- 2-3 search trees
- red-black BSTs
- B-trees

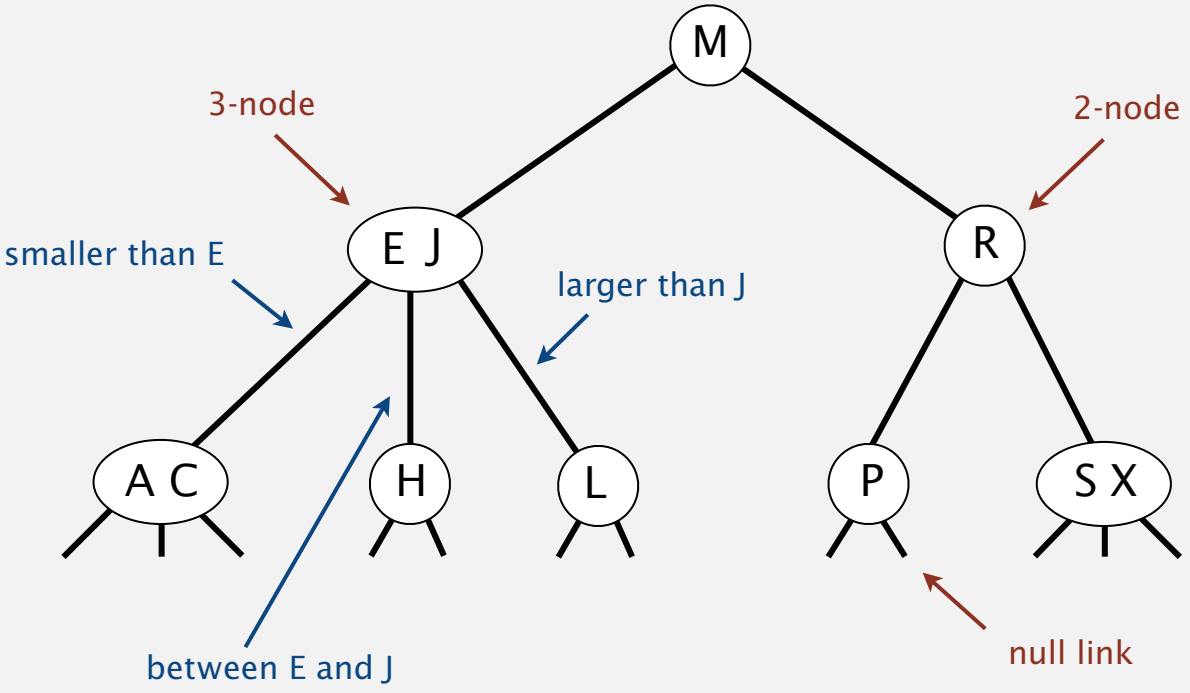
2-3 tree

Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

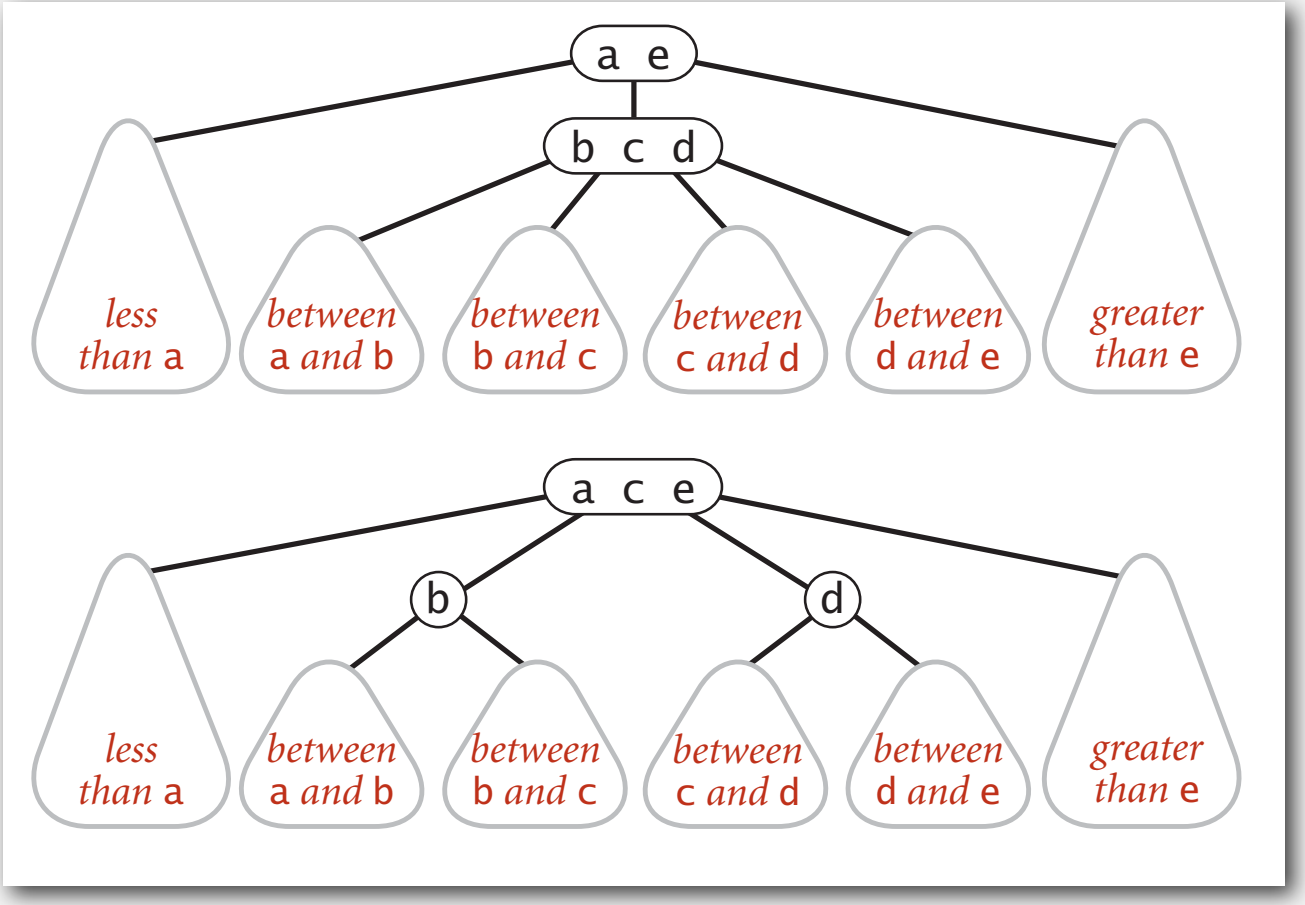
Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.



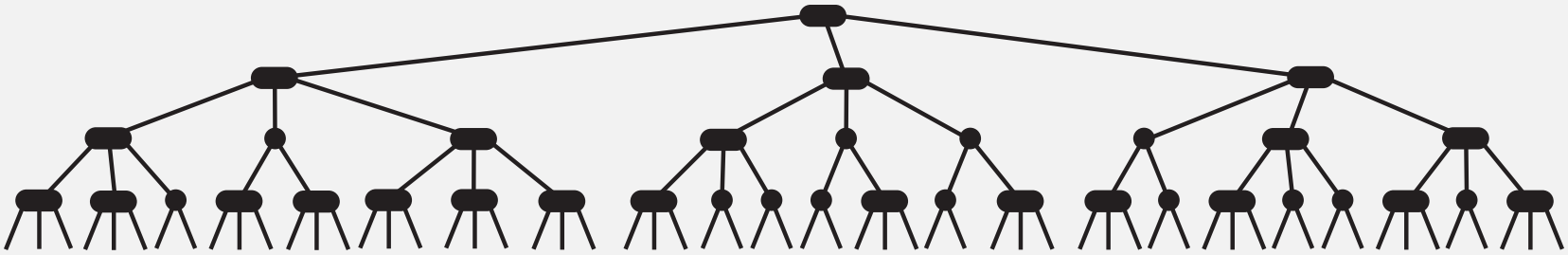
Local transformations in a 2-3 tree

Splitting a 4-node is a local transformation: constant number of operations.



