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3.3 BALANCED SEARCH TREES

- *2-3 search trees*
- *red-black BSTs*
- *B-trees*

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	$\frac{1}{2}\ N$	N	$\frac{1}{2}\ N$		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$\frac{1}{2}\ N$	$\frac{1}{2}\ N$	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
goal	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	$\log N$	✔	`compareTo()`

Challenge. Guarantee performance.

This lecture. 2-3 trees, left-leaning red-black BSTs, B-trees.

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3.3 Balanced Search Trees

- *2-3 search trees*
- *red-black BSTs*
- *B-trees*

2-3 tree

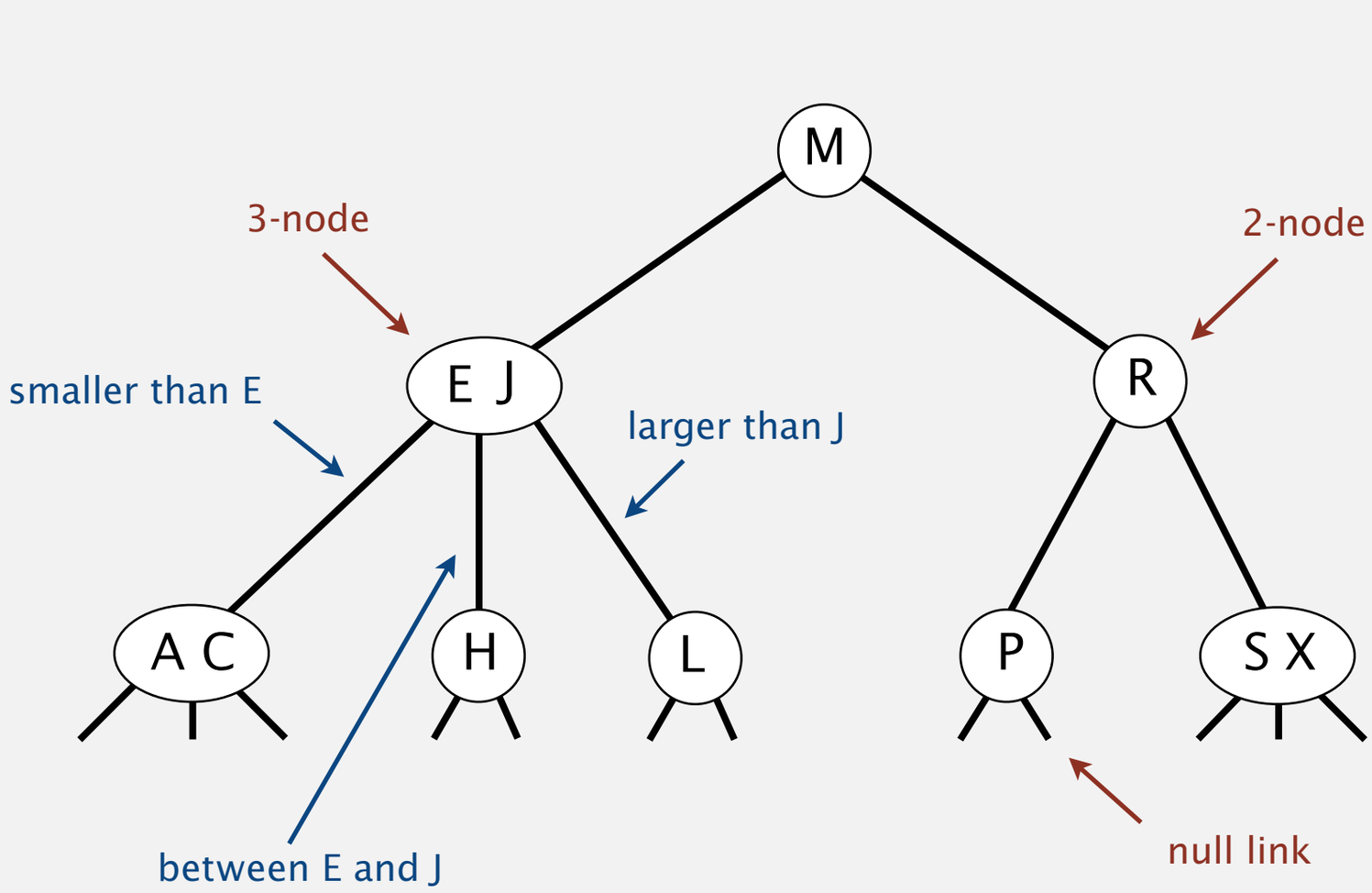
Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.

how to maintain?

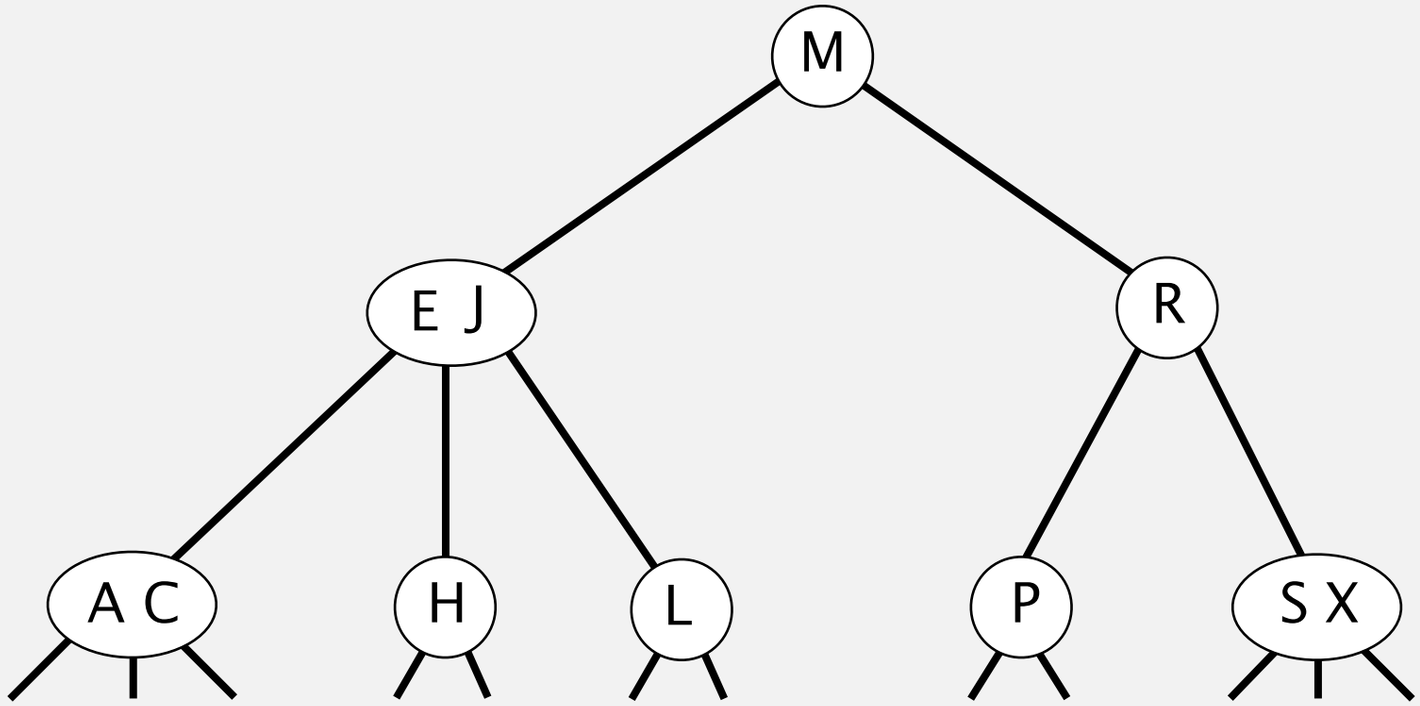


2-3 tree demo

Search.

- Compare search key against keys in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H

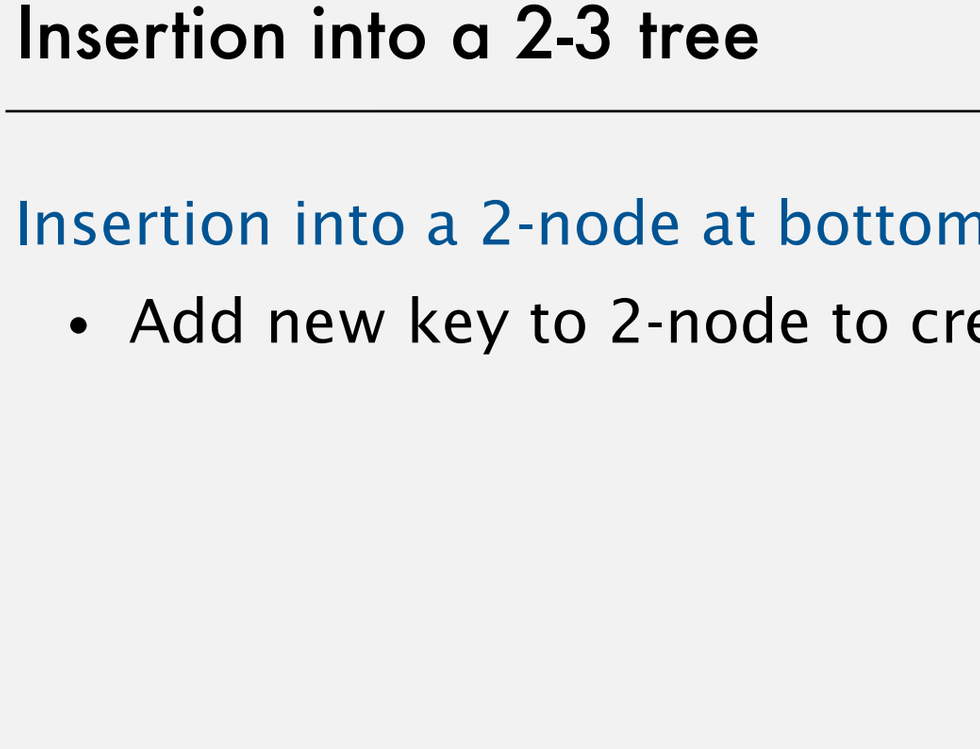


Insertion into a 2-3 tree

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G

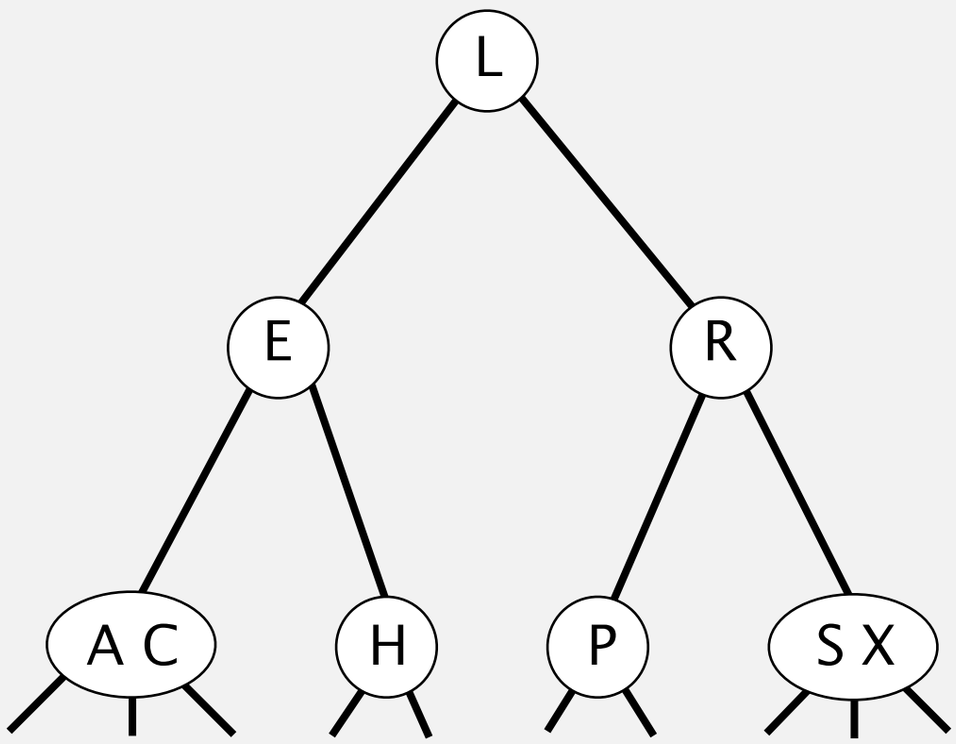


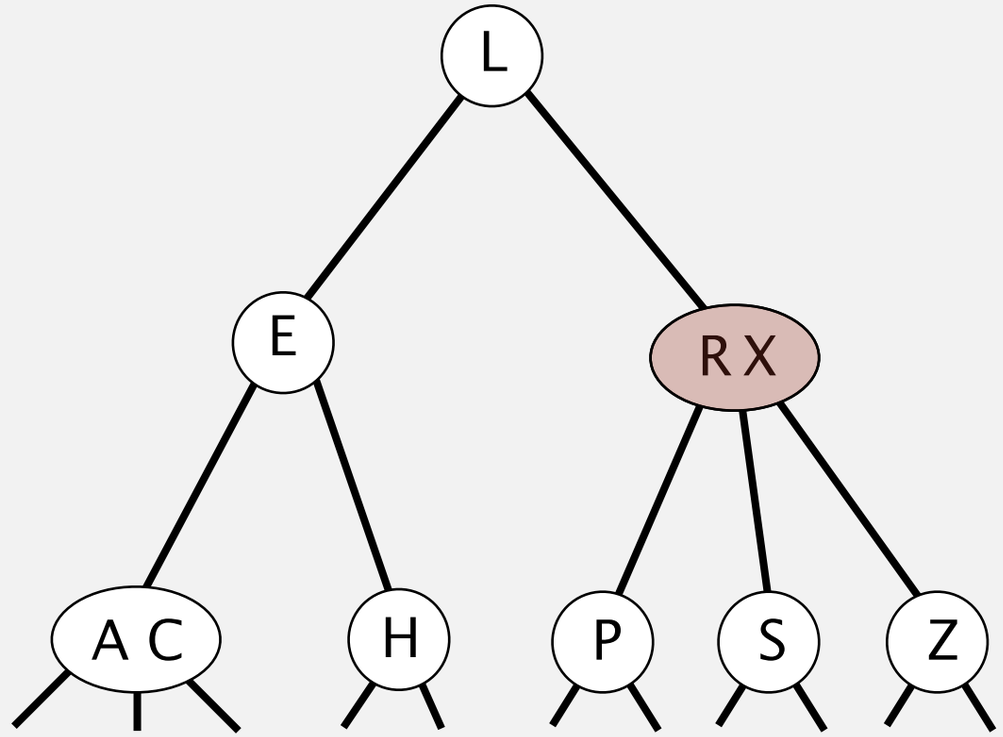
Insertion into a 2-3 tree

Insertion into a 3-node at bottom.

- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

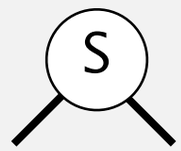
insert Z





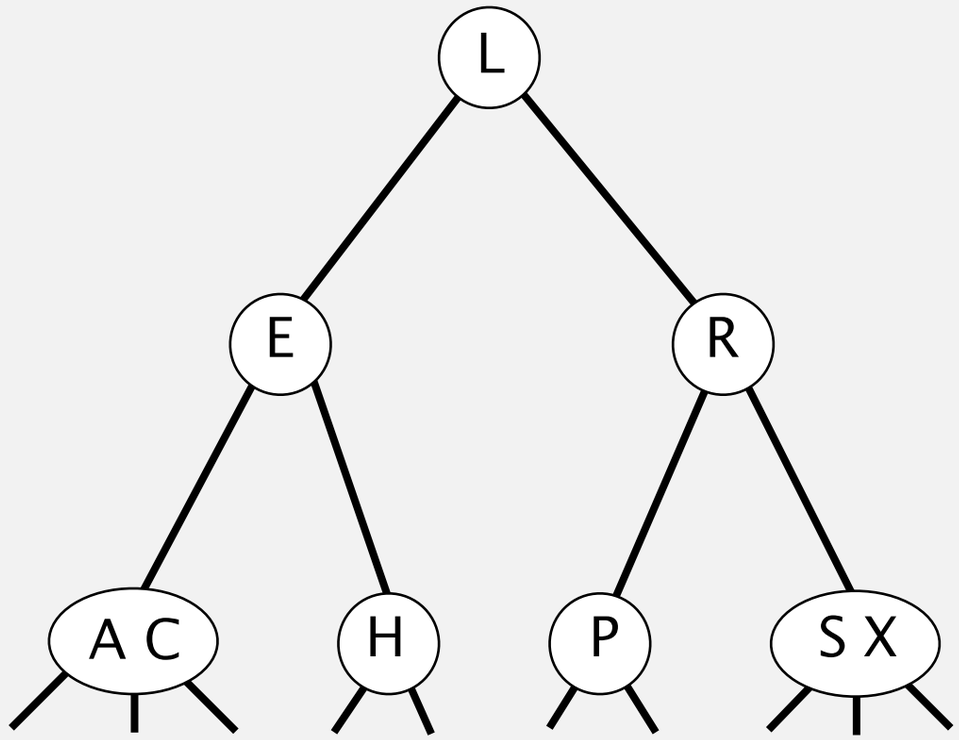
2-3 tree construction demo

insert S



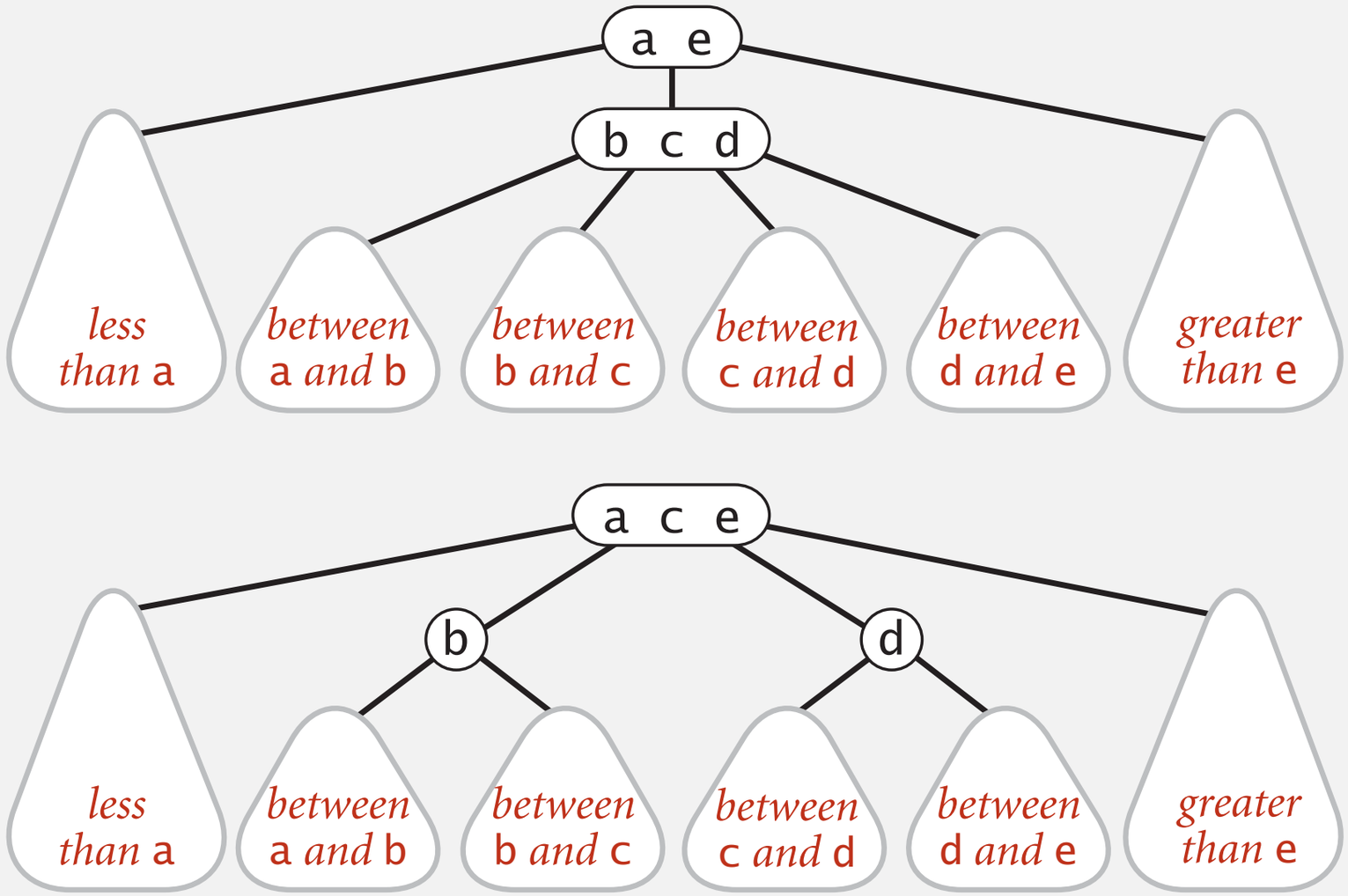
2-3 tree construction demo

2-3 tree



Local transformations in a 2-3 tree

Splitting a 4-node is a local transformation: constant number of operations.



