JRF (Quality, Reliability and Operations Research): 2013
INDIAN STATISTICAL INSTITUTE

INSTRUCTIONS

The test is divided into two sessions (i) Forenoon session and (ii) Afternoon session. Each session is for two hours. For the forenoon session question paper, the test code is **MMA** and for the afternoon session question paper, the test code is **QRB**. Candidates appearing for JRF (QROR) should verify and ensure that they are answering the right question paper.

For test **MMA**, see a different Booklet. For test **QRB**, refer to this Booklet only.

The test **QRB** is of short answer type. It has altogether 18 questions, 3 questions from each of the 6 groups as given in the syllabus. A candidate has to choose 3 groups (out of 6) and answer 2 questions from each chosen group.

OUTLINE OF THE SYLLABUS FOR QRB

The syllabus for JRF (QROR) will include the following subject groups: (A) Statistics, (B) Statistical Quality Control, (C) Reliability, (D) Operations Research, (E) Quality Management and Systems and (F) Mathematics. A broad coverage for each of the above subject groups is given below.

A. **Statistics**: Elementary probability and distribution theory, Bivariate distributions, Multivariate normal distribution, Regression and linear models, Estimation, Test of hypothesis, Design of experiments (block design, full and fractional factorial designs), Markov chain.

B. **Statistical Quality Control**: Statistical process control - attribute and variable control charts, Control chart with memory

(CUSUM, EWMA etc.) (univariate only), Multivariate control chart, Process capability analysis, QC tools, Acceptance sampling.

C. **Reliability**: Coherent systems and system reliability, Hazard function, Failure time distribution, Censoring schemes, Estimation and testing in reliability, Replacement models, Repairable system.

D. **Operations Research**: Linear programming (basic theory, simplex algorithm and its variants, duality theory, transportation and assignment problem), Non-linear programming-basic theory, Inventory control theory (EOQ models, dynamic demand model, concept of probabilistic models), Queuing theory (M/M/s), Game theory (two person zero-sum game).

E. **Quality Management and Systems**: Quality concepts, Service quality, Quality metrics, Quality dimensions, Quality loss, Evolution of quality, Quality planning, Quality costs, Total quality management, Six sigma, Quality management system concepts and standards.

F. **Mathematics**: Set theory, Calculus, Vectors and matrices, Differential equations - all at B.Sc. level.

Sample Questions for Group A (Statistics)

1. Suppose die A has 4 red faces and 2 green faces while die B has 2 red faces and 4 green faces. Assume that both the dice are unbiased. An experiment is started with the toss of an unbiased coin. If the toss results in a Head, then die A is rolled repeatedly while if the toss of the coin results in a Tail, then die B is rolled repeatedly. For $k = 1, 2, 3, \ldots$, define X_k to be the indicator random variable such that X_k takes the value 1 if the k-th roll of the die results in a red face, and takes the value 0 otherwise.

   i) Find the probability mass function of X_k.

   ii) Calculate the correlation between X_1 and X_7.

2. Fold over a 2^{5-2} design to construct a 2^{6-2} design. Write the complete defining relation of the resulting design. What is its resolution? Is the resulting design a minimal design?

3. (a) Let $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be the order statistics of a random sample of size 3 from

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta}, & \text{if } 0 \leq x \leq \theta \\ 0, & \text{otherwise,} \end{cases}$$

   for $0 < \theta < \infty$. Let $T_1 = \alpha_1 X_{(1)}$, $T_2 = \alpha_2 X_{(2)}$ and $T_3 = \alpha_3 X_{(3)}$ be unbiased estimators of θ where α_1, α_2 and α_3 are suitable positive numbers. Find the values of α_1, α_2 and α_3. Compare the efficiencies of T_1, T_2 and T_3.