2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

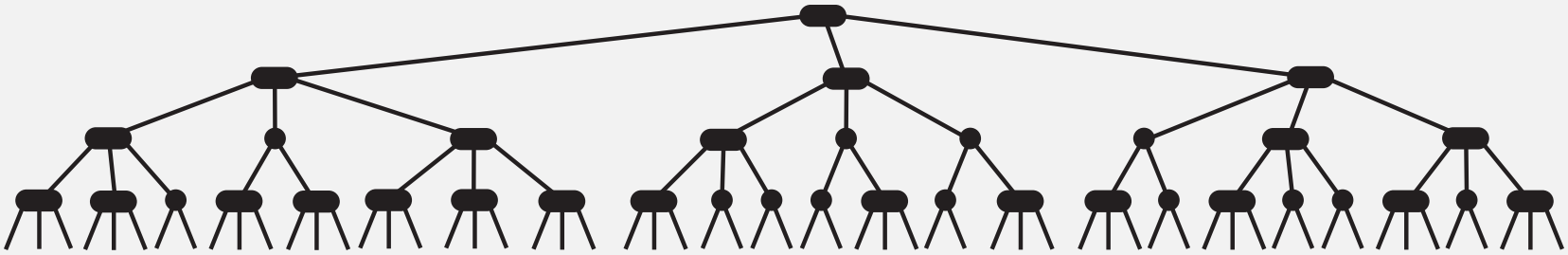


Tree height.

- Worst case:
- Best case:

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Guaranteed logarithmic performance for search and insert.

2-3 tree: implementation?

Direct implementation is complicated, because:

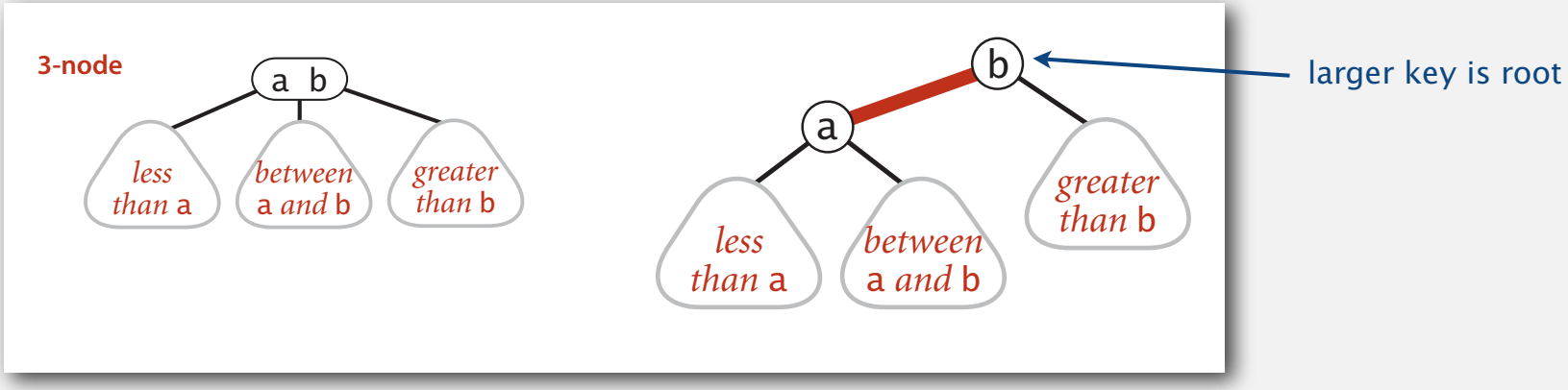
- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

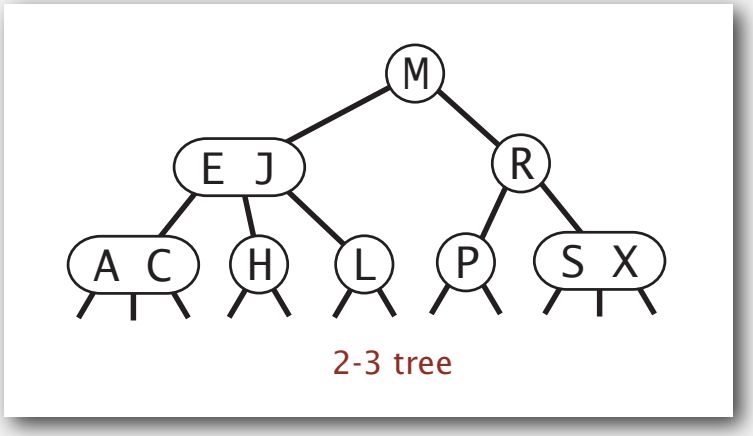
Bottom line. Could do it, but there's a better way.

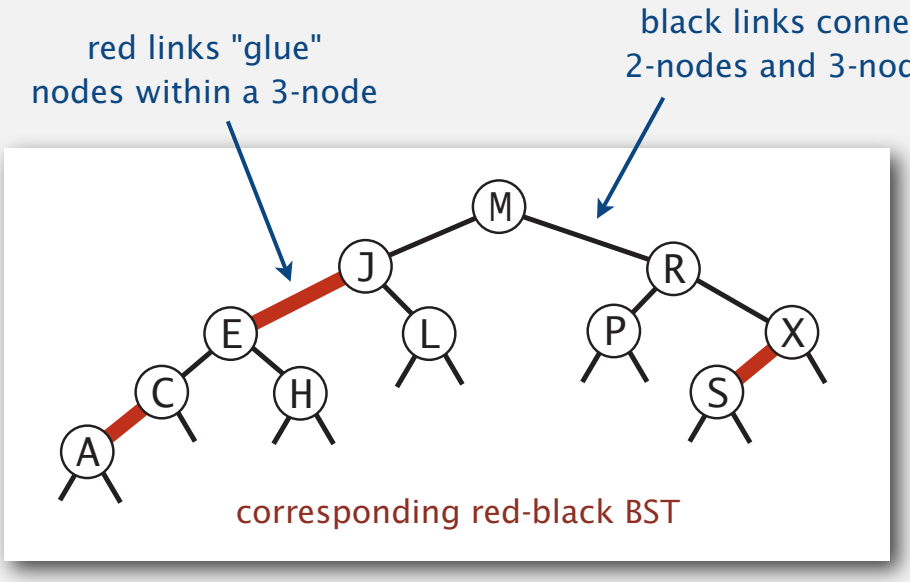
- 2-3 search trees
- **red-black BSTs**
- B-trees

Left-leaning red-black BSTs

1. Represent 2-3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3-nodes.





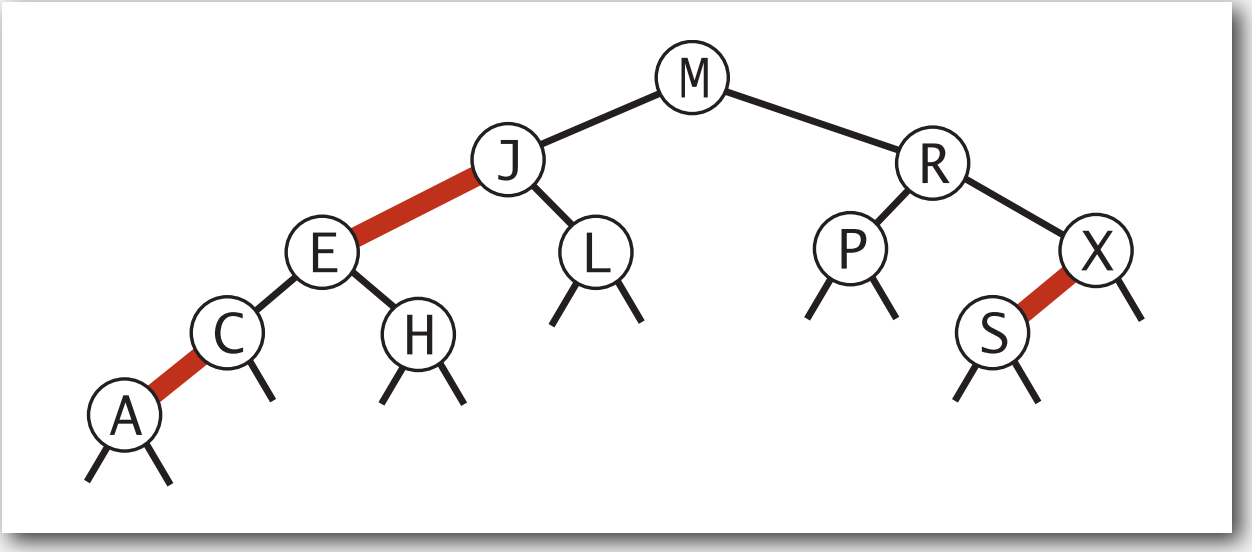


An equivalent definition

A BST such that:

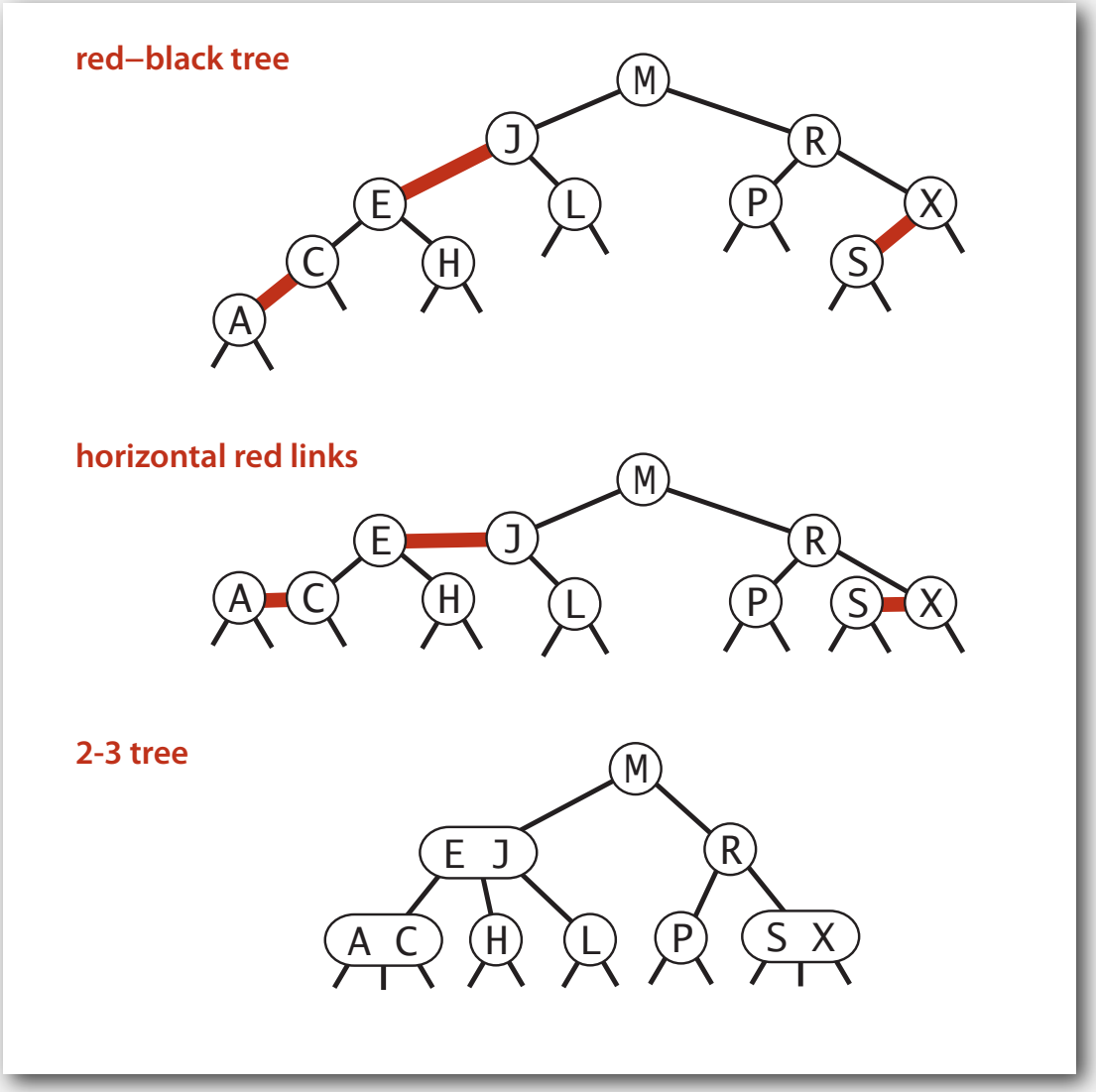
- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

"perfect black balance"



Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1-1 correspondence between 2-3 and LLRB.

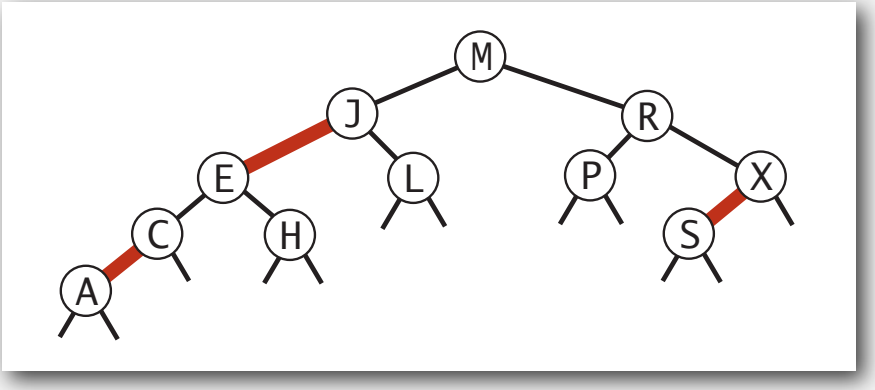


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



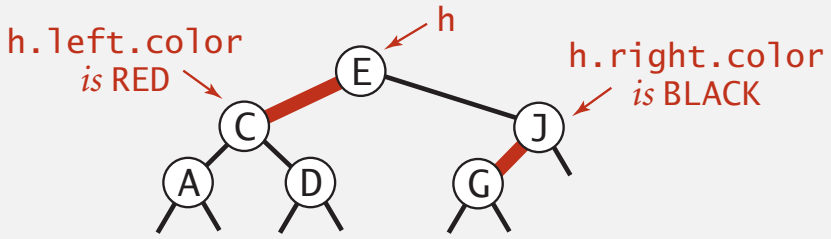
Remark. Most other ops (e.g., ceiling, selection, iteration) are also identical.