Global properties in a 2-3 tree

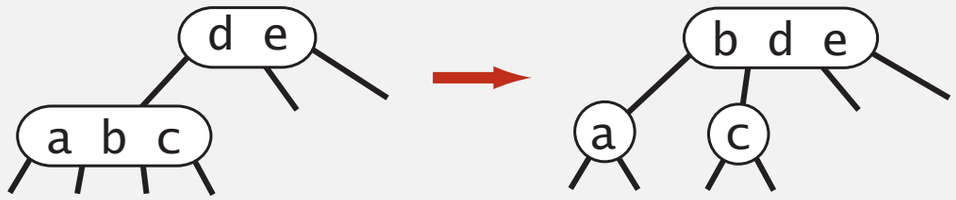
Invariants. Maintains symmetric order and perfect balance.

Pf. Each transformation maintains symmetric order and perfect balance.

root

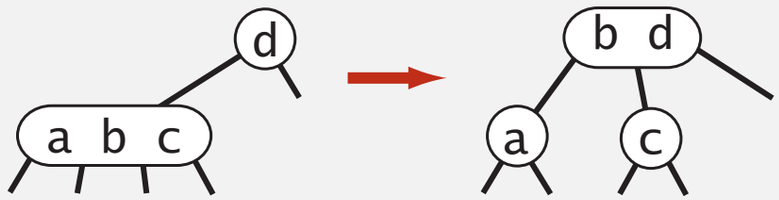
parent is a 3-node

left

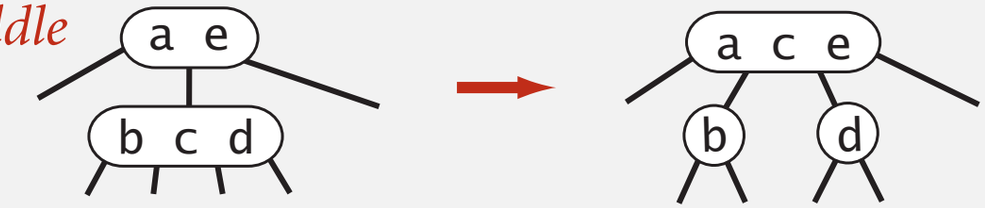


parent is a 2-node

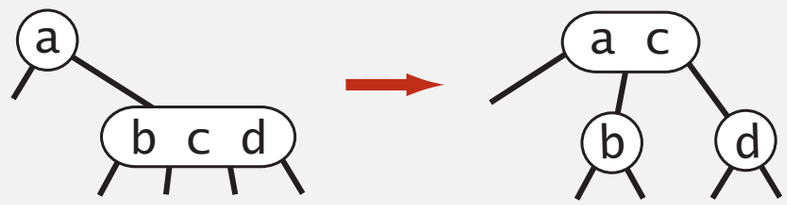
left



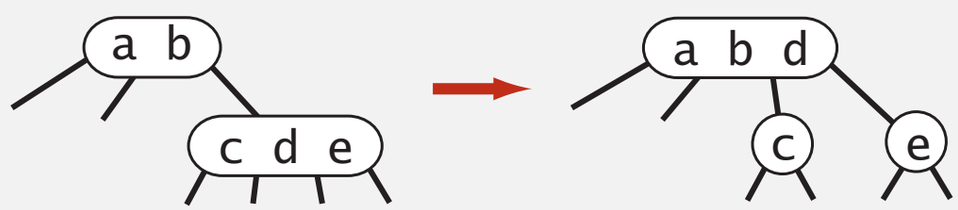
middle



right

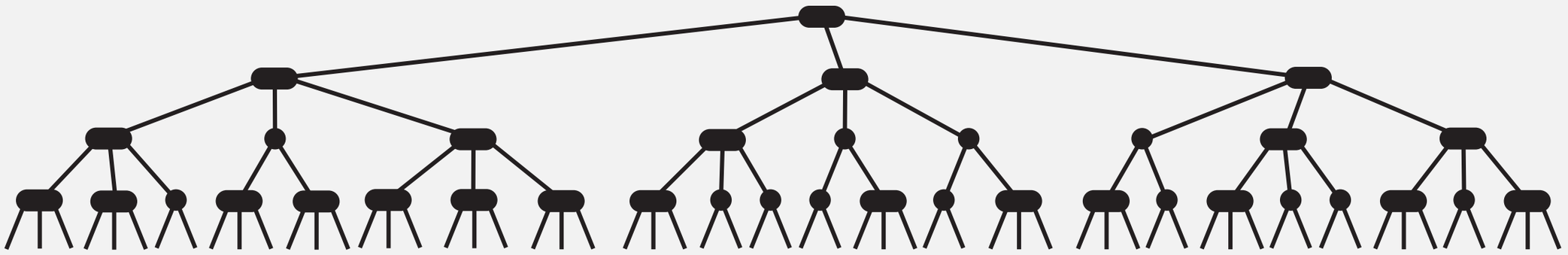


right



2-3 tree: performance

Perfect balance. Every path from root to null link has same length.

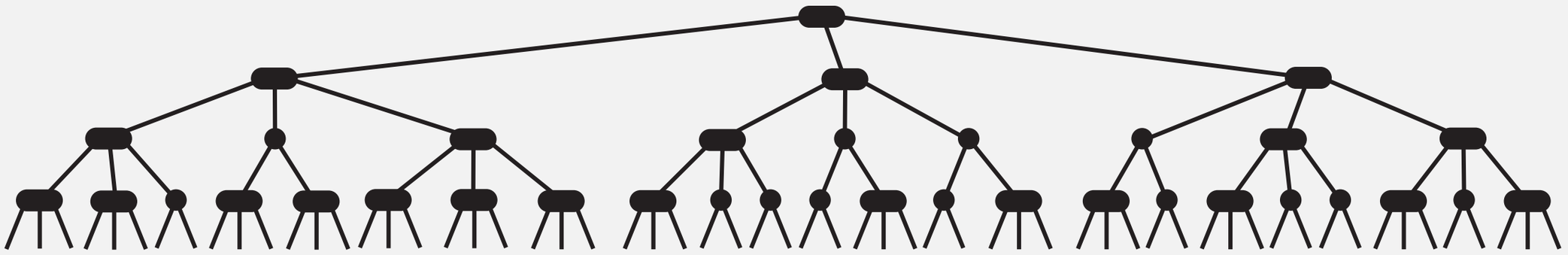


Tree height.

- Worst case:
- Best case:

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg N$. [all 2-nodes]
- Best case: $\log_3 N \approx .631 \lg N$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	$\frac{1}{2} N$	N	$\frac{1}{2} N$		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✔	`compareTo()`

constant c depend upon implementation

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

Bottom line. Could do it, but there's a better way.

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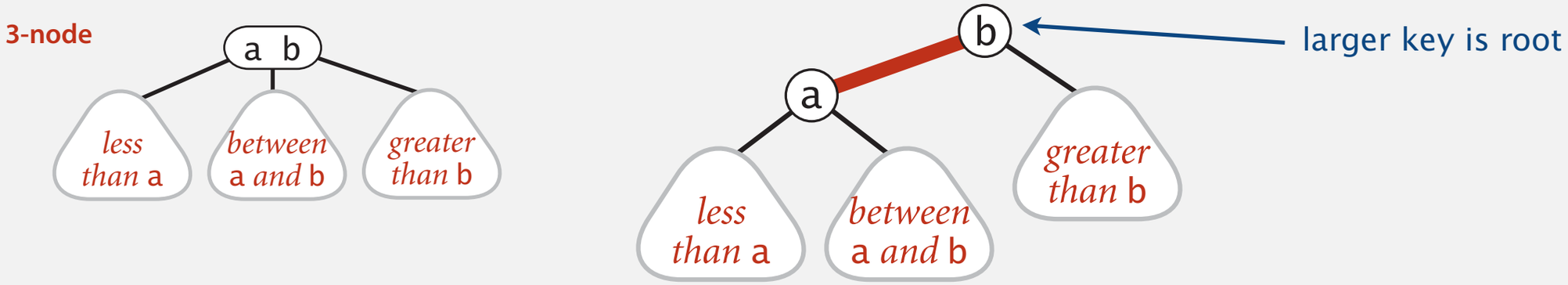
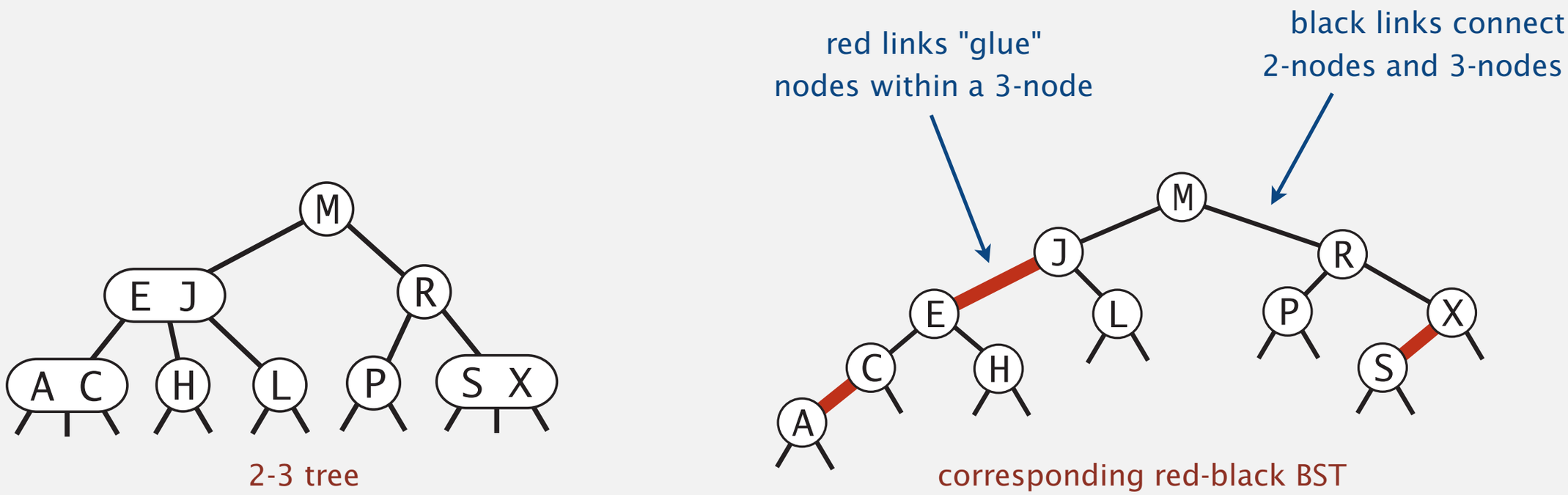
3.3 Balanced Search Trees

- 2-3 search trees
- red-black BSTs
- B-trees

Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2–3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3–nodes.





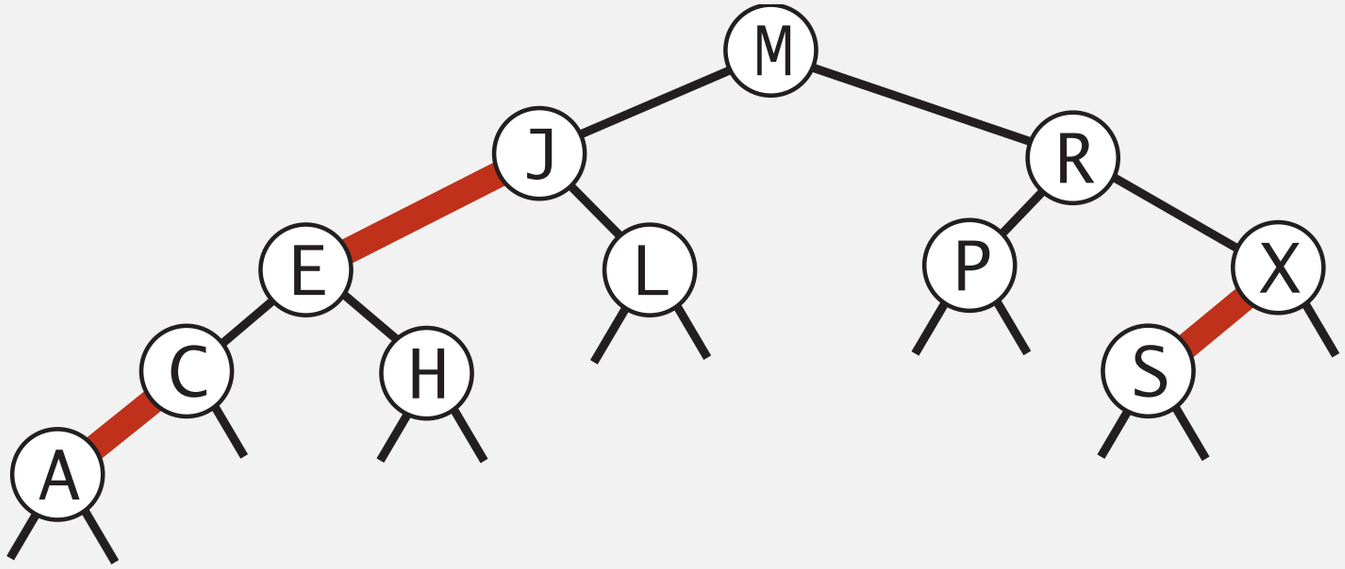
Take any 2-3 tree and represent it as a collection of BSTs held with red links

An equivalent definition

A BST such that:

- No node has two red links connected to it.
- Every path from root to null link has the same number of black links.
- Red links lean left.

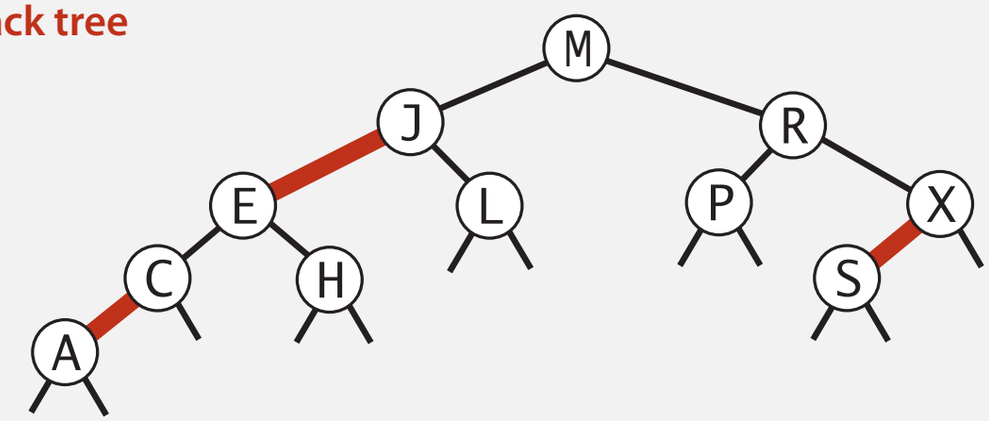
"perfect black balance"



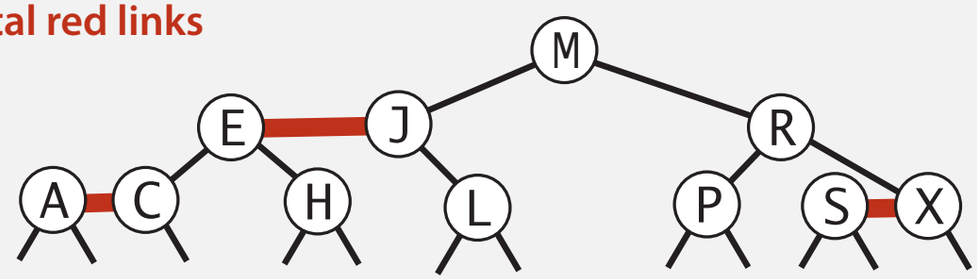
Left-leaning red-black BSTs: 1-1 correspondence with 2-3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