   (b) A manufacturer uses a certain type of electrical components from supplier A and supplier B. A previous study establishes that supplier A produces components whose resistances are Normally distributed with mean of 100 units and standard deviation of 0.1 unit and supplier B produces components whose resistances are Normally distributed with mean of 100.5 units and standard deviation of 0.5 unit. Components are supplied in large batches and are visually identical. A batch has come with no label. The value of the sample mean

$\bar{x}$ of resistance for a random sample of components is to be used to decide whether this batch comes from supplier A or B. This decision problem is regarded as a test of the null hypothesis H_0: Component comes from supplier A against H_1: Component comes from supplier B.

i) If the manufacturer insists that the probabilities of type I error and type II error are to be restricted to at most 5%, find the minimum number of components from the batch to be examined. [Given: $\Phi(1.282) = 0.90$, $\Phi(1.645) = 0.95$ and $\Phi(1.960) = 0.975$ where Φ is the c.d.f. of Standard Normal distribution].

ii) Using the minimum sample size obtained from (*i*), give an explicit expression for the power of the test.

4. (a) A secretary goes to work following one of the three routes A, B and C. Her choice of route is independent of weather. If it rains, the probability of arriving late following A, B and C are 0.06, 0.15 and 0.12, respectively. The corresponding probabilities if it does not rain are 0.05, 0.10 and 0.15. Assume that, on an average, one in every four days is rainy.

   i) Given that she arrives late on a day without rain, what is the probability that she took route C?

   ii) Given that she arrives late on a day, what is the probability that it was a rainy day?

   (b) Let (X, Y) have the joint probability density function given by

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x, y, x+y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the correlation coefficient between X and Y.

5. (a) Consider the linear model

$$Y_i = \beta \frac{i}{n} + \epsilon_i, \ i = 1, \ldots, n,$$

   where β is the unknown regression parameter of interest and ϵ_i's are independent error variables satisfying $E(\epsilon_i) = 0$ and

$Var(\epsilon_i) = \sigma^2$ (unknown). Find Best Linear Unbiased Estimator of β.

(b) Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

i) Calculate $P[X_3 = 1 | X_0 = 0]$.

ii) Check whether the chain is irreducible.

6. (a) Consider a random sample of size 1 from the population with density

$$f(x; \theta) = 2(\theta - x)/\theta^2, \; 0 \leq x < \theta, \; \theta > 0.$$

i) Obtain the maximum likelihood estimate of θ (by checking if it maximizes the likelihood).

ii) Find an unbiased estimator of θ.

(b) The average yield of grain in an experiment with treatment A is claimed to be greater than that with treatment B. Describe a statistical test for this claim, based on yield data from n_A and n_B experiments with treatment A and B, respectively. Clearly specify the null and alternative hypotheses and the underlying assumptions needed.

Sample Questions for Group B (Statistical Quality Control)

1. (a) For a single sampling plan with curtailed inspection associated with rejection, derive the expression for ASN as a function of product quality.

   (b) Derive the following properties of the binomial OC function of a single sampling plan with sample size n and acceptance number c at a process average p:

   $$B(c+1, n+1, p) - B(c, n, p) = qb(c+1, n, p)$$

   where

   $$B(c, n, p) = \sum_{x=0}^{c} b(x, n, p) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x}.$$

2. (a) Show that if $\lambda = 2(w+1)$ for the EWMA control chart, the chart is equivalent to a w period moving average control chart in the sense that the control limits are identical for large t.

   (b) For a control chart of a normally distributed quality characteristic if all of the next seven points fall on the same side of the center line, we conclude that the process is out of control. (i)What is the α risk? (ii)If the mean shifts by one standard deviation and remains there during collection of the next seven samples then find what is the β risk associated with this decision rule.

3. (a) Describe any one method for estimating the process capability indices C_p and C_{pk} when the quality characteristic is a non-normal variable.

   (b) A process operates at the mid-specification for a variable quality characteristic. Show that the extent of off-specification and the associated loss will be minimum assuming that the natural variation of the process is more than the desired engineering tolerance and the losses incurred per unit on both sides of the specification limits are identical.

4. (a) Describe a procedure for estimating process capability when each subgroup consists of an individual unit.

   (b) In designing a fraction nonconforming chart with centre line at $\bar{p} = 0.20$ and 3-sigma limits, (*i*) what sample size (n) is required to yield a positive lower control limit and (*ii*) what value of n is necessary to obtain a probability of 0.50 for detecting a shift in the process $\bar{p}$ to 0.26?

