magnetic moment and the energy, assuming the system is in thermal equilibrium at temperature T.

10. a) Two levels in an atom whose nuclear spin is $I = 3$, have the designations $^2D_{3/2}$ and $^2P_{1/2}$. Find the expected number of components in the hyperfine structure of the corresponding spectral line.

b) A beam of electrons with kinetic energy 1 keV is diffracted as it passes through a polycrystalline metal foil. The metal has a cubic crystal structure with a spacing of $1A^0$. Calculate the wave length of electron and the Bragg angle for the first order diffraction. Take m, h, c the mass of the electron, Planck's constant and speed of light respectively as follows: $mc^2 = .5$Mev, $c = 3 \times 10^8$m/s, $h = 6.6 \times 10^{-34}$Js. Also take $1eV = 1.6 \times 10^{-19}$J.

11. a) Consider the Dirac Hamiltonian $H = c\vec{\alpha}.\vec{p} + \beta mc^2 + V(r)$ where the symbols have their usual meaning. Show that $[H, \vec{L}] = -i\hbar c(\vec{\alpha} \times \vec{p})$.

b) Consider a particle of mass m in a one dimensional infinite potential well of width a:

$$\begin{aligned} V(x) &= 0, \qquad 0 \leq x \leq a \\ &= \infty, \qquad \text{otherwise} \end{aligned}$$

The particle is subjected to a perturbation of the form $W(x) = a\omega_0\delta(x - a/2)$. Calculate the changes in the energy levels to first order in ω_0.

12. Draw the momentum space Feynman diagrams for electron-electron scattering in lowest order in electromagnetic coupling (Møller scattering). Write down the momentum space scattering amplitudes. (Note: Explicit expressions of the propagators need not be given but the momentum arguments should be clearly indicated.)

13. Consider the real free Klein-Gordon Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - m^2\phi^2\right).$$

Using the expansion,

$$\phi(x) = \sum_k \sqrt{\frac{\hbar c^2}{2V\omega_k}}\left(a(\vec{k})e^{-ik^\mu x_\mu} + a^\dagger(\vec{k})e^{ik^\mu x_\mu}\right)$$

show that the Hamiltonian is given by

$$H = \sum_k \hbar\omega_k\left[a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}\right].$$

Note: $k^{\mu}x_{\mu} = \omega_k t - \vec{k}.\vec{x}$.

14. a) A particle is initially in its ground state in a one-dimensional harmonic oscillator potential, $V(x) = \frac{1}{2}\omega x^2$. If the coupling constant ω is suddenly doubled, calculate the probability of finding the particle in the ground state of the new potential.

b) Consider a particle of mass m inside a one-dimensional box of length a. Let at $t = 0$, the state of the particle is given by the following:

$$\begin{aligned} \psi(x,0) &= A\cos(\frac{2\pi x}{a})\sin(\frac{3\pi x}{a}), \qquad 0 \leq x \leq a \\ &= 0, \qquad \text{everywhere else.} \end{aligned}$$

i) Calculate the value of A.

ii) What is the average energy of the system at $t = 0$?

iii) What is the smallest positive time t_0 for which $\psi(x, t_0)$ would be orthogonal to $\psi(x, 0)$?

iv) If a measurement for energy is performed at $t = 0$, what is the probability of getting the value of energy as $\frac{h^2}{2ma^2}$?

c) Let $S_{\pm} = S_x \pm iS_y$ where S_x, S_y and S_z are Pauli spin matrices. If $|\pm, \frac{1}{2}>$ are eigenvectors of S_z, then find $S_{\pm}|\pm, \frac{1}{2}>$.

15. a) In the following Strong Interactions identify, with proper analysis, the unknown particle X :

(i) $\pi^- + p \rightarrow K^0 + X$,

(ii) $\bar{K}^- + p \rightarrow K^+ + X$.

b) State whether in the following processes parity, strangeness, isospin and third component of isospin is conserved. Also, clarify the process as Strong, Electromagnetic, Weak or totally forbidden.

(i) $\Sigma^0 \rightarrow \Lambda^0 + \gamma$,

(ii) $\pi^- + p \rightarrow n + \pi^0$.

c) Explain why the following processes are forbidden:

(i) $p \rightarrow e^+ + \gamma$,

(ii) $\pi^0 + n \rightarrow \bar{K}_0 + \Sigma^0$.