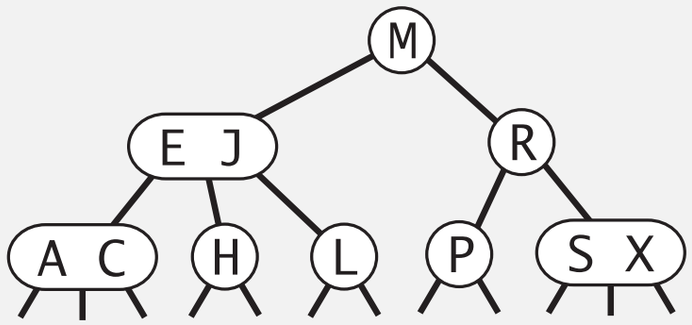
red–black tree



horizontal red links



2-3 tree

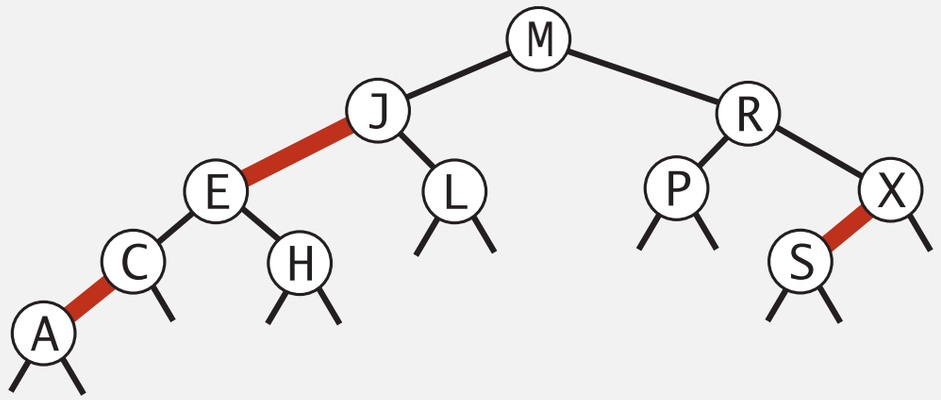


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
because of better balance

```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Most other ops (e.g., floor, iteration, selection) are also identical.

Red-black BST representation

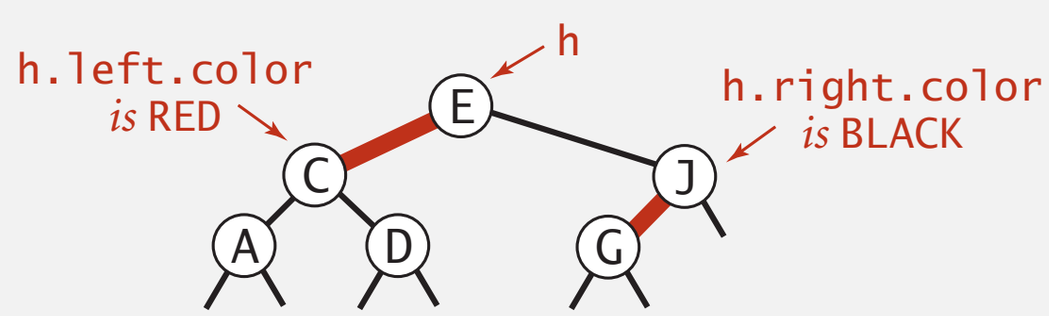
Each node is pointed to by precisely one link (from its parent) ⇒
can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



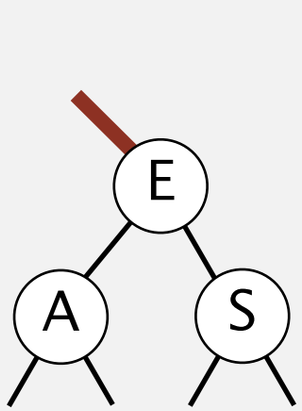
Insertion in a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

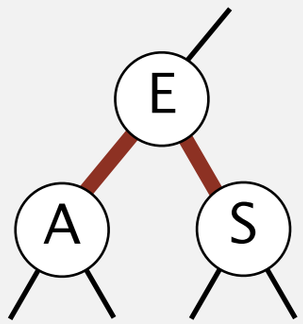
During internal operations, maintain:

- Symmetric order.
- Perfect black balance.

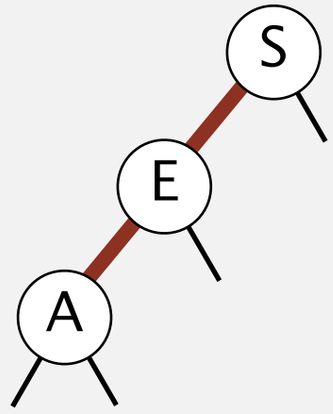
  [but not necessarily color invariants]



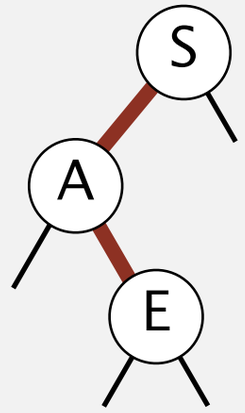
right-leaning red link



two red children (a temporary 4-node)



left-left red (a temporary 4-node)

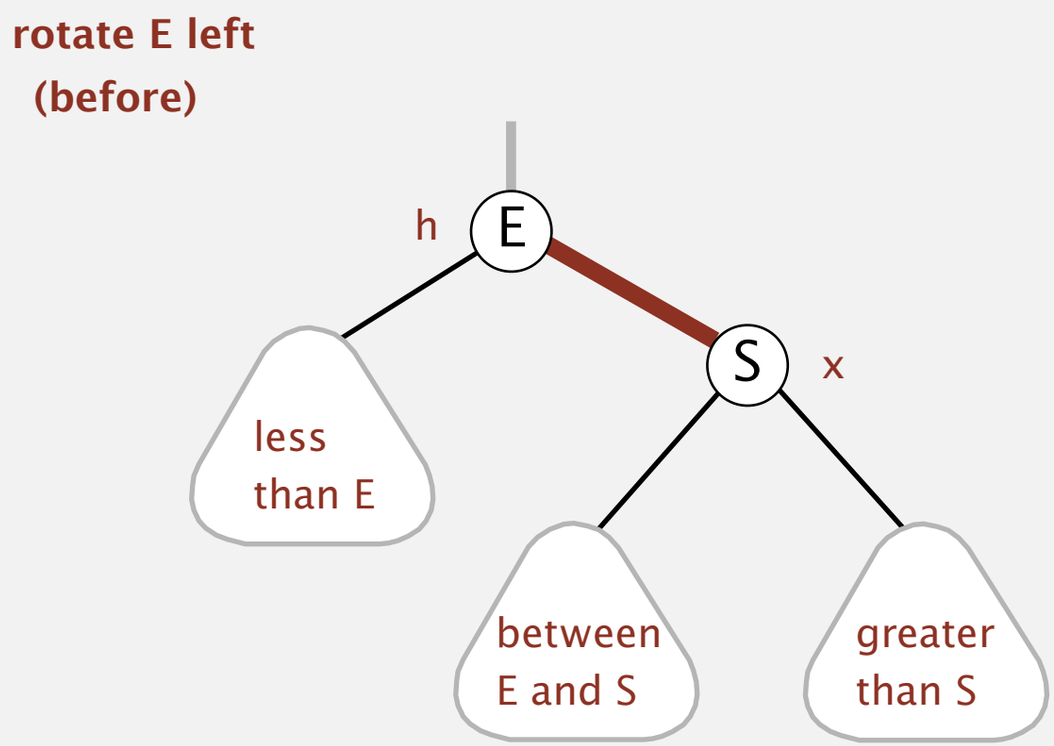


left-right red (a temporary 4-node)

How? Apply elementary red-black BST operations: rotation and color flip.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

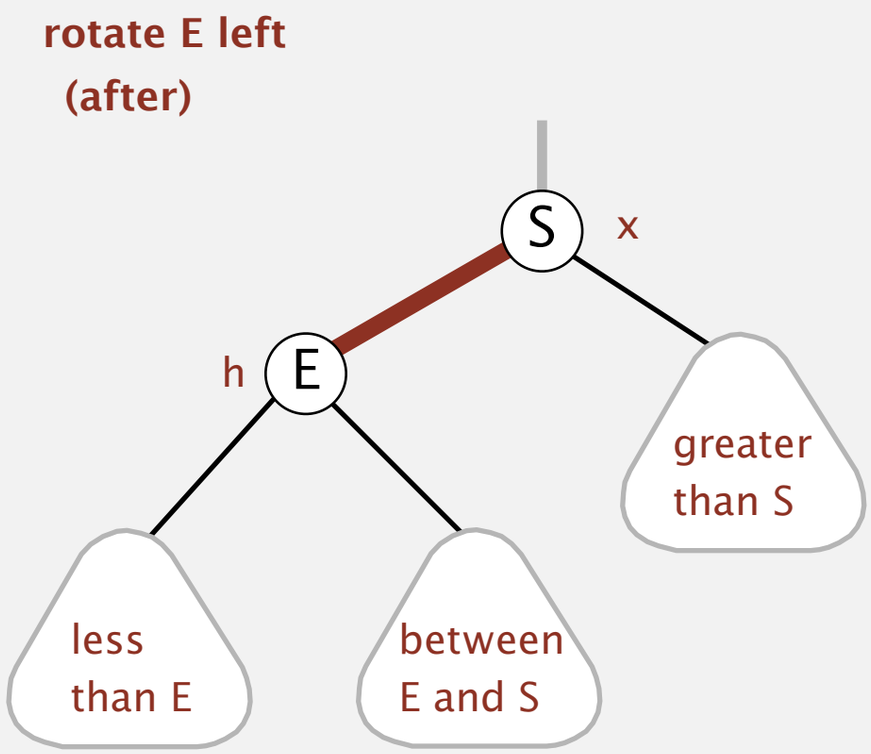



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

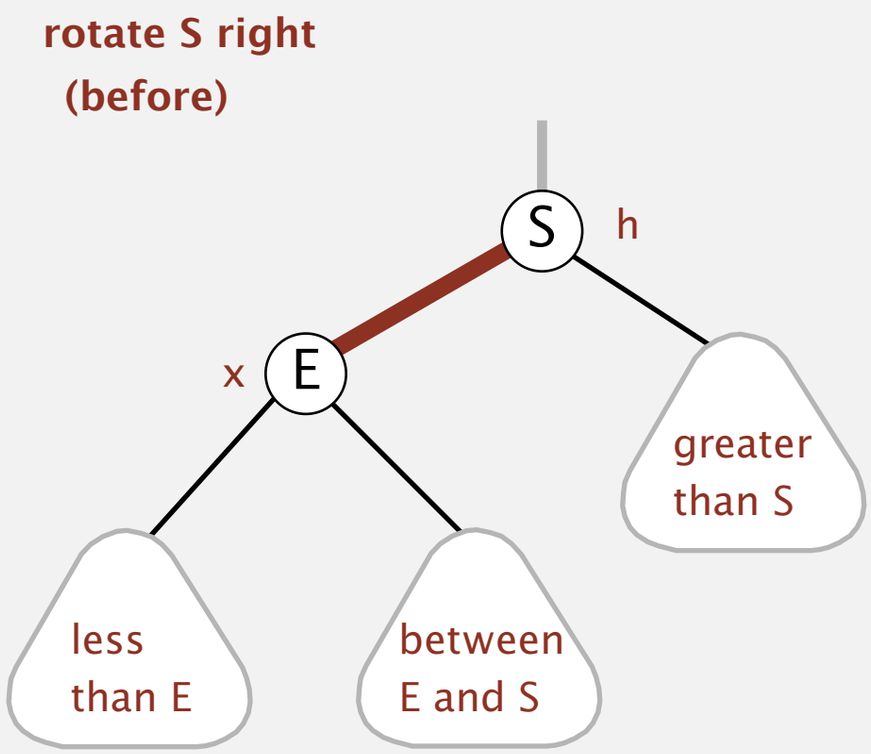



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

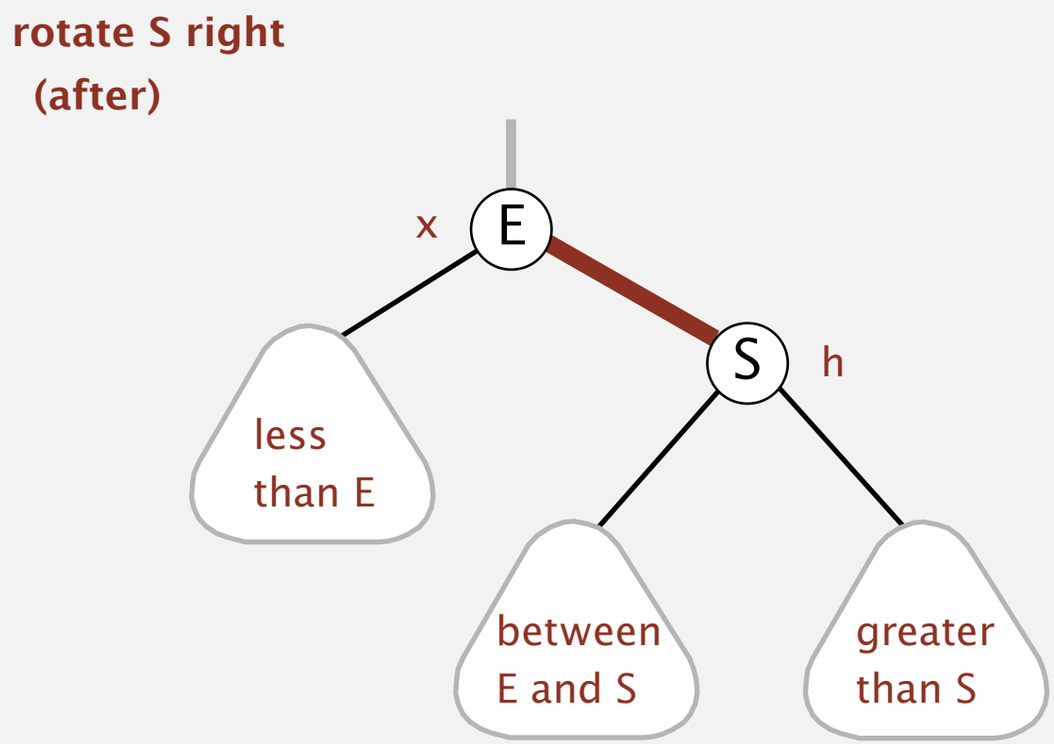



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

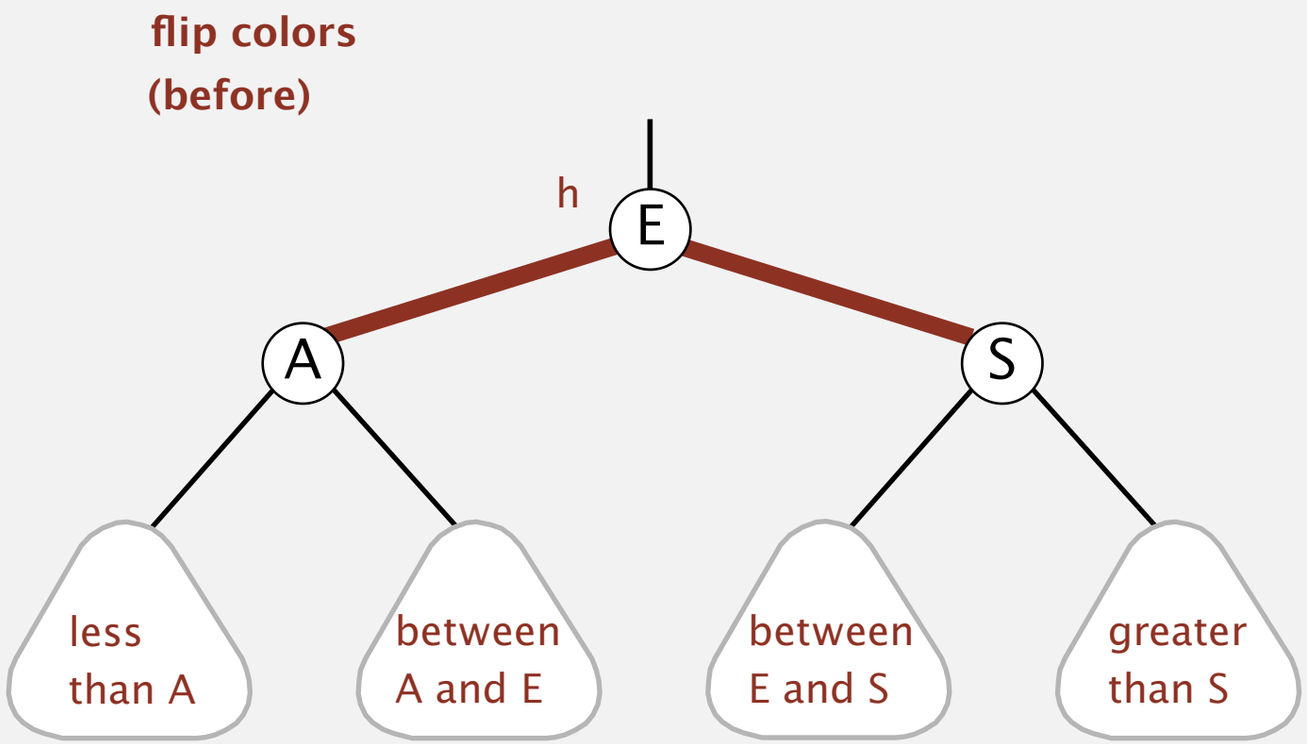



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

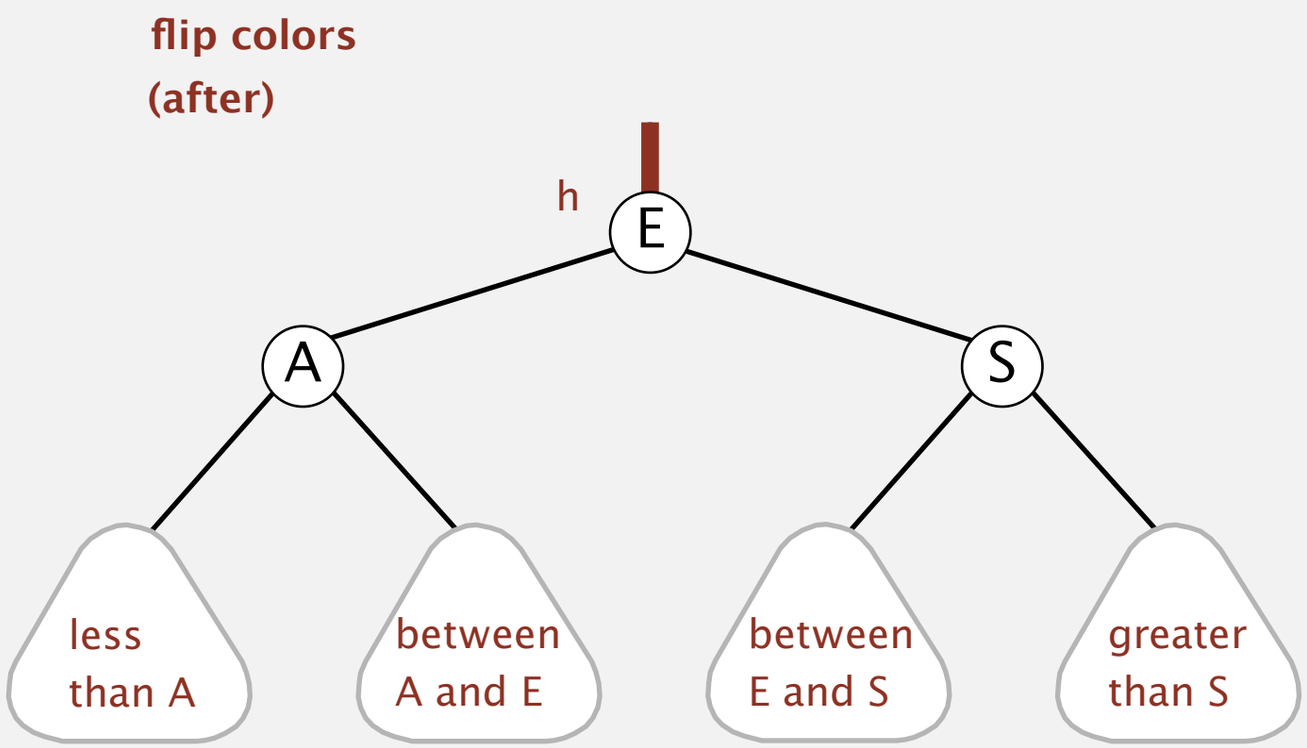



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

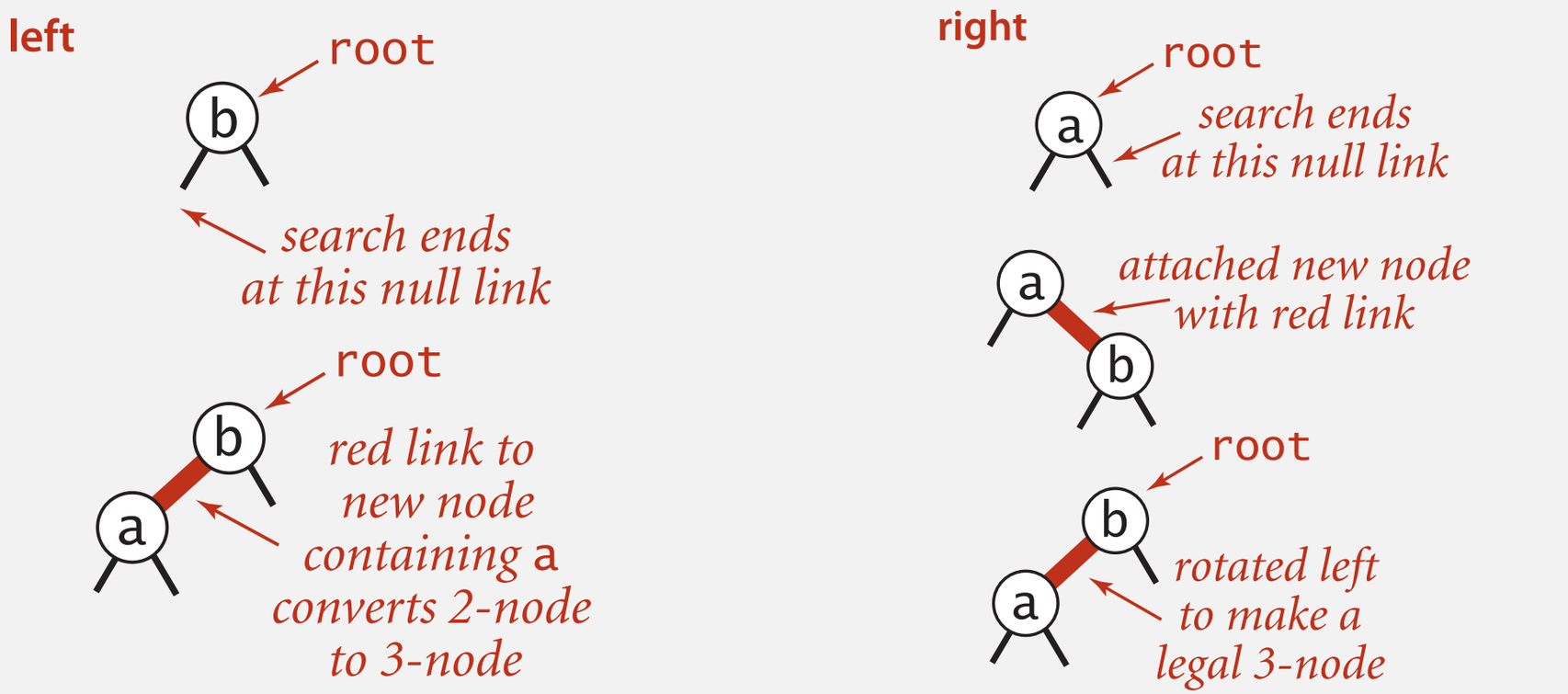



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Insertion in a LLRB tree

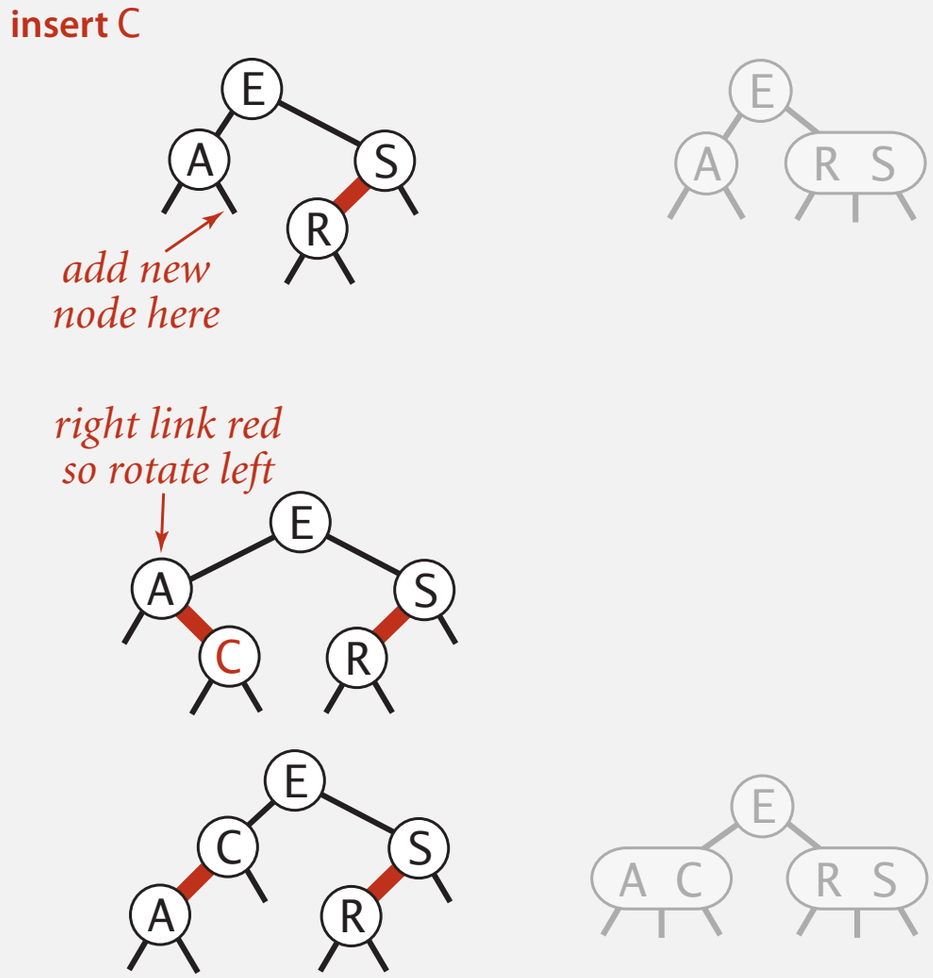
Warmup 1. Insert into a tree with exactly 1 node.



Insertion in a LLRB tree

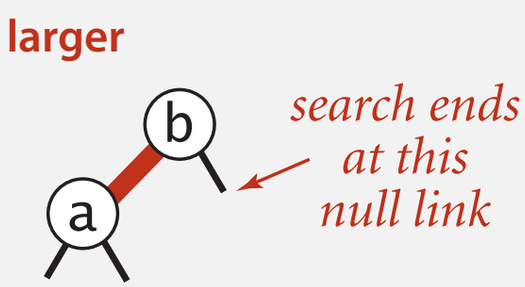
Case 1. Insert into a 2-node at the bottom.

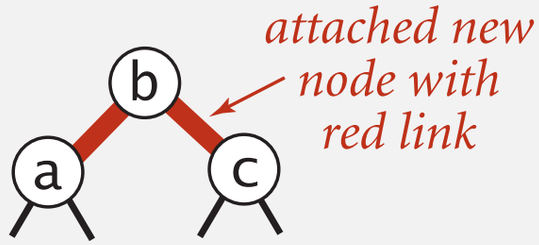
- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- If new red link is a right link, rotate left. ← to fix color invariants

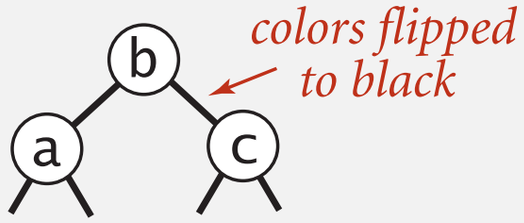


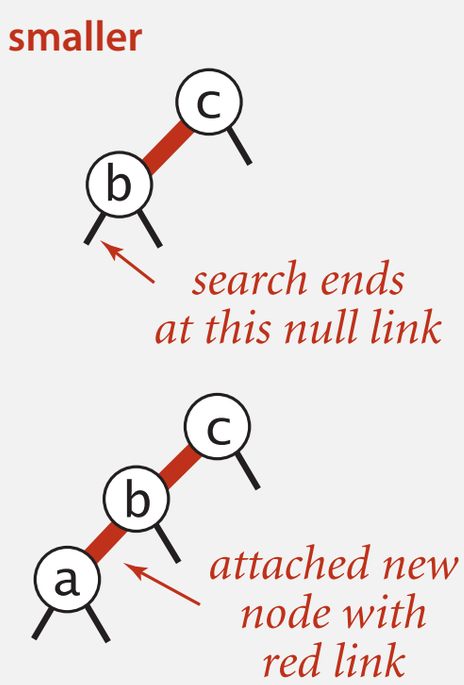
Insertion in a LLRB tree

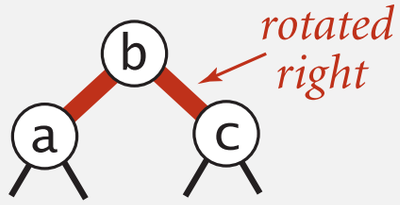
Warmup 2. Insert into a tree with exactly 2 nodes.

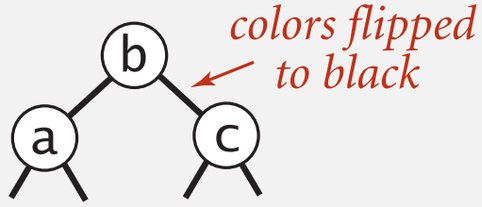


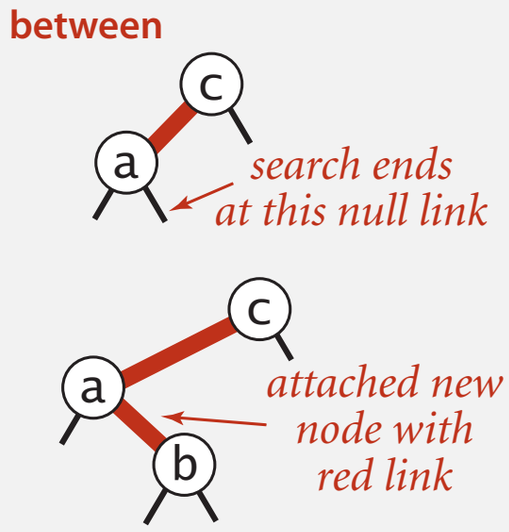


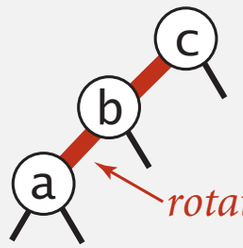


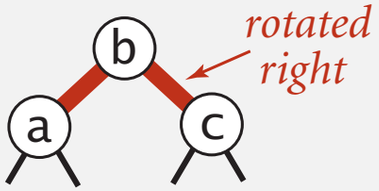


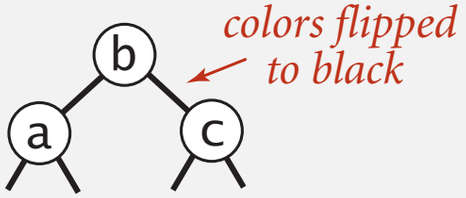








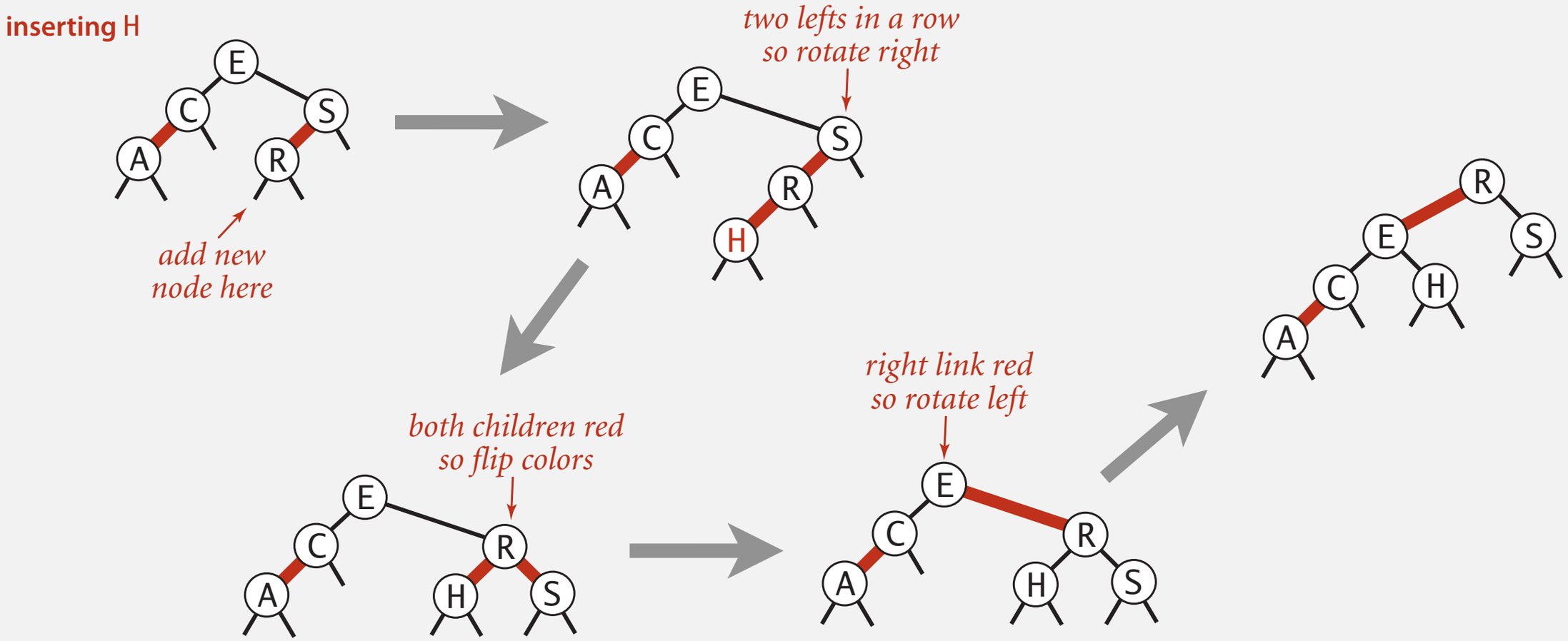




Insertion in a LLRB tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red. ← to maintain symmetric order and perfect black balance
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed). ← to fix color invariants



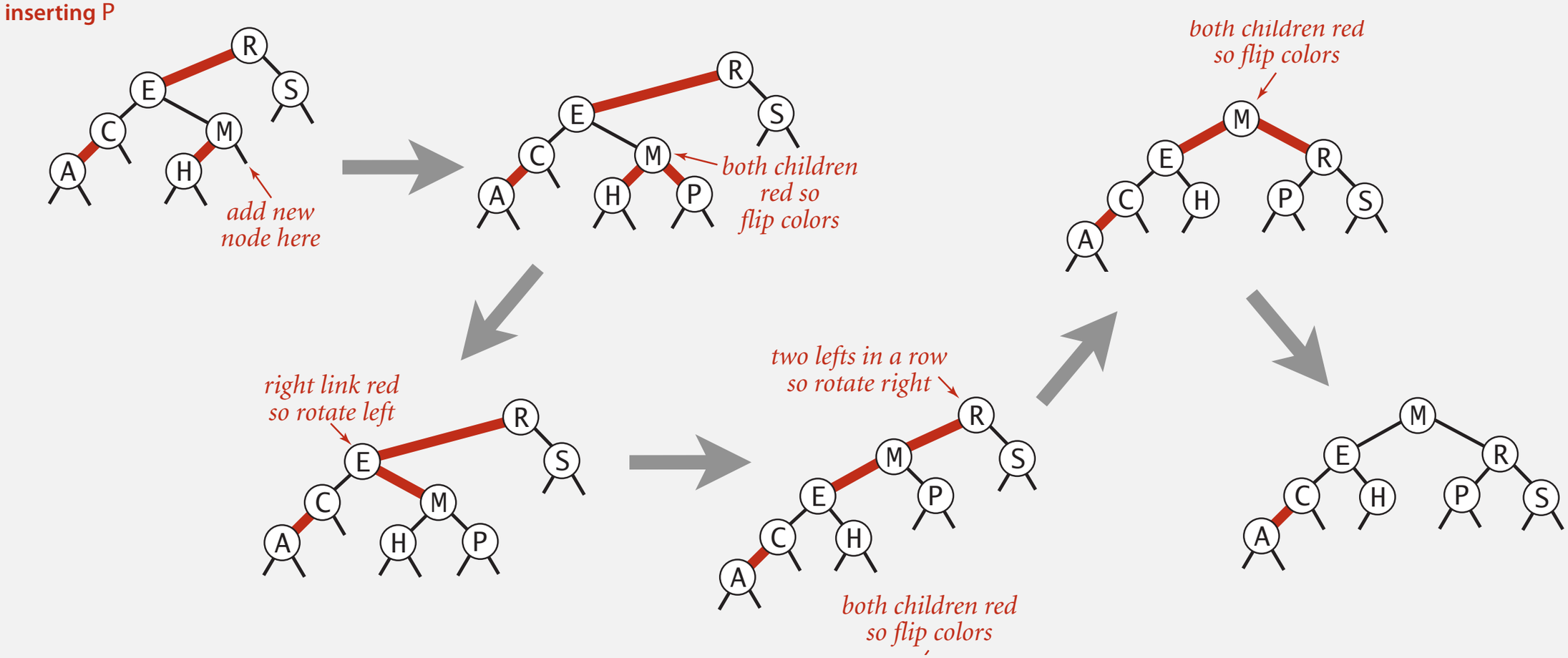
Insertion in a LLRB tree: passing red links up the tree

Case 2. Insert into a 3-node at the bottom.

- Do standard BST insert; color new link red.
- Rotate to balance the 4-node (if needed).
- Flip colors to pass red link up one level.
- Rotate to make lean left (if needed).
- Repeat case 1 or case 2 up the tree (if needed).

to maintain symmetric order and perfect black balance

to fix color invariants

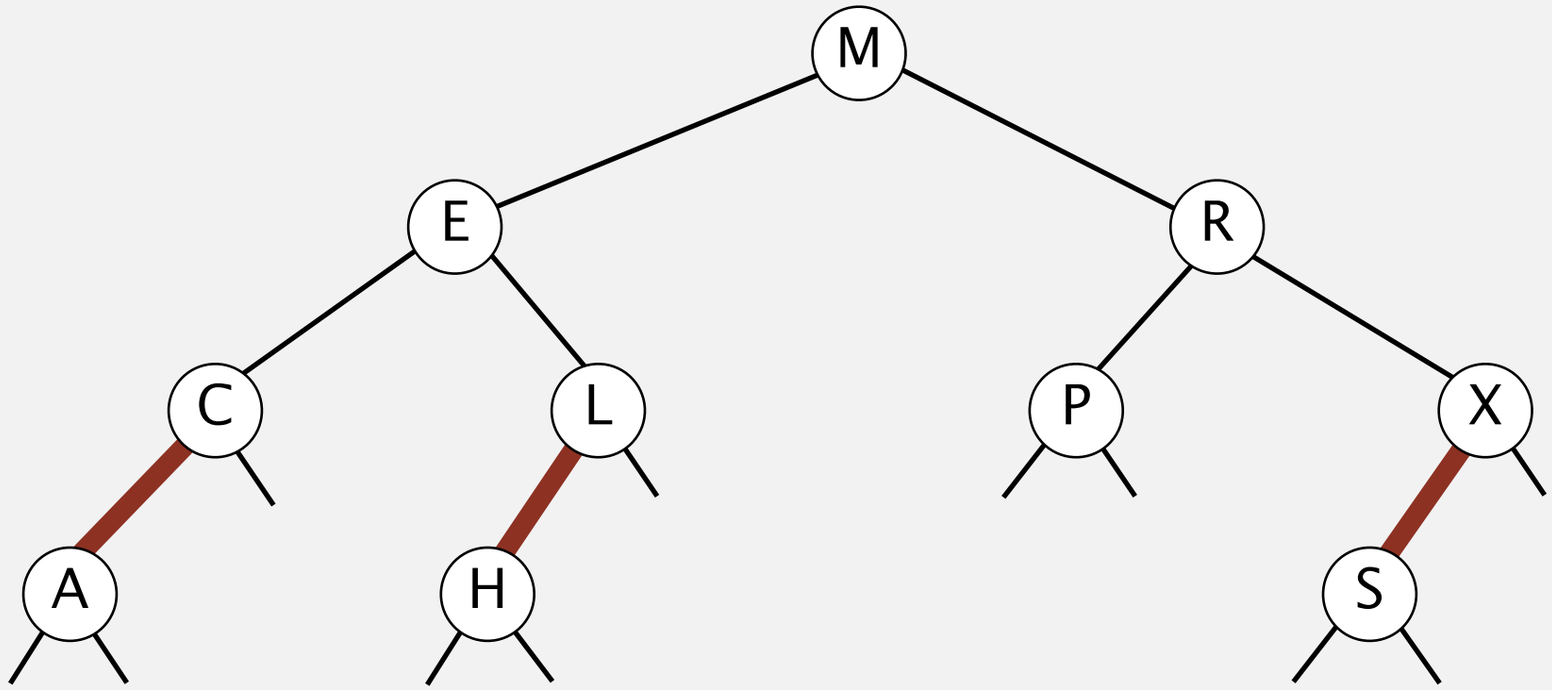


Red-black BST construction demo

insert S

Red-black BST construction demo

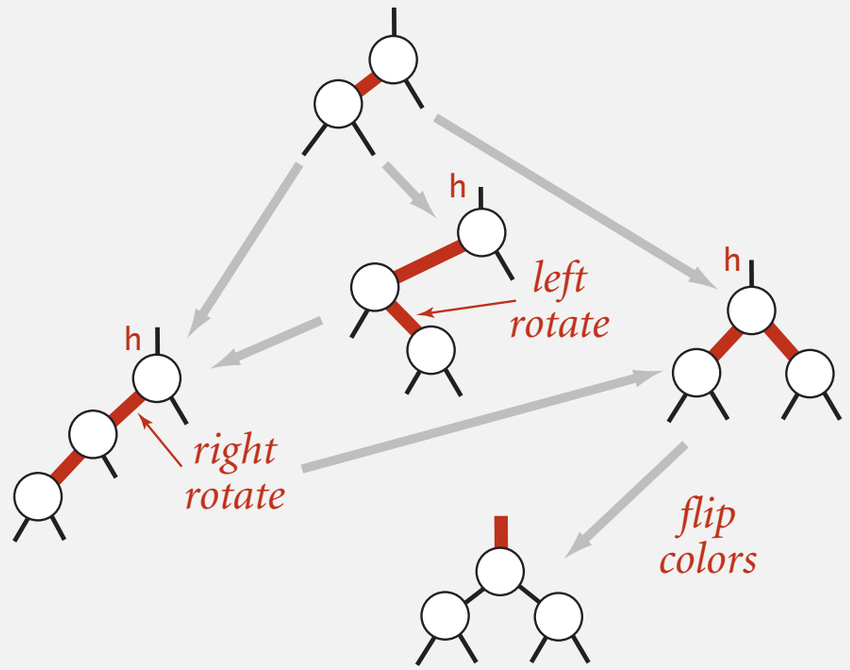
red-black BST



Insertion in a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

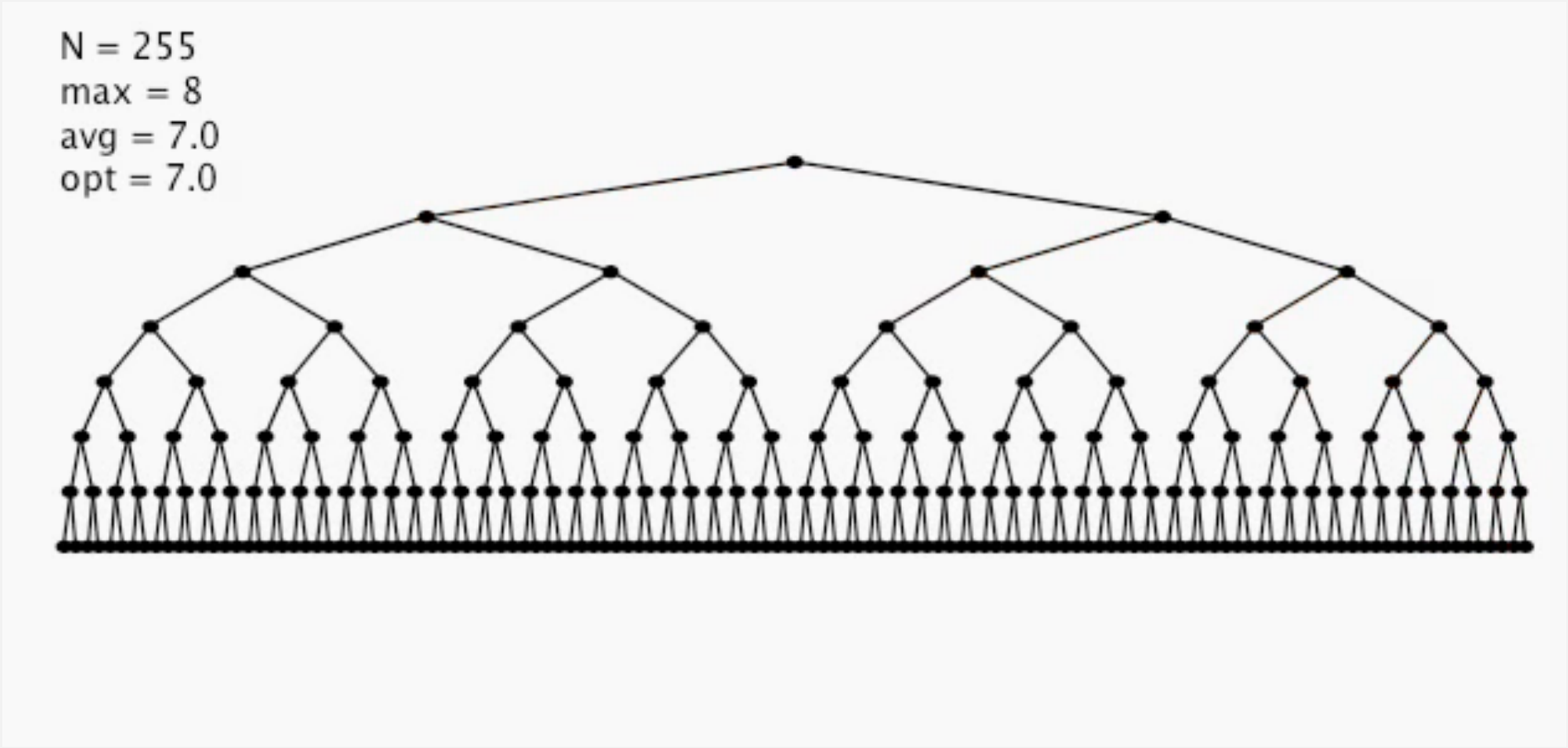
lean left

balance 4-node

split 4-node

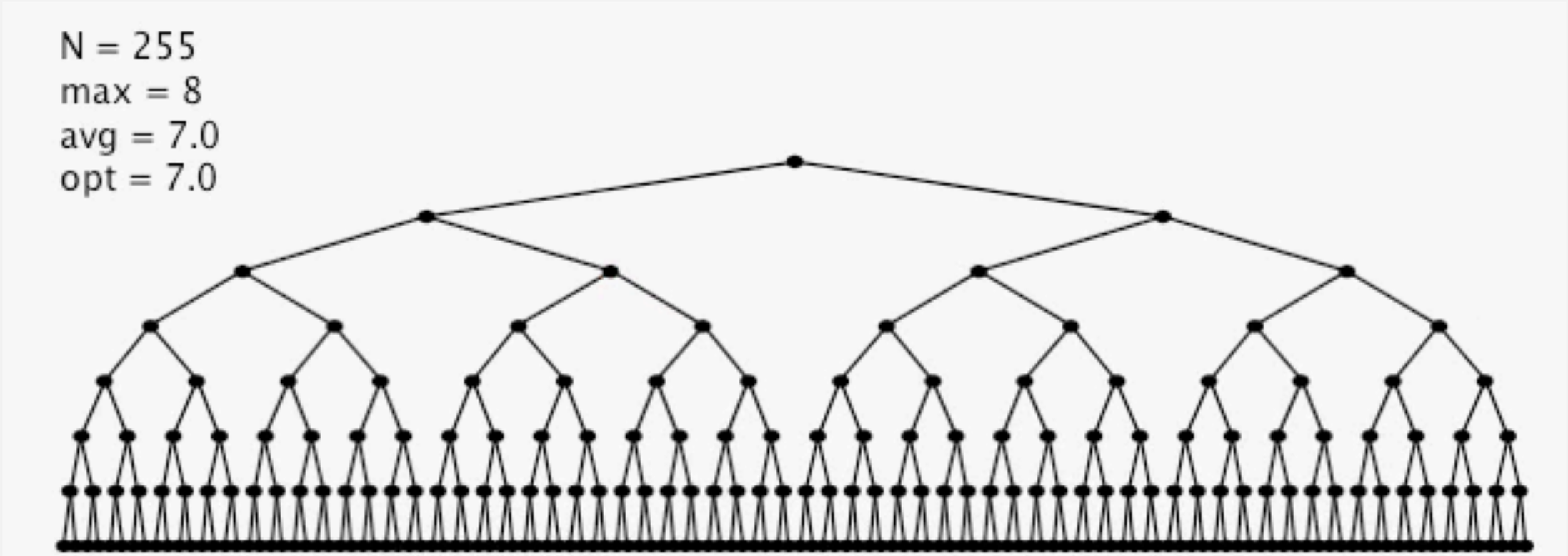
only a few extra lines of code provides near-perfect balance

Insertion in a LLRB tree: visualization



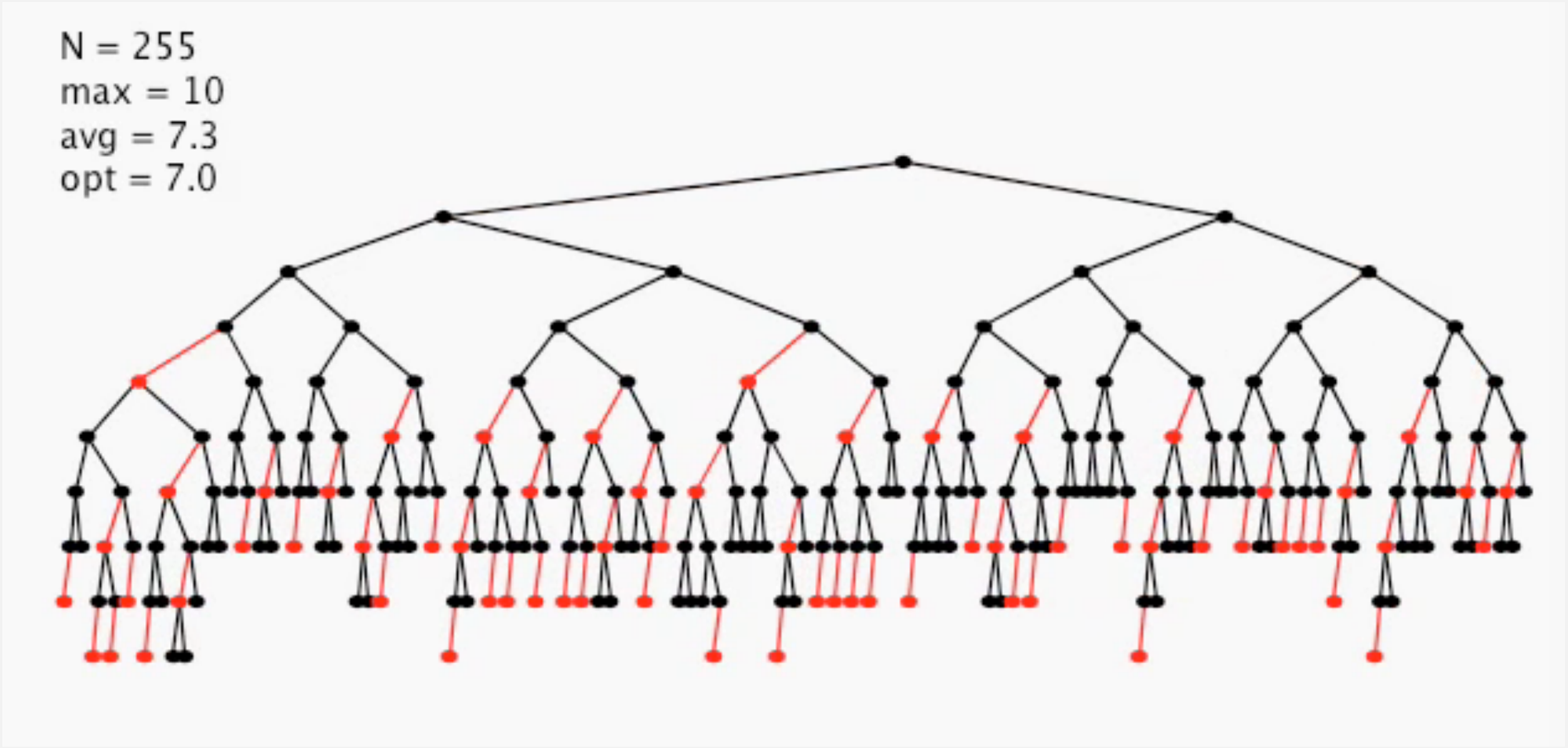
255 insertions in ascending order

Insertion in a LLRB tree: visualization



255 insertions in descending order

Insertion in a LLRB tree: visualization



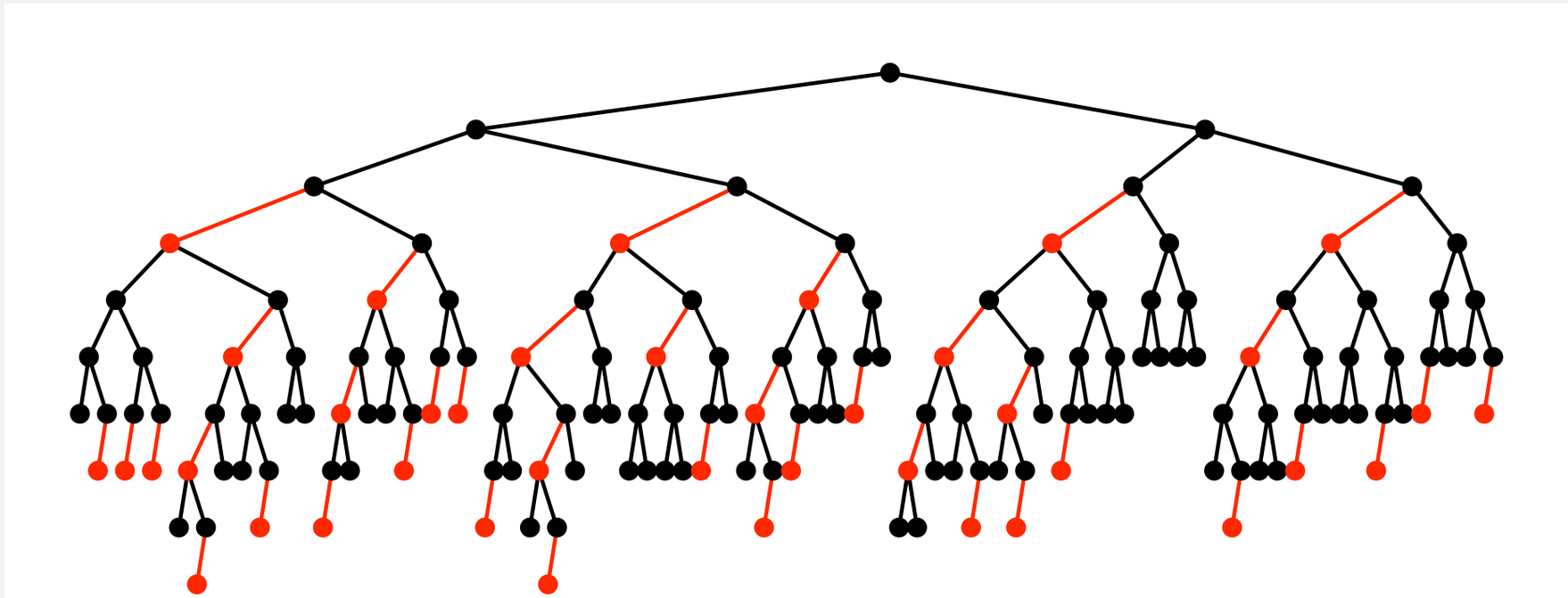
255 random insertions

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg N$ in the worst case.

Pf.

- Every path from root to null link has same number of black links.
- Never two red links in-a-row.