Sample Questions for Group C (Reliability)

1. (a) The UN security council consists of five permanent members having 7 points each and ten temporary members having 1 point each. In order to get a resolution passed, at least 39 points are needed in favour. Present this problem (of getting a resolution passed) as a coherent system with 15 components and obtain the structural importance of the components. Find a non-trivial modular decomposition of the system.

   (b) Prove that the min path sets of a coherent system are the min cut sets of its dual and vice versa.

2. (a) The hazard rate for an item is given by

$$h(x) = \begin{cases} a, & \text{if } 0 < x \le x_0 \\ a + b(x - x_0), & \text{if } x > x_0, \end{cases}$$

   where a, b, x_0 are positive constants. Derive the reliability function. For known x_0 and based on random right censored lifetime data, write down the likelihood function for estimating a and b.

   (b) Consider the Weibull distribution with reliability function

$$R(t) = e^{-\alpha t^{\beta}}, t > 0, \alpha > 0, \beta > 0.$$

   i) When does the distribution have decreasing failure rate?

   ii) Obtain an expression, possibly in the form of an integral, for the mean remaining life function for this distribution.

3. (a) Suppose the n components of a parallel system have independent exponential lifetimes with failure rates as $\lambda_1, \cdots, \lambda_n$. Compute the system reliability function and the mean system lifetime.

   (b) Suppose that two independent systems, with identical and exponentially distributed lifetime with failure rate 3×10^{-7} per hour, are either to be placed in active parallel or a standby configuration. For $t = 7$ years, derive the reliability gain of one over the other configuration.

4. (a) Consider a series system with two independent components each with strength having exponential distribution with parameter α. The system is subjected to a random stress following exponential distribution with parameter β. A component fails when the stress exceeds its strength. Find the reliability of the system.

   (b) Suppose n independent units each having *exponential* (λ) life distribution are put on a life test for a pre-fixed time T_0. Find the distribution of the number of failures during the test. Derive the maximum likelihood estimate of expected number of such failures.

5. (a) An equipment is replaced by a spare in the event of its failure or having reached the age of T_0, whichever is earlier. Let F be the lifetime distribution of the equipment. Derive the expectation of this replacement time.

   (b) Consider a one-unit device and suppose that there are n spares stocked at time 0. When the operating unit fails, it is replaced by a spare having independent and identical life distribution given by the density $f(x) = \lambda e^{-\lambda x},\ \lambda > 0, x > 0$. Assume that all the replacements are instantaneous. Derive an expression to determine the minimum value of n such that the probability of uninterrupted service over time interval $(0, t]$ is more than 0.95.

Sample Questions for Group D (Operations Research)

1. (a) Consider the following problem:

$$\text{minimize } c'x, \text{ subject to } Ax \geq b, x \geq 0,$$

where $A \in R^{m \times n}, c \in R^n$ and $b \in R^m$.

i) Show that the optimal solution to this linear programming problem is equivalent to solving a system of equations in non-negative variables.

ii) If the optimal solution is not unique then show that there cannot be finitely many optimal solutions to the linear programming problem.

(b) The set of all feasible solutions to a linear programming problem is a convex set.

2. (a) If the i^{th} row of the payoff matrix of an $m \times n$ rectangular game be strictly dominated by a convex combination of the other rows of the matrix, then show that the deletion of the i^{th} row from the matrix does not affect the set of optimal strategies for the row player.

(b) Derive the waiting time distribution in an $M/M/1$ queue.

(c) Show that a balanced transportation problem always has a feasible solution.

3. (a) A contractor has to supply a certain product at the rate of R units of per/day. He sets up the production run for k units per day at a cost of Rs. c_1 per set up. Let the cost of holding one unit in inventory per day be is Rs. c_2. What would be the optimal interval between production runs?

(b) Describe a situation where the problem of machine interference arises. How can we compute the operative utilization and machine utilization from the Finite Queue table when 16 automatic machines are looked after by 3 operatives with service factor of 0.25? From Finite Queue table we find the value of F as 0.733 for M=3. What are the assumptions we have made?

4. (a) Consider the problem:

$$\text{minimize } \frac{p^t x + \alpha}{q^t x + \beta}, \;\; q^t x + \beta \neq 0$$

$$\text{subject to } Ax = b$$
$$x \geq 0,$$

where $p, q \in R^n$, $b \in R^m$, $A \in R^{m \times n}$ and $\alpha, \beta \in R$.
Formulate the above problem as linear programming problem.