Red-black BST representation

Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

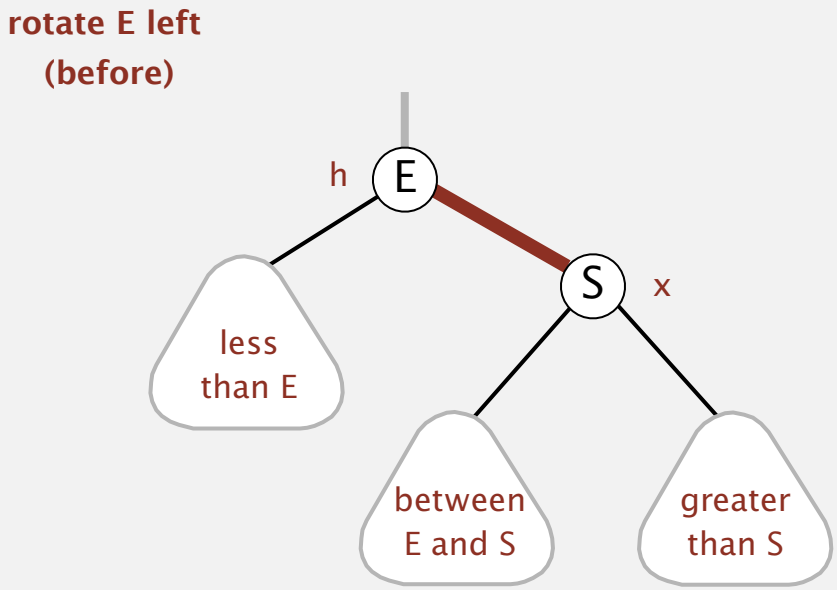
```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

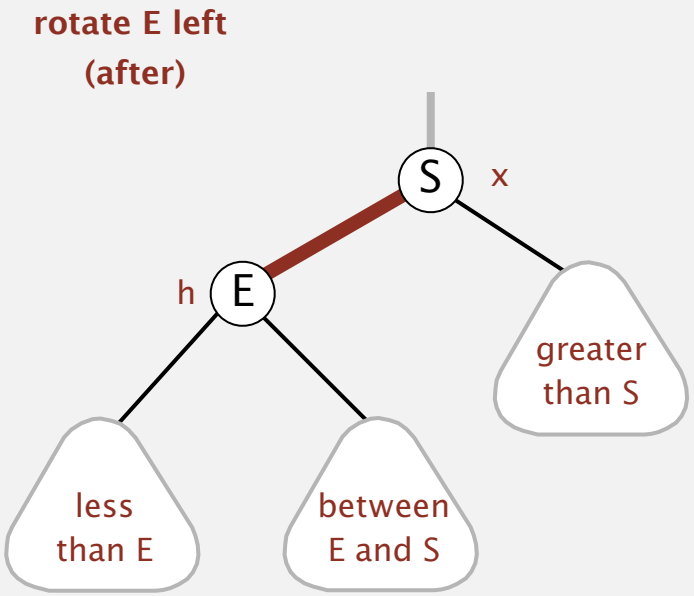



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

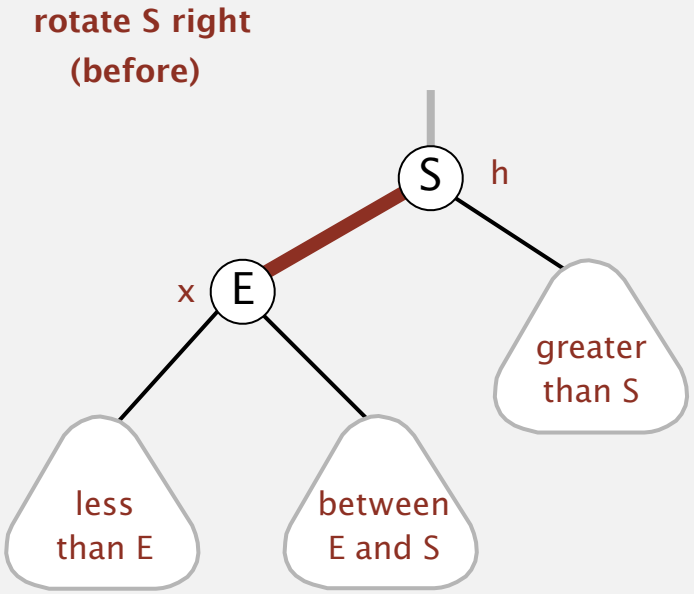



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

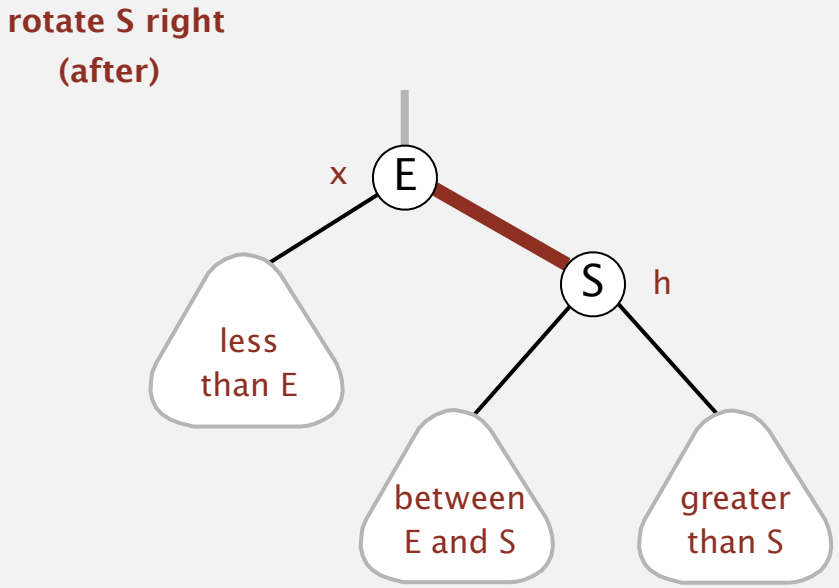



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

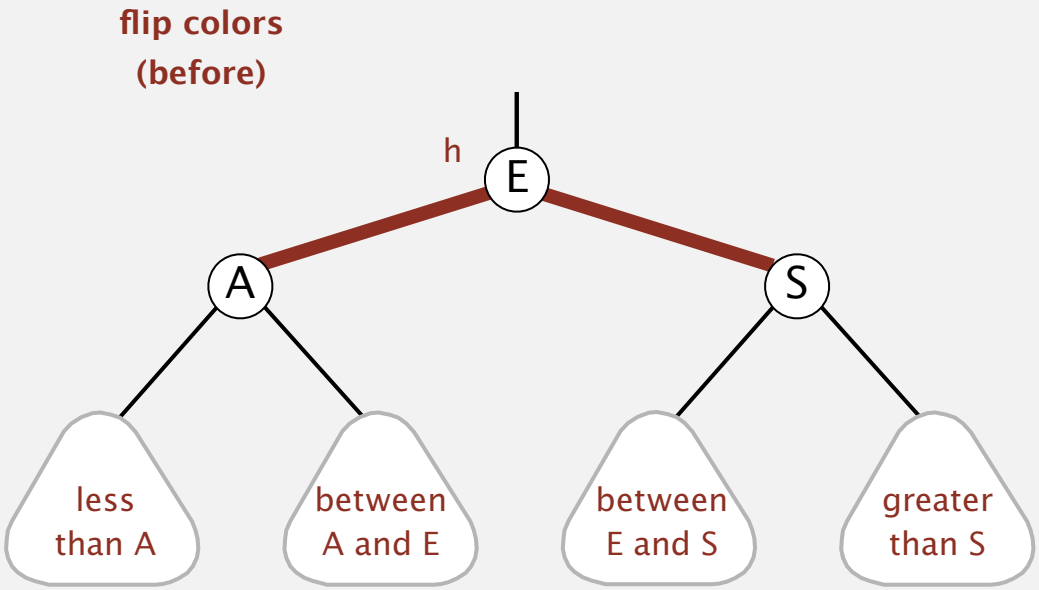



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

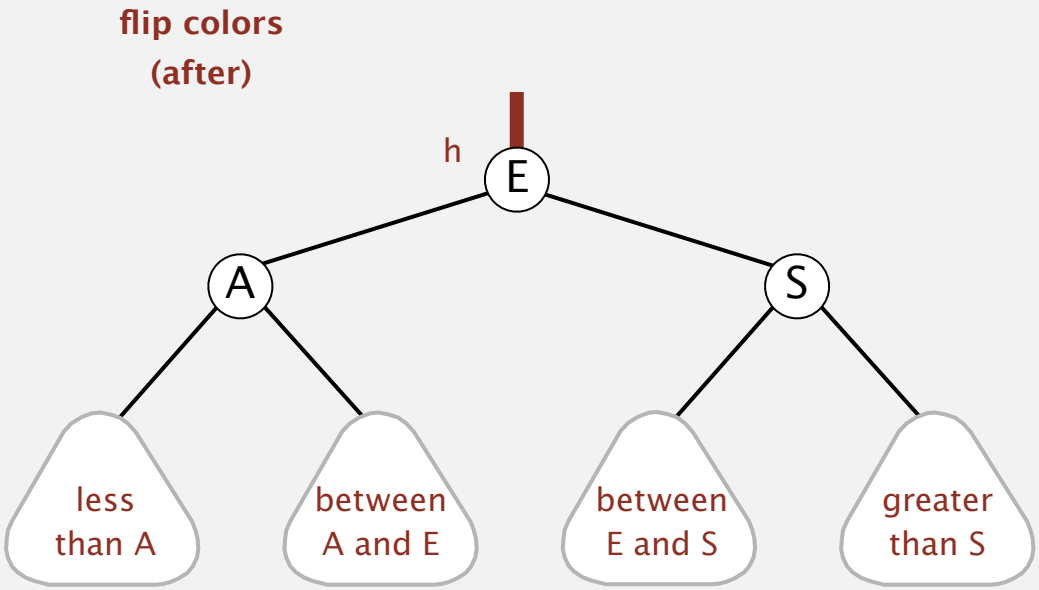



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