Property. Height of tree is ~ $1.0 \lg N$ in typical applications.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search hit	insert	delete		
sequential search (unordered list)	N	N	N	$\frac{1}{2} N$	N	$\frac{1}{2} N$		`equals()`
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✔	`compareTo()`
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	`compareTo()`
2-3 tree	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	$c \lg N$	✔	`compareTo()`
red-black BST	$2 \lg N$	$2 \lg N$	$2 \lg N$	$1.0 \lg N$ *	$1.0 \lg N$ *	$1.0 \lg N$ *	✔	`compareTo()`

* exact value of coefficient unknown but extremely close to 1

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3.3 Balanced Search Trees

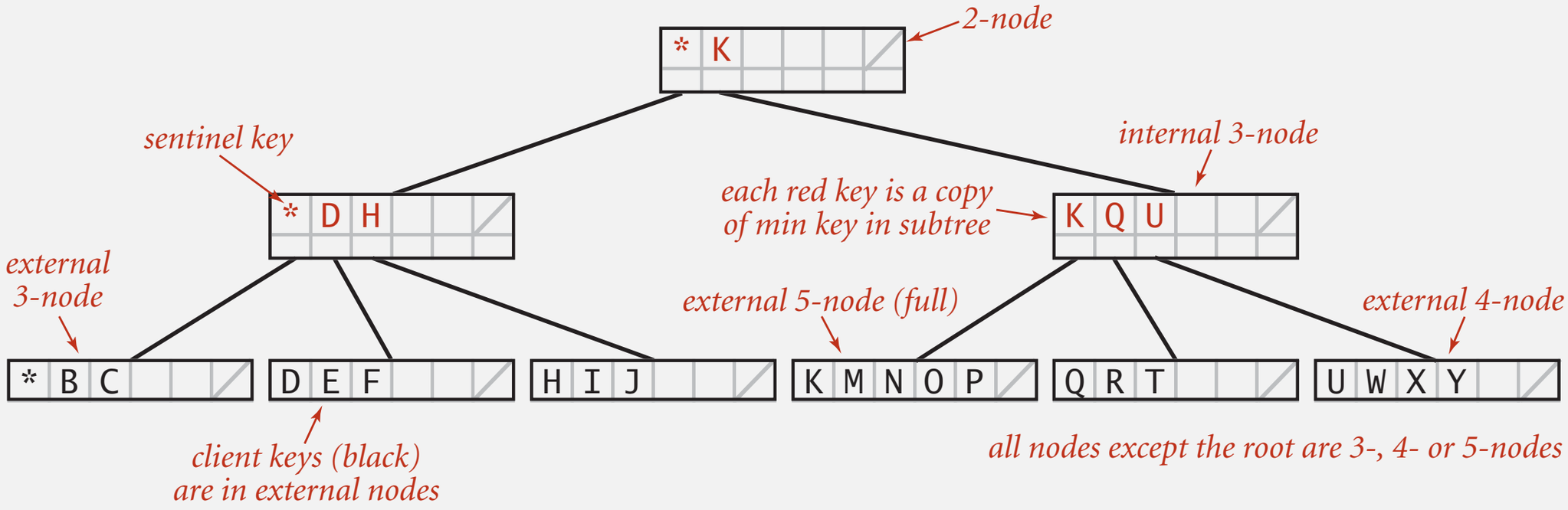
- *2-3 search trees*
- *red-black BSTs*
- *B-trees*

B-trees (Bayer-McCreight, 1972)

B-tree. Generalize 2-3 trees by allowing up to $M - 1$ key-link pairs per node.

- At least 2 key-link pairs at root.
- At least $M / 2$ key-link pairs in other nodes.
- External nodes contain client keys.
- Internal nodes contain copies of keys to guide search.

choose M as large as possible so that M links fit in a page, e.g., M = 1024



Anatomy of a B-tree set (M = 6)

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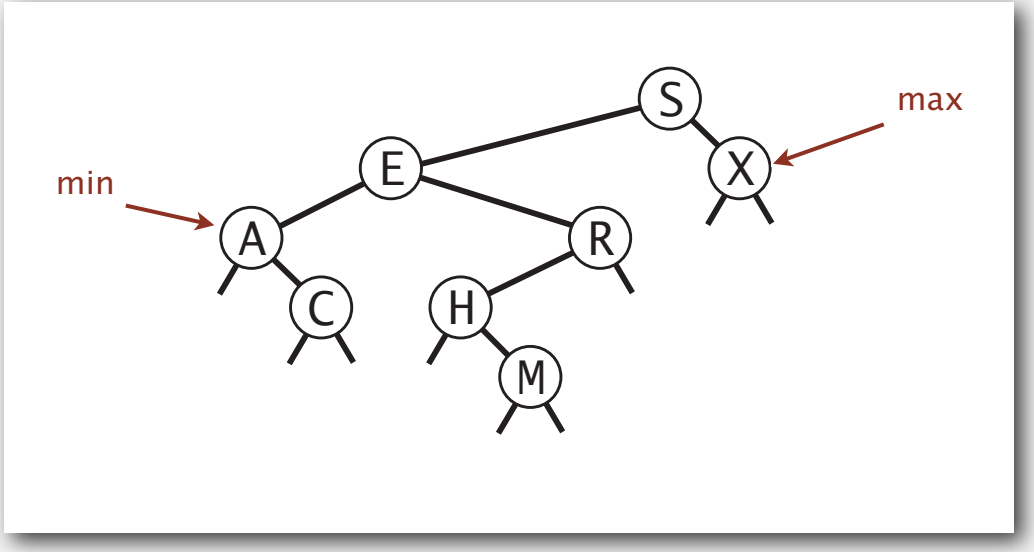
3.2 Binary Search Trees

- *BSTs*
- *ordered operations*
- *deletion*

Minimum and maximum

Minimum. Smallest key in table.

Maximum. Largest key in table.

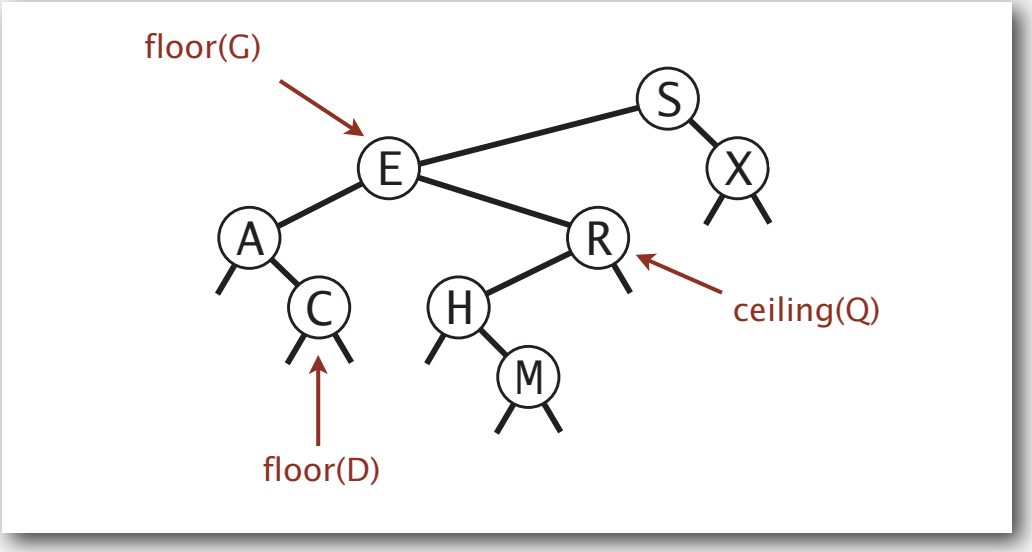


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ a given key.

Ceiling. Smallest key ≥ a given key.



Q. How to find the floor / ceiling?

Computing the floor

Case 1. [k equals the key at root]

The floor of k is k.

Case 2. [k is less than the key at root]

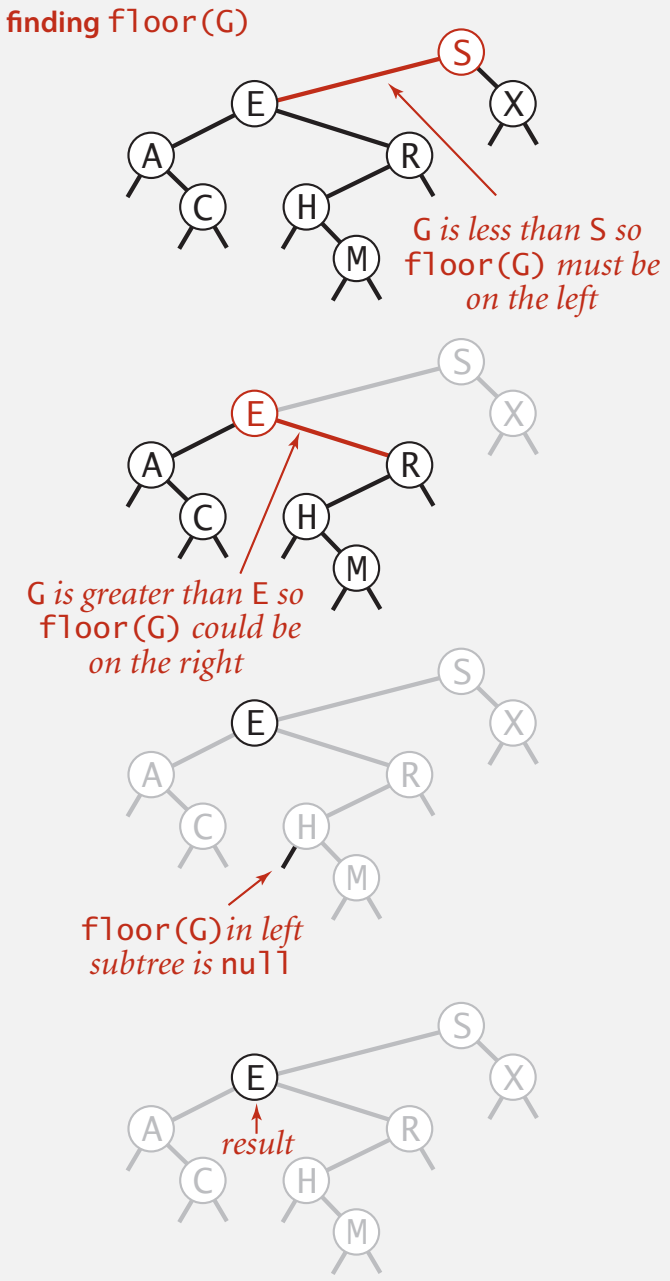
The floor of k is in the left subtree.

Case 3. [k is greater than the key at root]

The floor of k is in the right subtree

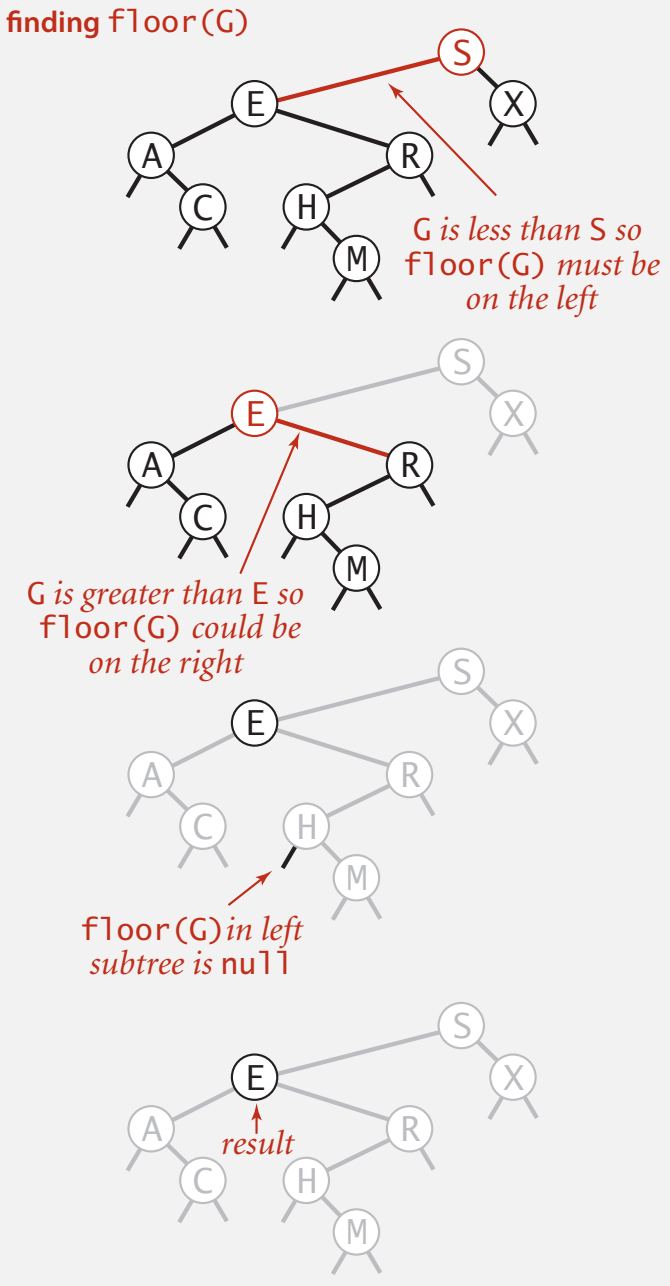
(if there is any key $\leq k$ in right subtree);

otherwise it is the key in the root.



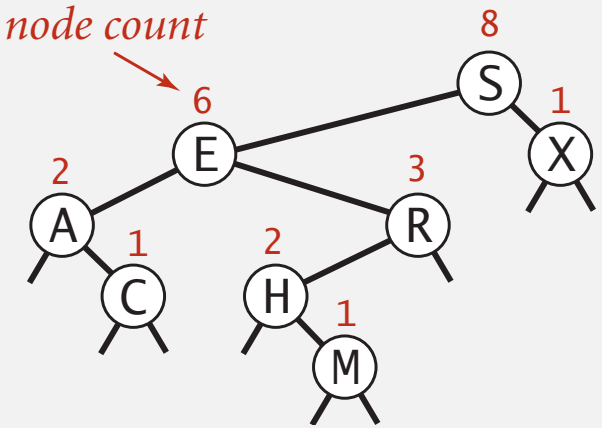
Computing the floor