(b) Let $A \in R^{m \times n}$. Show that exactly one of the following two systems has a solution:

System 1 : $Ax < 0$ for some $x \in R^n$
System 2 : $A^t y = 0, \; y \geq 0$ for some $y \in R^m$.

(c) Solve the following problem:

$$\text{maximize } f(x_1, x_2, x_3) = x_1 x_2^2 x_3$$
$$\text{subject to } x_1 + x_2 + x_3^2 = k,$$

where $k > 0$ is fixed and x_1, x_2, x_3 are positive real numbers.

Sample Questions for Group E (Quality Management and Systems)

1. International Standards of the QMS promote the adoption of a process approach when developing, implementing and improving the effectiveness of a quality/environment management system, to enhance customer satisfaction by meeting customer requirements. What is meant by process approach and what are the generic requirements of the QMS in this context?

2. "Six Sigma initiative has given the organizations a well defined, disciplined problem solving framework" - justify the statement citing the commonly used templates of problem solving.

3. (a) Define Quality costs and identify its elements.

   (b) State and briefly explain Deming's 14 points for improvement of quality in an organization.

4. Discuss the eight basic principles of quality management as considered in the ISO 9000:2008 Quality Management System Standards.

Sample Questions for Group F (Mathematics)

1. (a) If a set contains $(2n+1)$ elements, show that the number of subsets containing more than n elements of this set is 2^{2n}.

   (b) Let $x < 1$, Then, show that

$$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\cdots\infty=\frac{1+2x}{1+x+x^2}.$$

2. (a) Let B be a nonsingular matrix of order n and $D = B'B$. Show that if $x'Dx = 0$, then $x = 0$.

   (b) Let $f : Z \to R$ where Z is the set of all integers and R is the set of all real numbers such that $f(n)+f(n-1)+f(n-2) = 0$ for all $n \in Z$ and $f(0) = 1$. Find the value of $\sum_{k=1}^{101} f(k)$.

   (c) Let $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$. Show that

$$A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix},$$

   where n is a positive integer.

3. (a) $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers such that $2x_n^2 + x_n y_n + 2y_n^2 \to 0$ as $n \to \infty$. Show that $x_n \to 0$ and $y_n \to 0$.

   (b) Let $a_{n+1} = \sqrt{2 + a_n}$. Prove that the sequence is increasing or decreasing according as $a_1 < 2$ or $a_1 > 2$. Hence, show that $\lim_{n\to\infty} a_n = 2$.

   (c) Prove that if $a_1, a_2, ..., a_n$ are positive numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

4. (a) Consider the sequence $\{(5^n+4^n)^{\frac{1}{n}},\ n = 1, 2, \ldots\}$. Show that the sequence is bounded and decreasing.

(b) Let $A \in R^{m \times n}$ and $D = AA^t$. Define

$$\tilde{A} = \begin{bmatrix} D & -A \\ A^t & 0 \end{bmatrix}.$$

Show that $x^t \tilde{A} x \geq 0 \ \forall x \in R^{m+n}$.

(c) Find the global minimizers (if any) for the following functions:

$$i)\ f(x,y) = e^{x-y} + e^{y-x}, \text{ and}$$
$$ii)\ f(x,y) = e^{x-y} + e^{x+y}.$$

5. (a) Let Q denote the set of rational numbers. For given constants a, b, c, define the function f on real numbers:

$$f(x) = \begin{cases} ax & \text{if } x \in Q \\ c - bx & \text{if } x \notin Q. \end{cases}$$

Derive necessary and sufficient conditions on the constants a, b, c for each of the following cases:

i) f is a continuous function,
ii) f is continuous exactly at one point and
iii) f is nowhere continuous.

(b) Let

$$g(t) = \int_0^t \frac{1}{x} \sin\left(\frac{tx}{\pi}\right) dx, \quad \text{for } t > 0.$$

Find $g'(\pi)$.

(c) Find the total number of functions $f : \{1, 2, \ldots, 7\} \to \{-1, 1\}$ satisfying

$$\sum_{i=1}^{7} f(i) > 1.$$