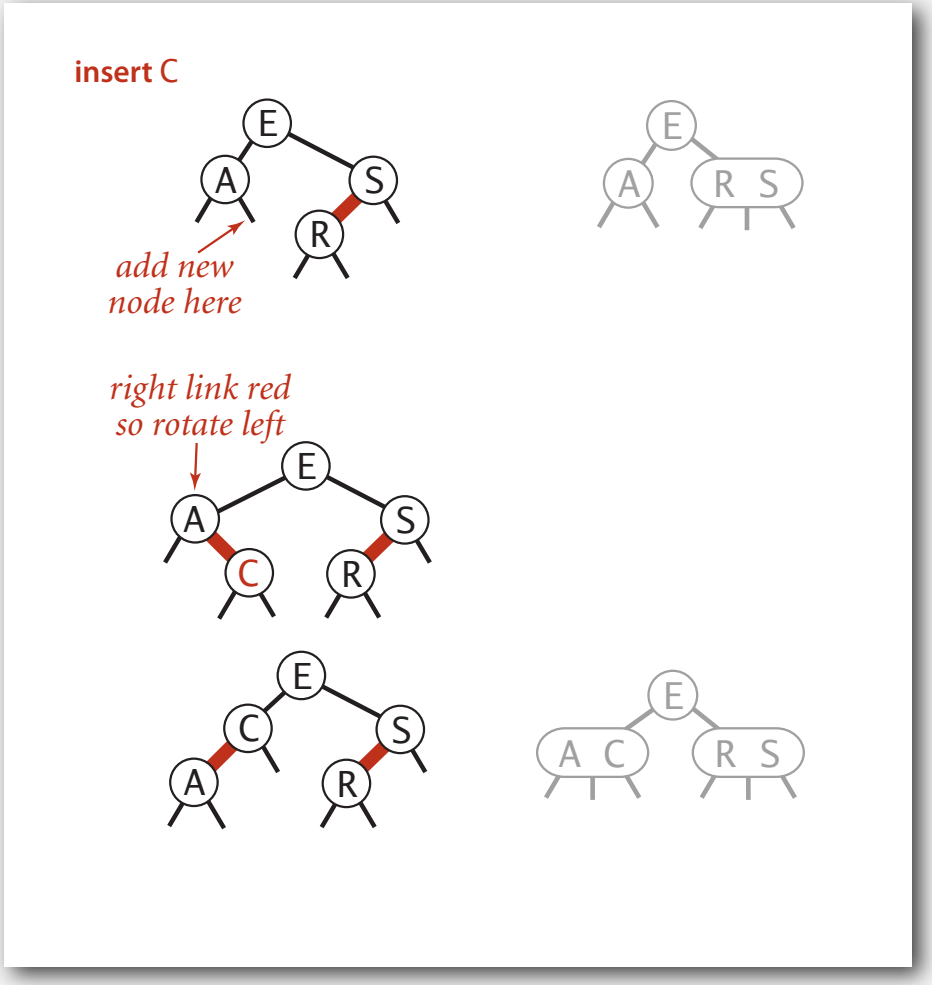



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   asset isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

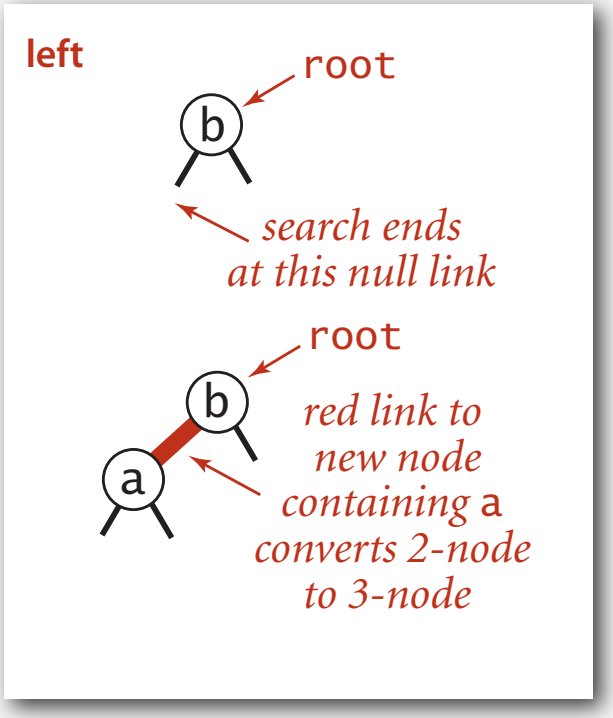
Insertion in a LLRB tree: overview

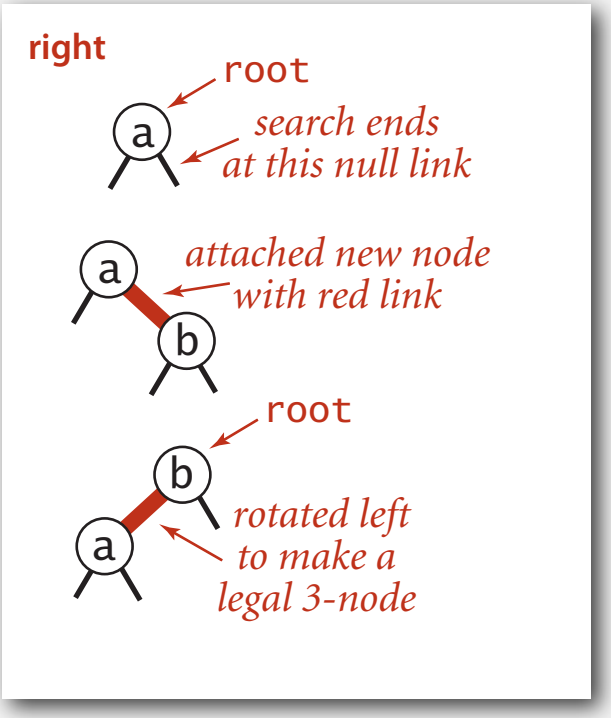
Basic strategy. Maintain 1-1 correspondence with 2-3 trees by applying elementary red-black BST operations.



Insertion in a LLRB tree

Warmup 1. Insert into a tree with exactly 1 node.

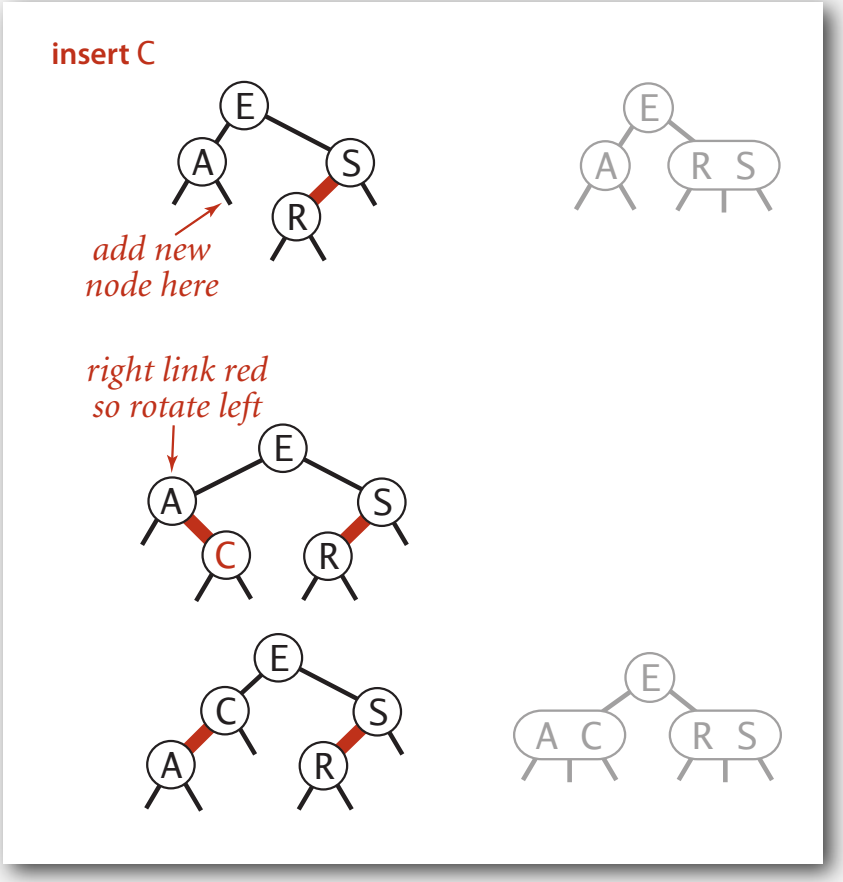




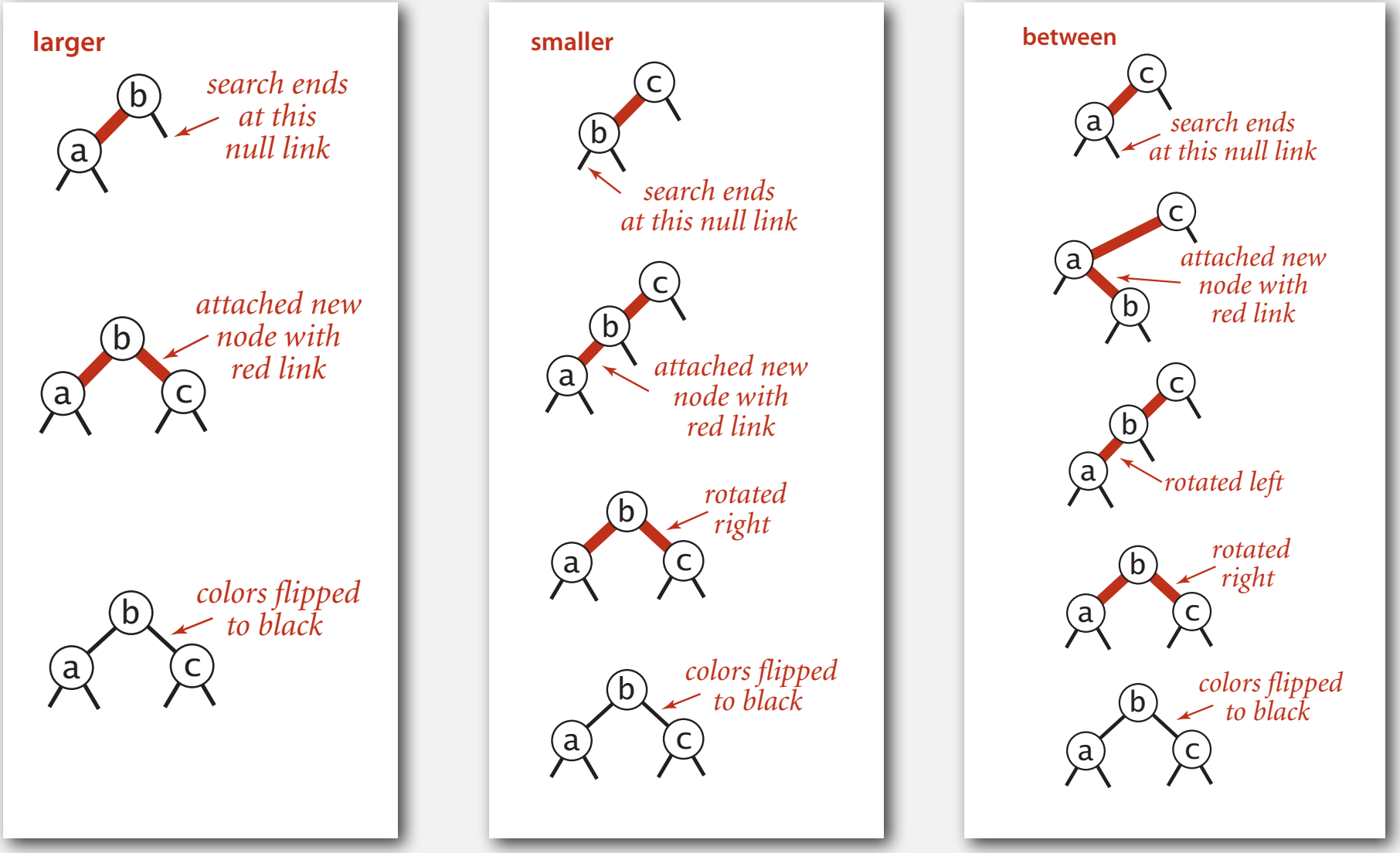
Insertion in a LLRB tree

Case 1. Insert into a 2-node at the bottom.

- Do standard BST insert; color new link red.