```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Subtree counts

In each node, we store the number of nodes in the subtree rooted at that node; to implement `size()`, return the count at the root.



Remark. This facilitates efficient implementation of `rank()` and `select()`.

BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{  return size(root);  }

private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

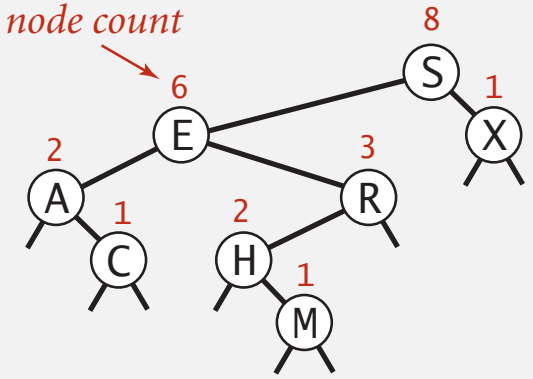
ok to call when x is null

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

Rank

Rank. How many keys < k ?

Easy recursive algorithm (3 cases!)




```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

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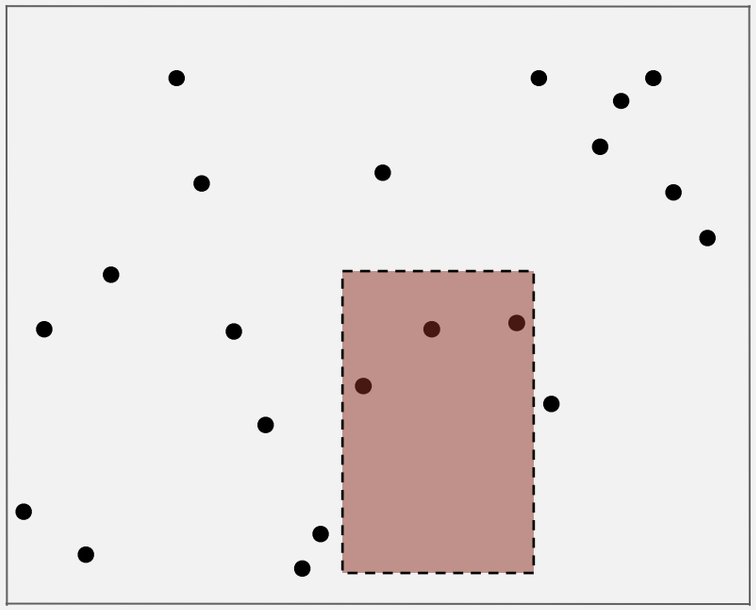
GEOMETRIC APPLICATIONS OF BSTS

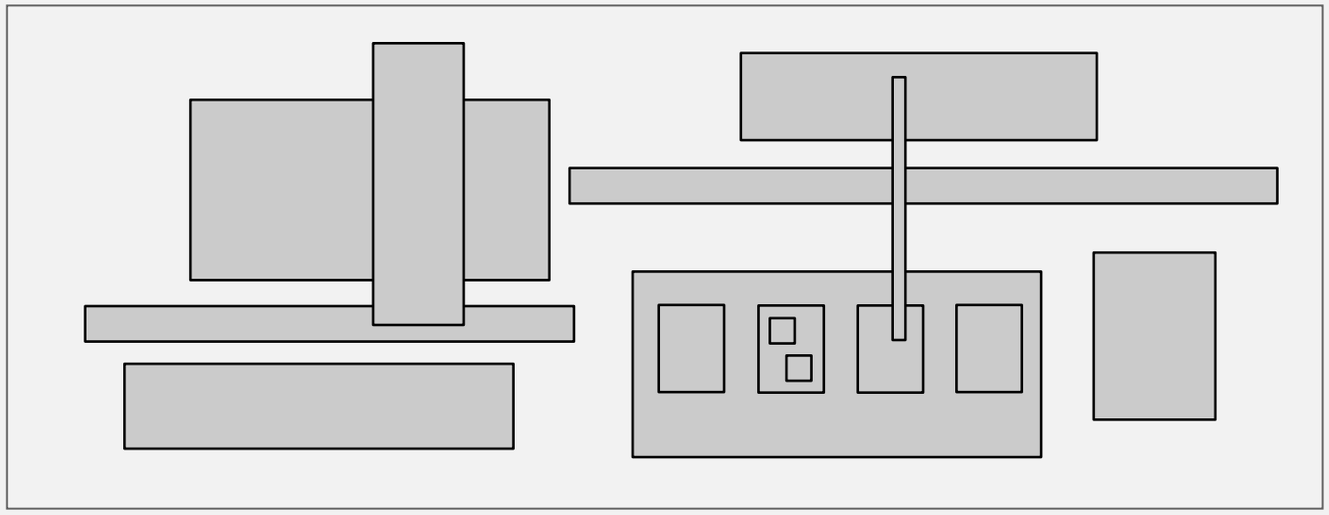
- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

Overview

This lecture. Intersections among geometric objects.



2d orthogonal range search



orthogonal rectangle intersection

Applications. CAD, games, movies, virtual reality, databases, GIS, ...

Efficient solutions. Binary search trees (and extensions).

Geometric Applications of BSTs

- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

1d range search

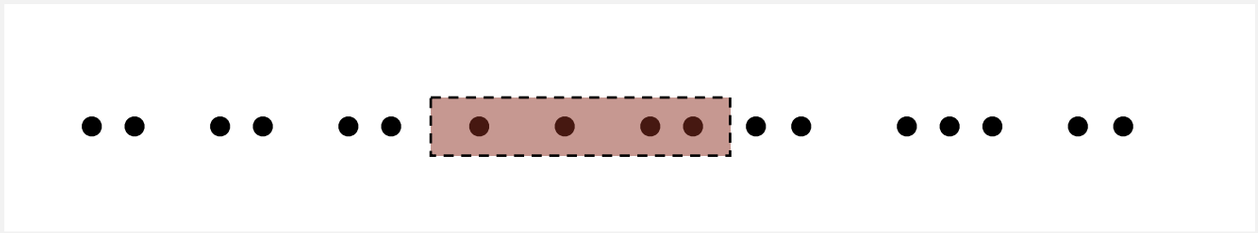
Extension of ordered symbol table.

- Insert key-value pair.
- Search for key k.
- Delete key k.
- Range search: find all keys between k_1 and k_2.
- Range count: number of keys between k_1 and k_2.

Application. Database queries.

Geometric interpretation.

- Keys are point on a line.
- Find/count points in a given 1d interval.



insert B	B
insert D	B D
insert A	A B D
insert I	A B D I
insert H	A B D H I
insert F	A B D F H I
insert P	A B D F H I P
search G to K	H I
count G to K	2

1d range search: elementary implementations

Unordered list. Fast insert, slow range search.

Ordered array. Slow insert, binary search for k_1 and k_2 to do range search.

order of growth of running time for 1d range search

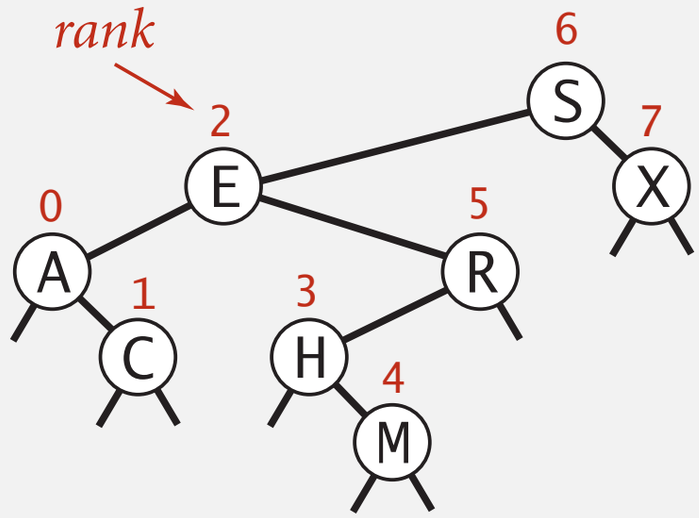
data structure	insert	range count	range search
unordered list	1	N	N
ordered array	N	$\log N$	$R + \log N$
goal	$\log N$	$\log N$	$R + \log N$

N = number of keys

R = number of keys that match

1d range count: BST implementation

1d range count. How many keys between `lo` and `hi` ?