- If new red link is a right link, rotate left.



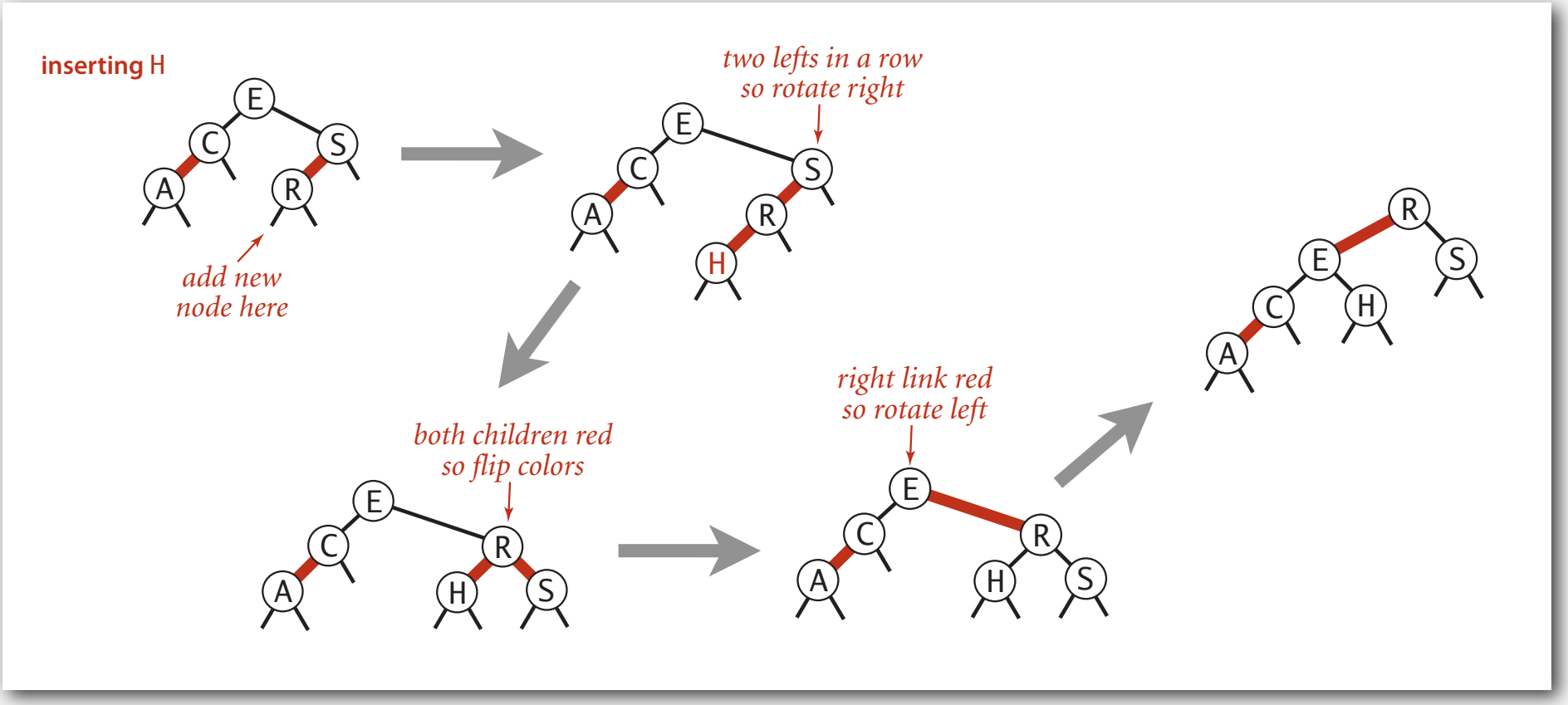
Warmup 2. Insert into a tree with exactly 2 nodes.



Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

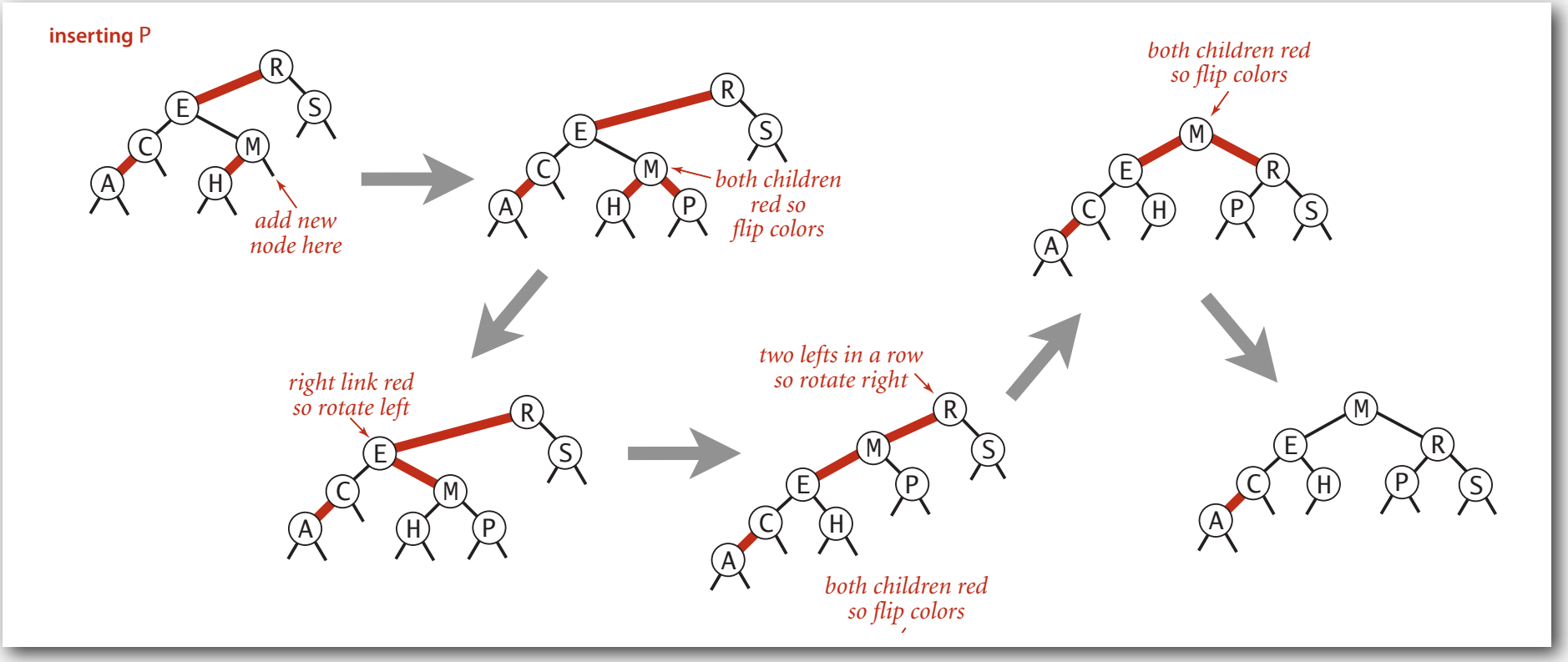
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).



Insertion in a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

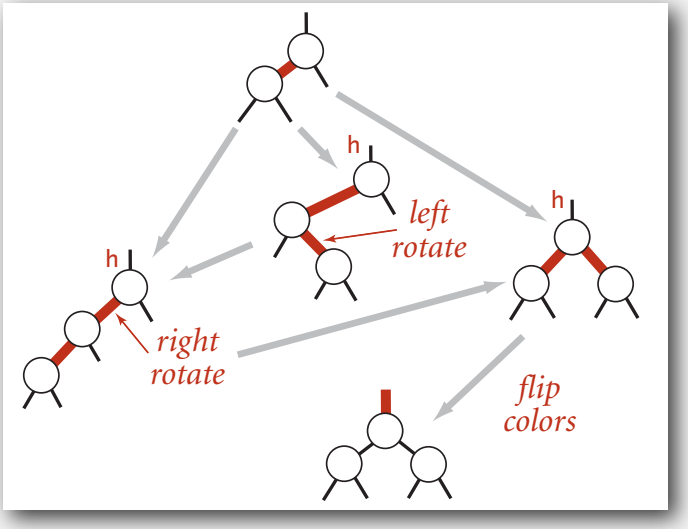
- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).



Insertion in a LLRB tree: Java implementation

Same code for both cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))      h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))      flipColors(h);

   return h;
}
```

insert at bottom (and color red)

lean left

balance 4-node

split 4-node

only a few extra lines of code provides near-perfect balance

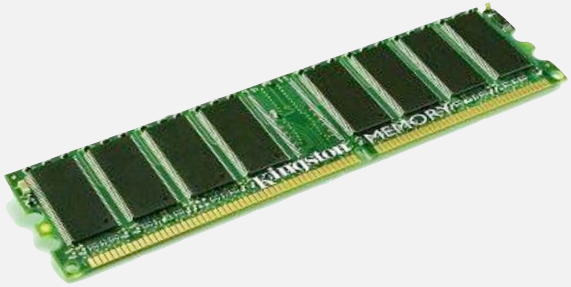
- 2-3 search trees
- red-black BSTs
- **B-trees**

File system model

Page. Contiguous block of data (e.g., a file or 4,096-byte chunk).

Probe. First access to a page (e.g., from disk to memory).

slow



fast

Property. Time required for a probe is much larger than time to access data within a page.

Cost model. Number of probes.

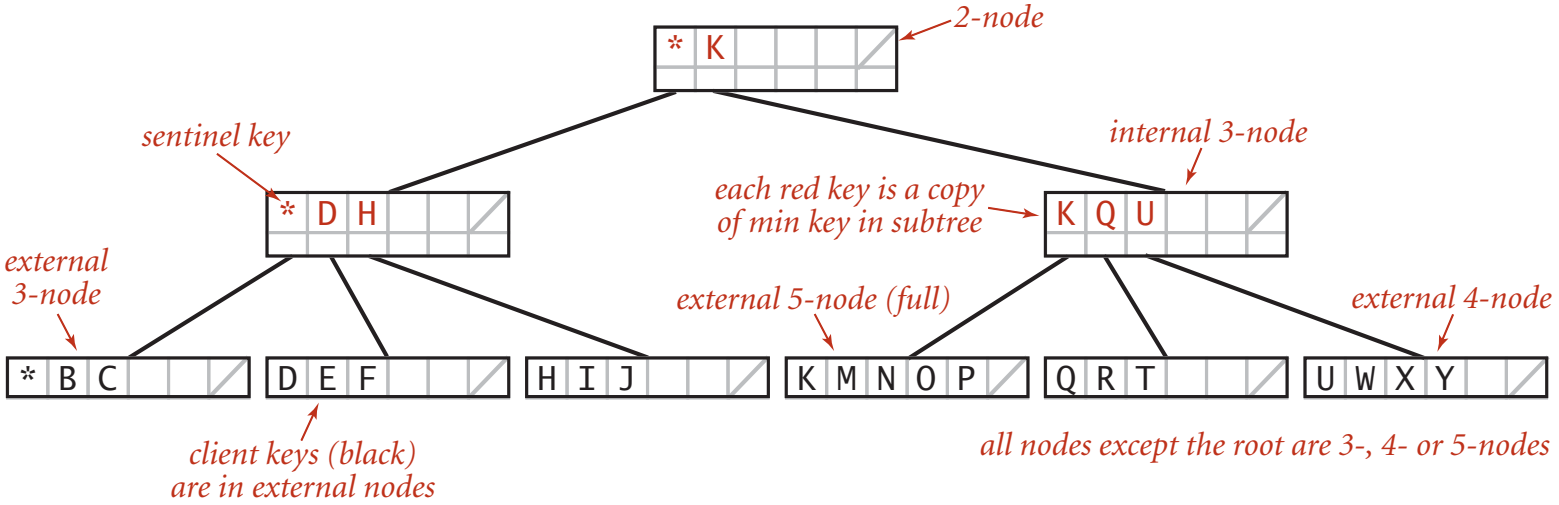
Goal. Access data using minimum number of probes.

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

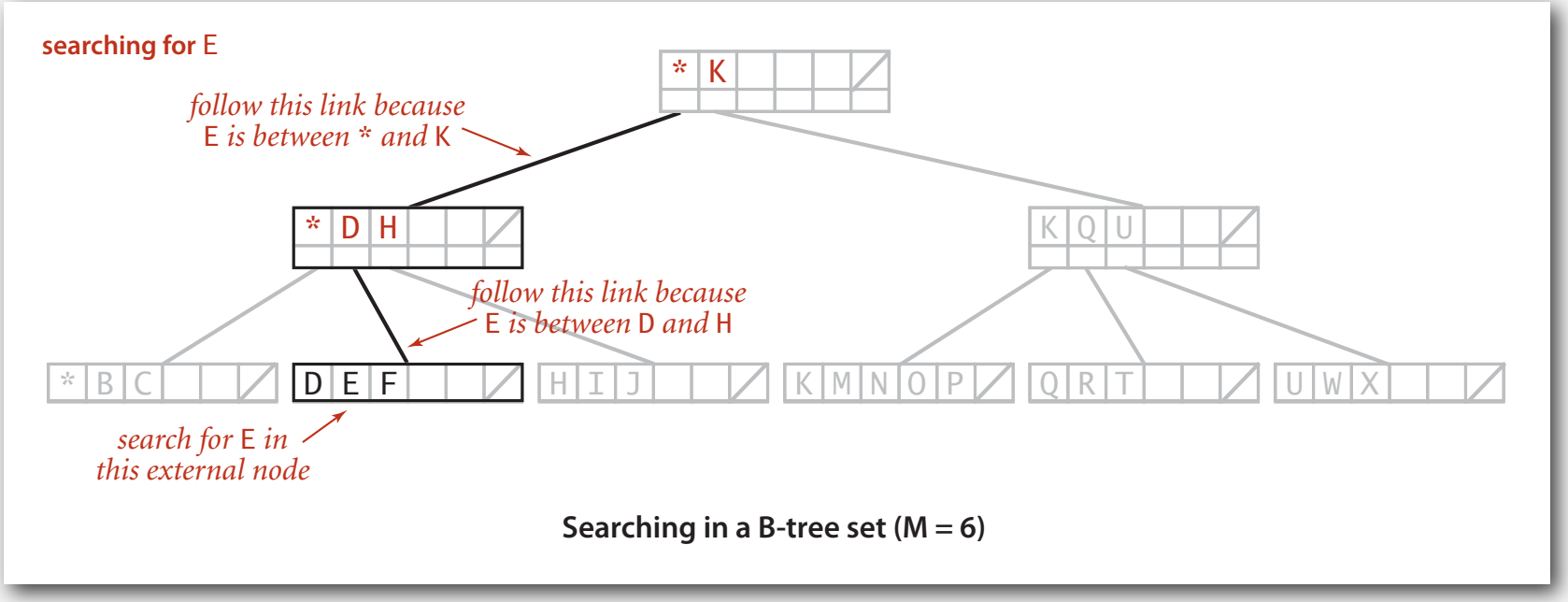
choose M as large as possible so that M links fit in a page, e.g., M = 1024