```
public int size(Key lo, Key hi)
{
   if (contains(hi)) return rank(hi) - rank(lo) + 1;
   else              return rank(hi) - rank(lo);
}
```

number of keys < hi

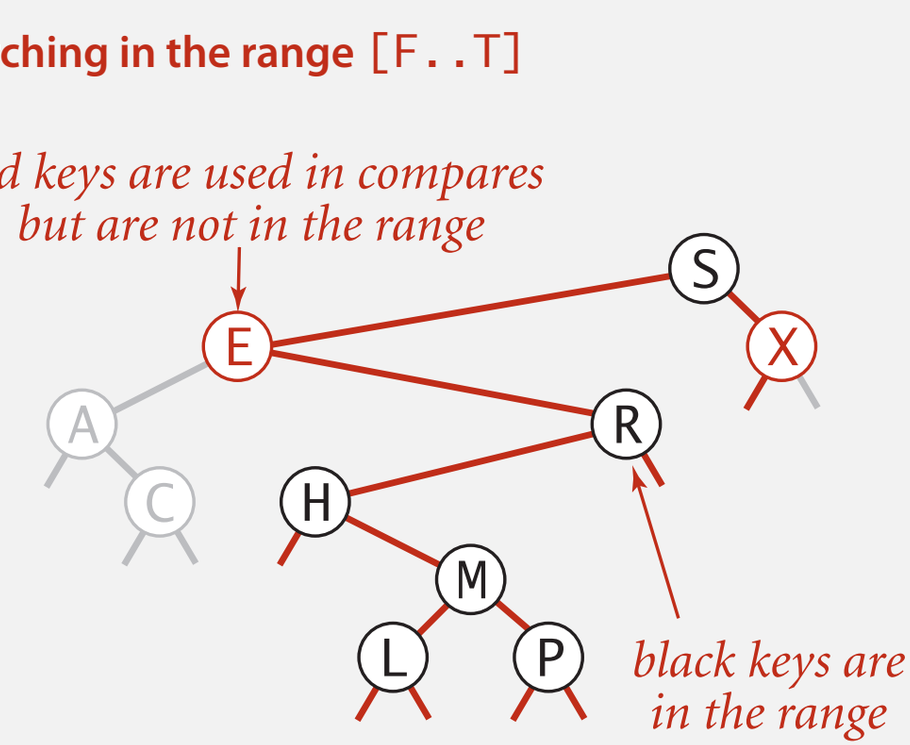
Proposition. Running time proportional to $\log N$.

Pf. Nodes examined = search path to `lo` + search path to `hi`.

1d range search: BST implementation

1d range search. Find all keys between `lo` and `hi`.

- Recursively find all keys in left subtree (if any could fall in range).
- Check key in current node.
- Recursively find all keys in right subtree (if any could fall in range).



Proposition. Running time proportional to $R + \log N$.

Pf. Nodes examined = search path to `lo` + search path to `hi` + matches.

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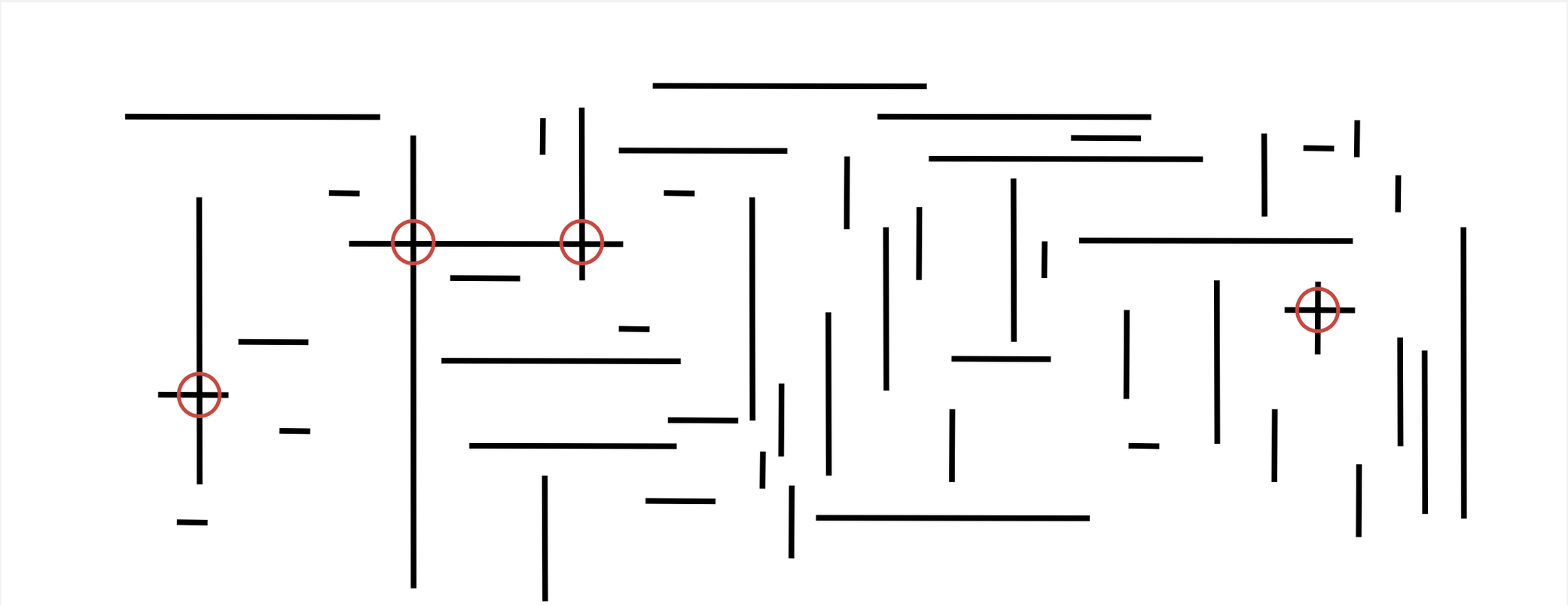
Geometric Applications of BSTs

- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

Orthogonal line segment intersection

Given N horizontal and vertical line segments, find all intersections.



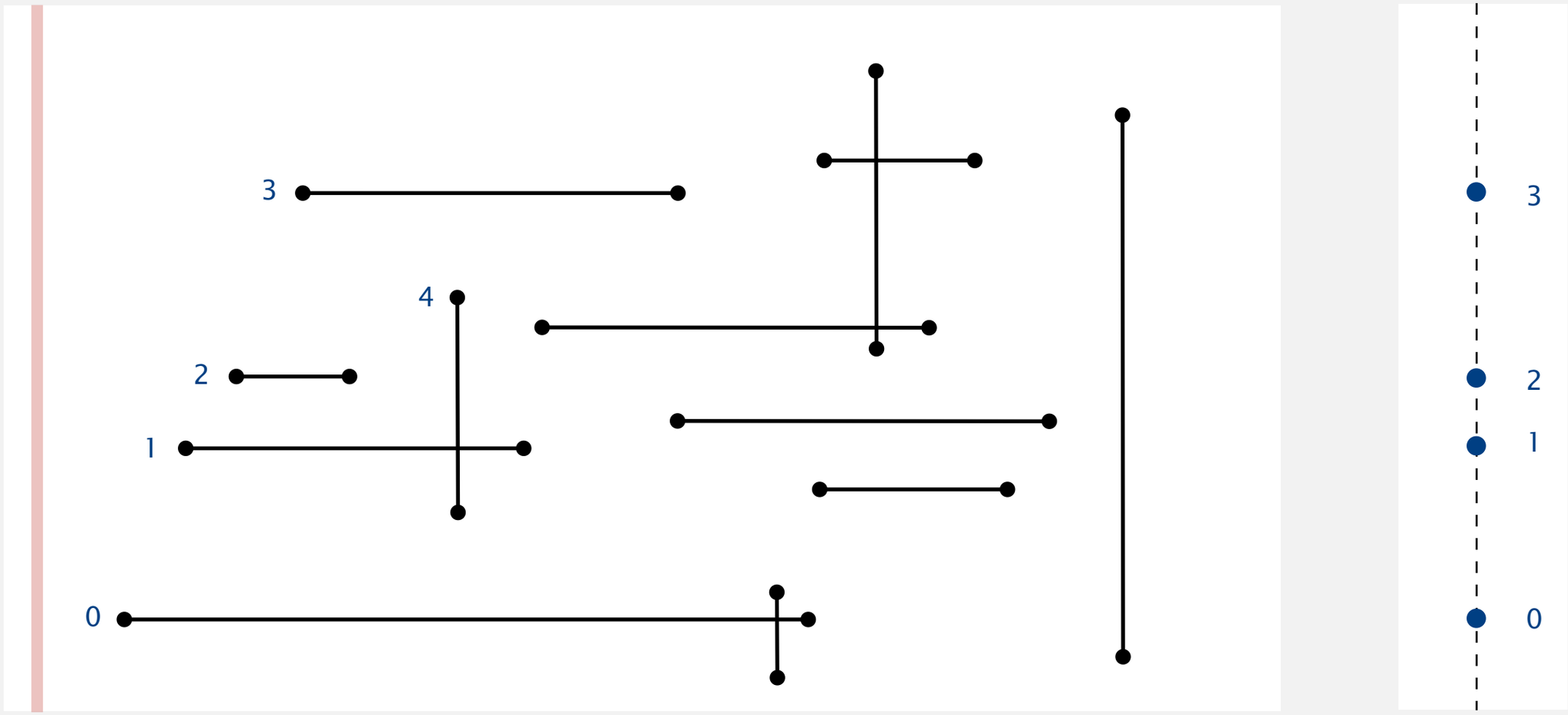
Quadratic algorithm. Check all pairs of line segments for intersection.

Nondegeneracy assumption. All x- and y-coordinates are distinct.

Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

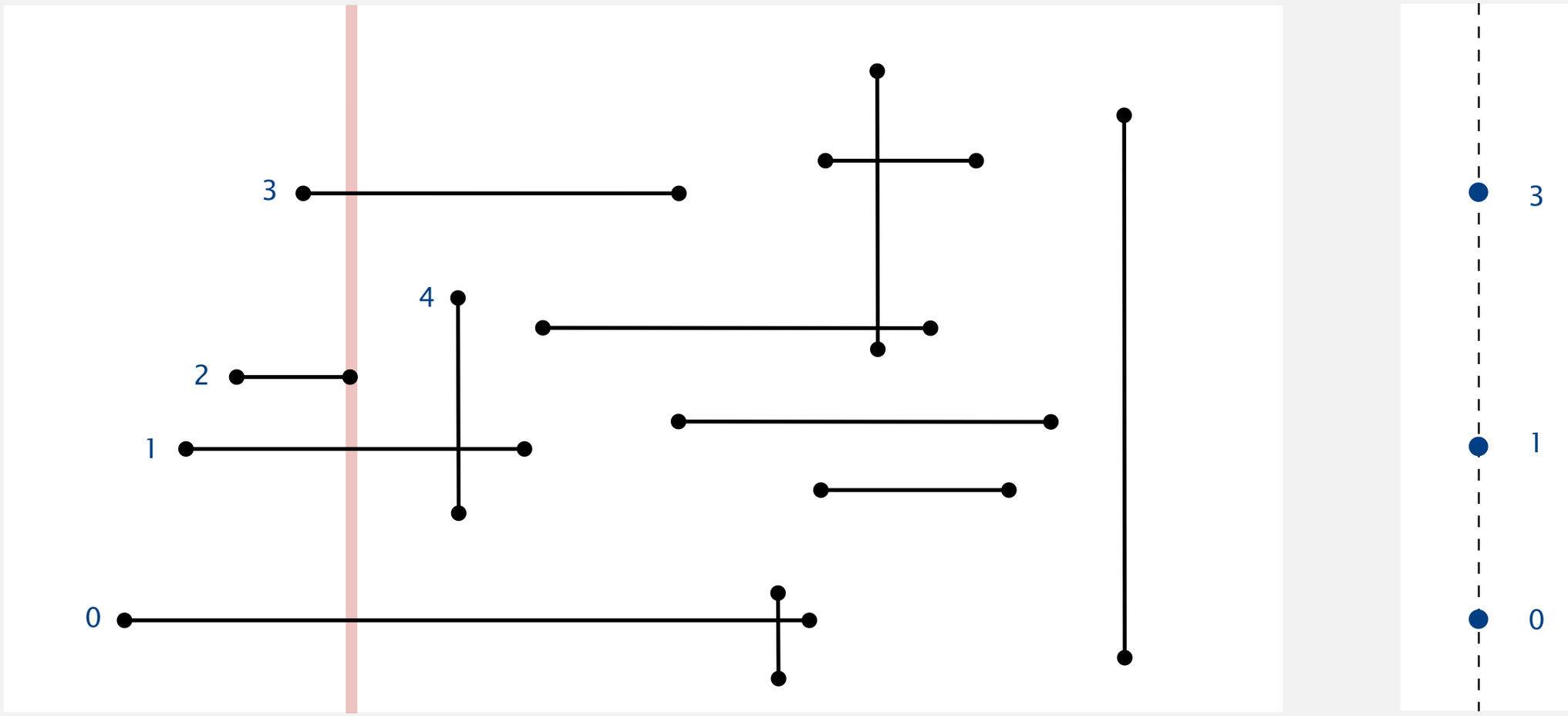
- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.



Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

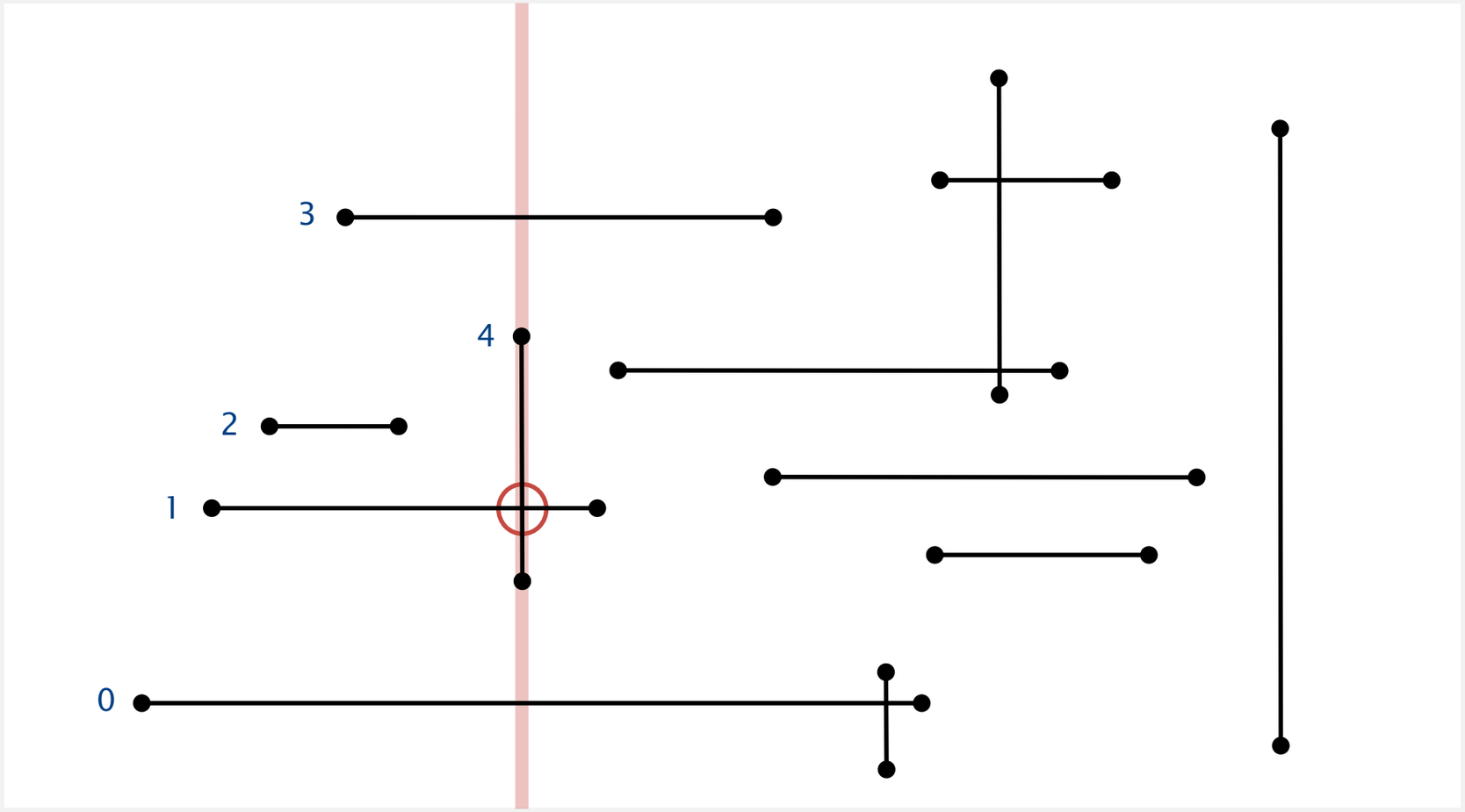
- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.
- h-segment (right endpoint): remove y-coordinate from BST.

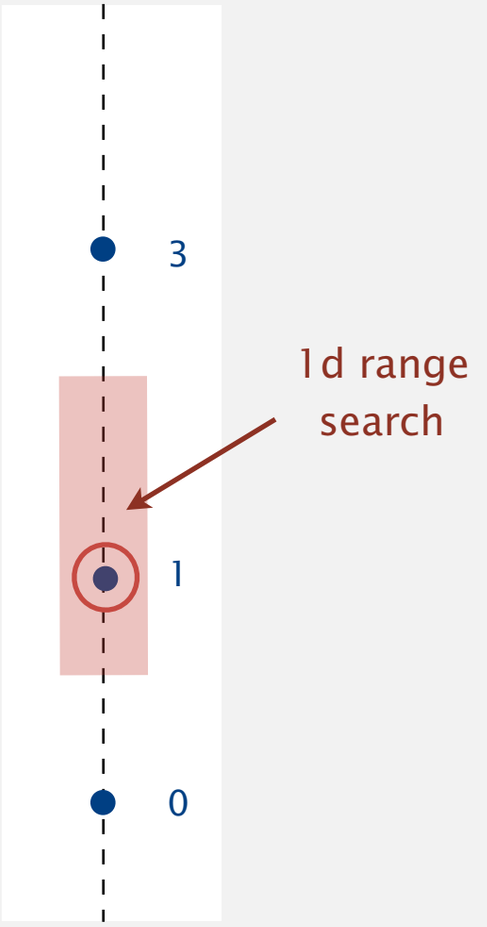


Orthogonal line segment intersection: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates define events.
- h-segment (left endpoint): insert y-coordinate into BST.
- h-segment (right endpoint): remove y-coordinate from BST.
- v-segment: range search for interval of y-endpoints.





Orthogonal line segment intersection: sweep-line analysis

Proposition. The sweep-line algorithm takes time proportional to $N \log N + R$ to find all R intersections among N orthogonal line segments.

Pf.

- Put x-coordinates on a PQ (or sort). ← N log N
- Insert y-coordinates into BST. ← N log N
- Delete y-coordinates from BST. ← N log N
- Range searches in BST. ← N log N + R

Bottom line. Sweep line reduces 2d orthogonal line segment intersection search to 1d range search.

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Geometric Applications of BSTs

- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

2-d orthogonal range search

Extension of ordered symbol-table to 2d keys.

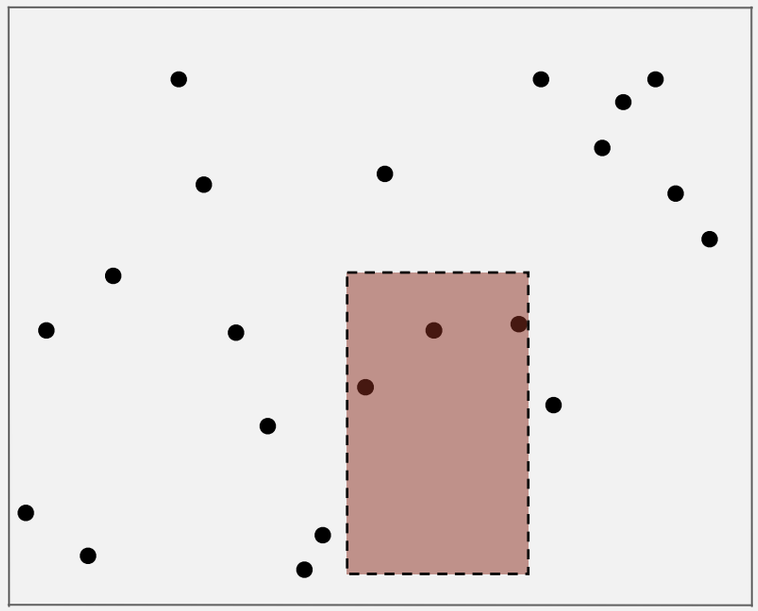
- Insert a 2d key.
- Delete a 2d key.
- Search for a 2d key.
- Range search: find all keys that lie in a 2d range.