Anatomy of a B-tree set (M = 6)

Searching in a B-tree

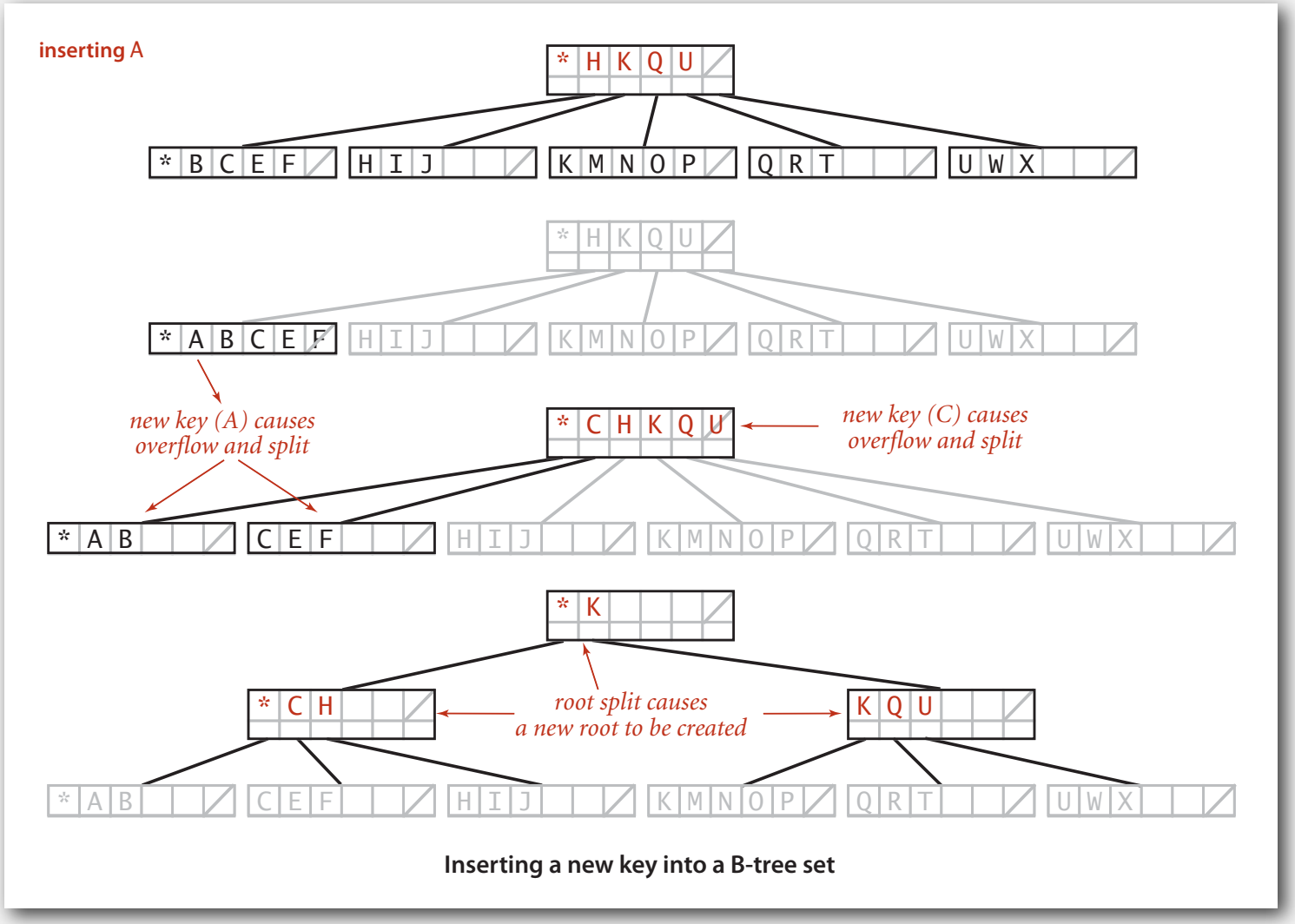
- Start at root.
- Find interval for search key and take corresponding link.
- Search terminates in external node.



Searching in a B-tree set (M = 6)

Insertion in a B-tree

- Search for new key.
- Insert at bottom.
- Split nodes with M key-link pairs on the way up the tree.



Inserting a new key into a B-tree set

Balance in B-tree

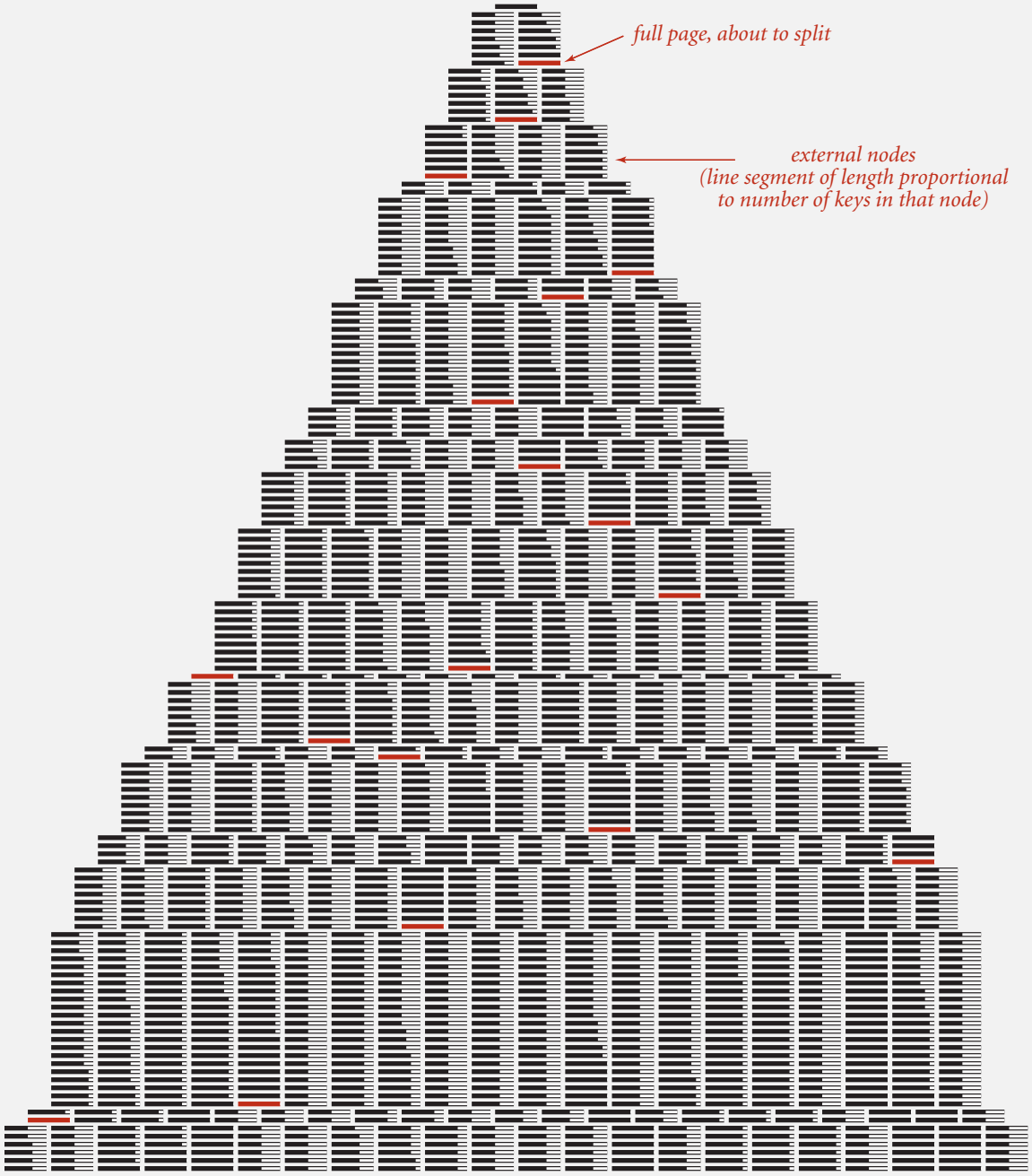
Proposition. A search or an insertion in a B-tree of order M with N keys requires between $\log_{M-1} N$ and $\log_{M/2} N$ probes.

Pf. All internal nodes (besides root) have between $M / 2$ and $M - 1$ links.

In practice. Number of probes is at most 4. ← M = 1024; N = 62 billion
$\log_{M/2} N \leq 4$

Optimization. Always keep root page in memory.

Building a large B tree



Balanced trees in the wild

Red-black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.

B-tree variants. B+ tree, B*tree, B# tree, ...

B-trees (and variants) are widely used for file systems and databases.

- Windows: HPFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.