- Range count: number of keys that lie in a 2d range.

Applications. Networking, circuit design, databases, ...

Geometric interpretation.

- Keys are point in the plane.
- Find/count points in a given h–v rectangle

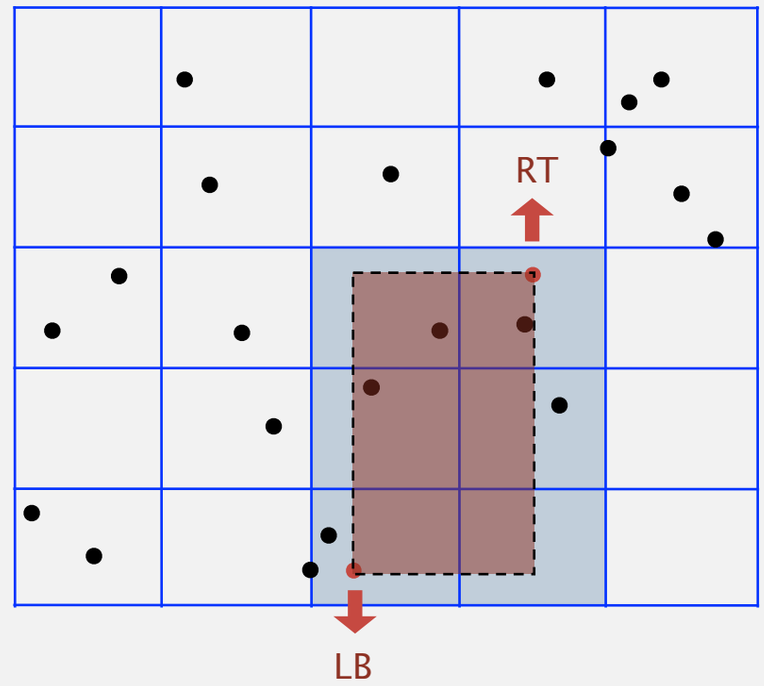
rectangle is axis-aligned



2d orthogonal range search: grid implementation

Grid implementation.

- Divide space into M-by-M grid of squares.
- Create list of points contained in each square.
- Use 2d array to directly index relevant square.
- Insert: add (x, y) to list for corresponding square.
- Range search: examine only squares that intersect 2d range query.



2d orthogonal range search: grid implementation analysis

Space-time tradeoff.

- Space: $M^2 + N$.
- Time: $1 + N / M^2$ per square examined, on average.

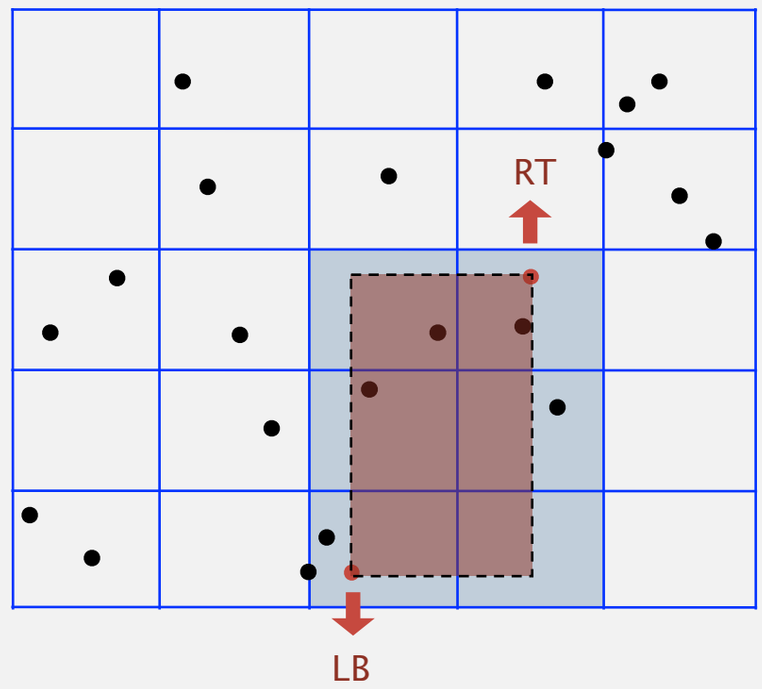
Choose grid square size to tune performance.

- Too small: wastes space.
- Too large: too many points per square.
- Rule of thumb: $\sqrt{N}$-by-$\sqrt{N}$ grid.

Running time. [if points are evenly distributed]

- Initialize data structure: N.
- Insert point: Constant time
- Range search: 1 per point in range.

choose M ~ $\sqrt{N}$

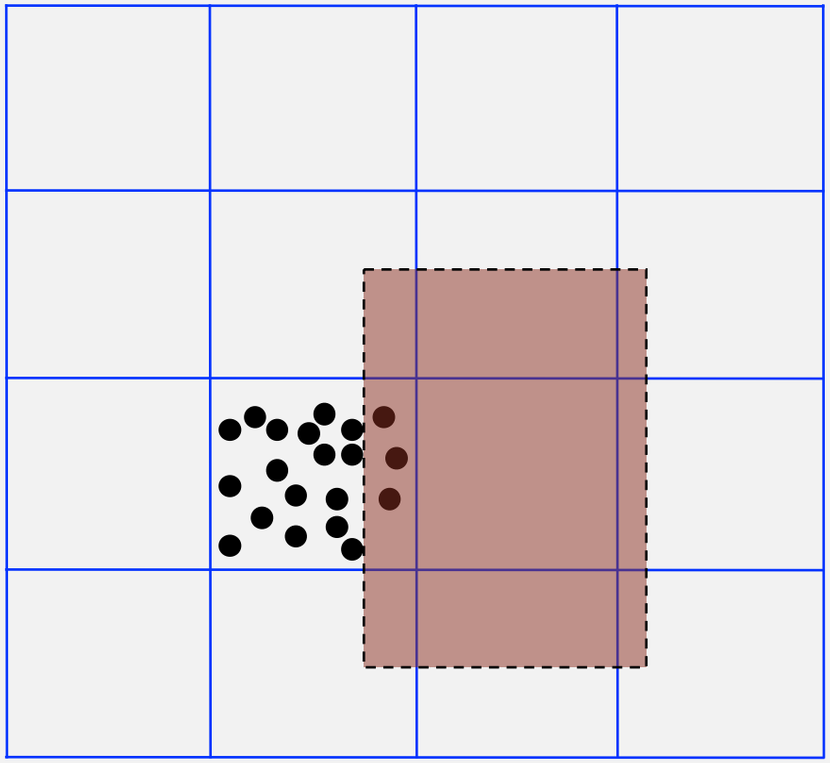


Clustering

Grid implementation. Fast, simple solution for evenly-distributed points.

Problem. Clustering a well-known phenomenon in geometric data.

- Lists are too long, even though average length is short.
- Need data structure that adapts gracefully to data.



Space-partitioning trees

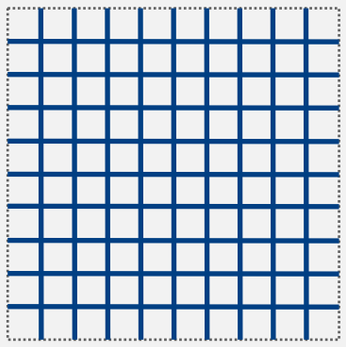
Use a tree to represent a recursive subdivision of 2d space.

Grid. Divide space uniformly into squares.

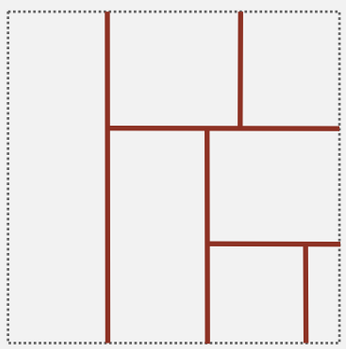
2d tree. Recursively divide space into two halfplanes.

Quadtree. Recursively divide space into four quadrants.

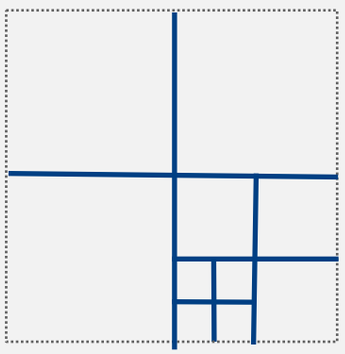
BSP tree. Recursively divide space into two regions.



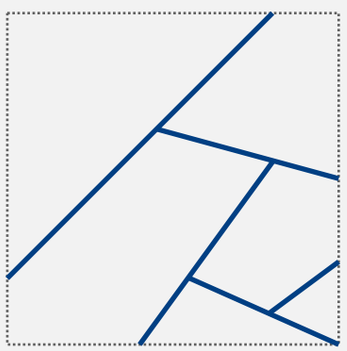
Grid



2d tree



Quadtree

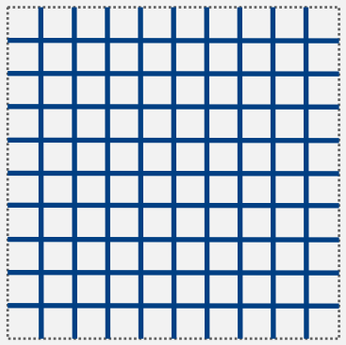


BSP tree

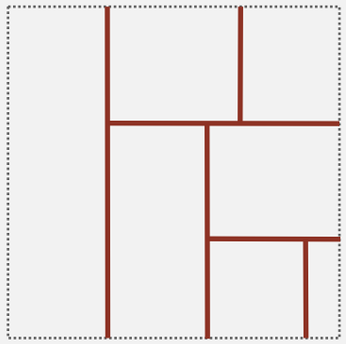
Space-partitioning trees: applications

Applications.

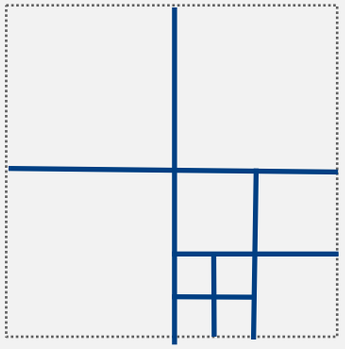
- Ray tracing.
- 2d range search.
- Flight simulators.
- N-body simulation.
- Collision detection.
- Astronomical databases.
- Nearest neighbor search.
- Adaptive mesh generation.
- Accelerate rendering in Doom.
- Hidden surface removal and shadow casting.



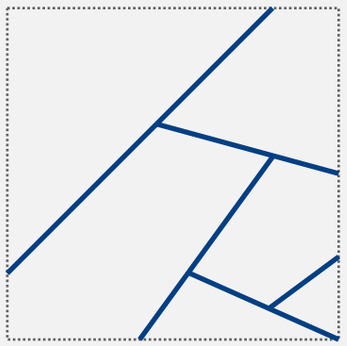
Grid



2d tree



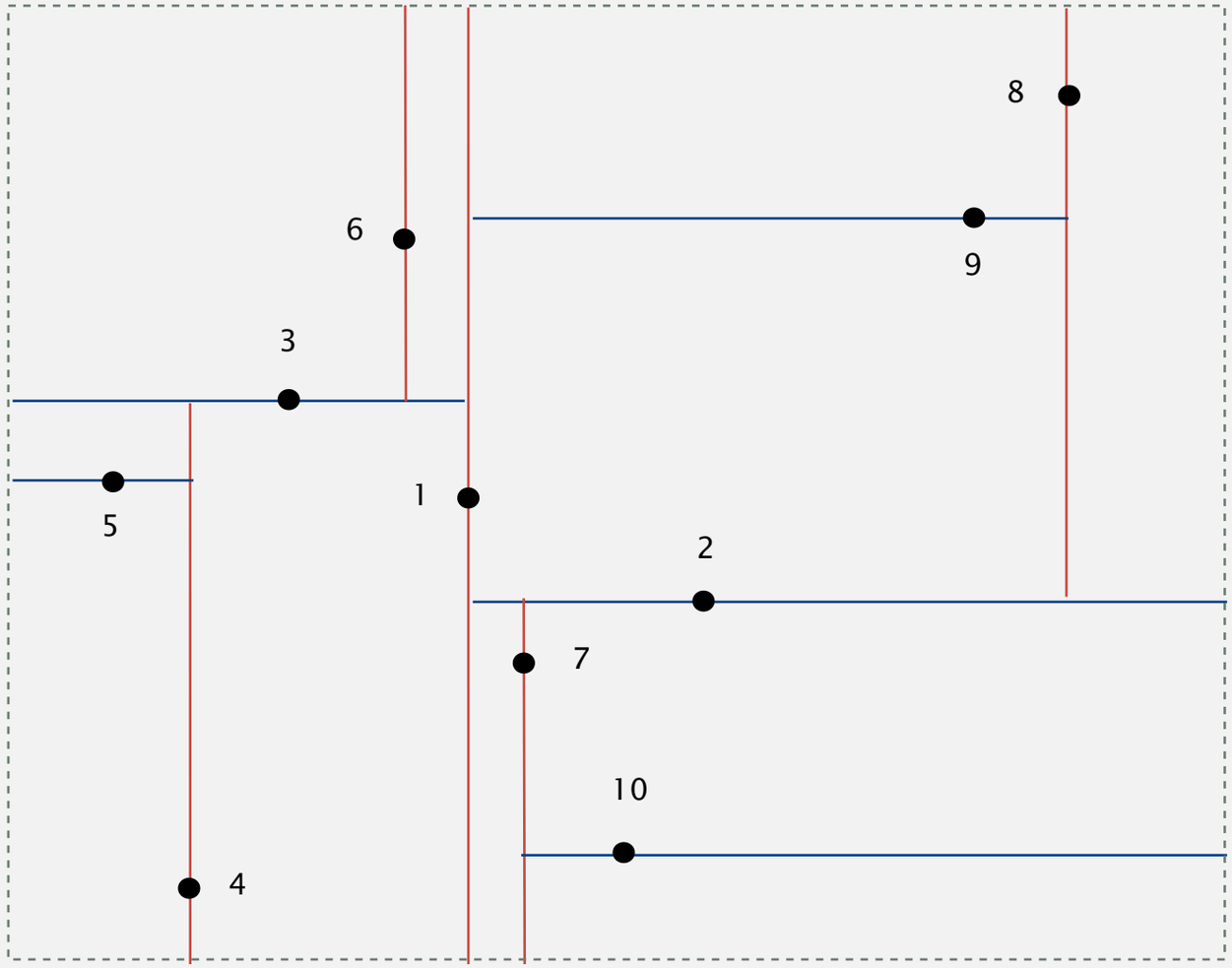
Quadtree

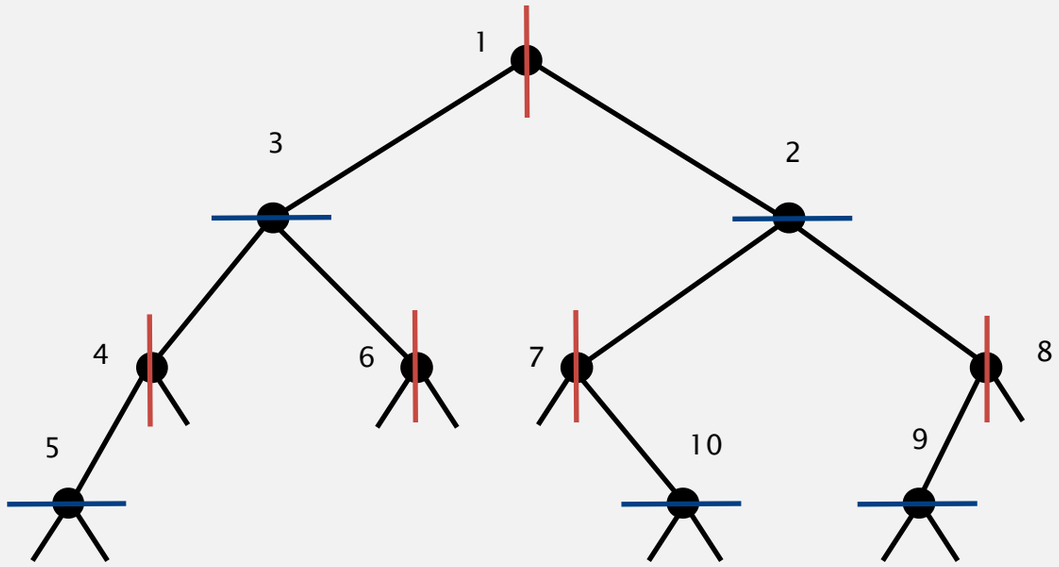


BSP tree

2d tree construction

Recursively partition plane into two halfplanes.

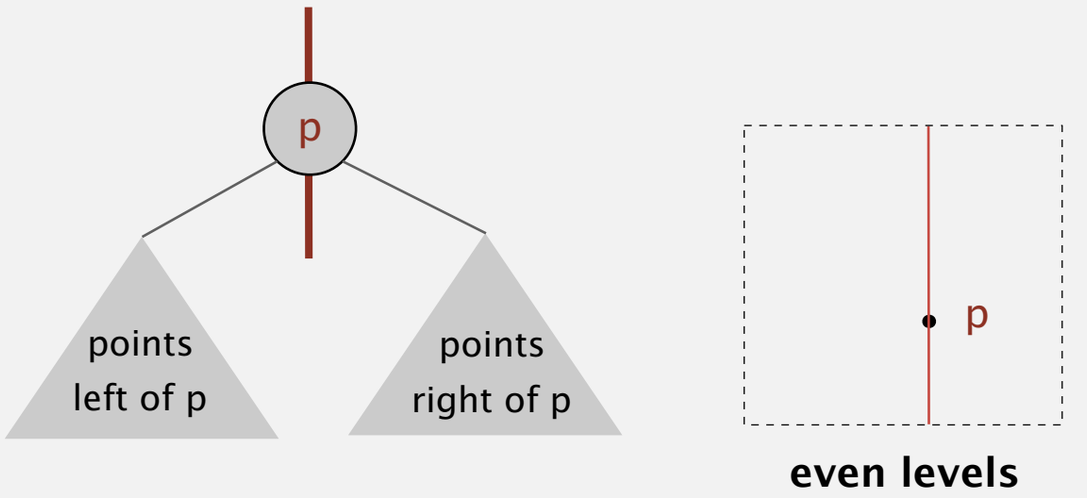


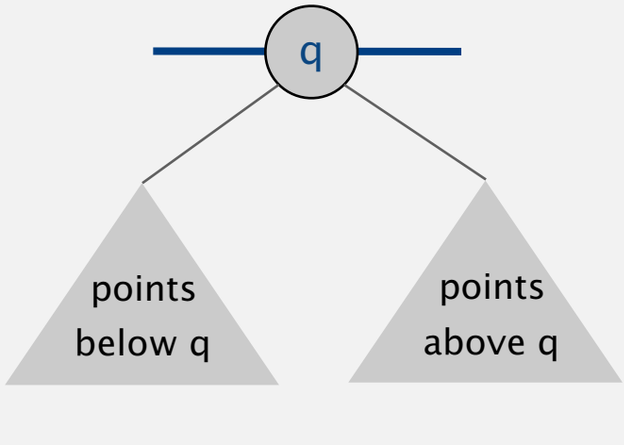


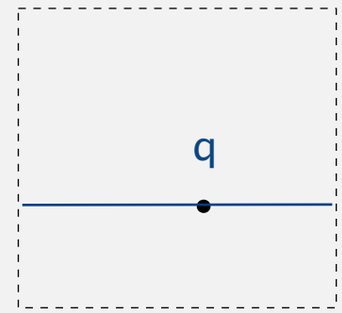
2d tree implementation

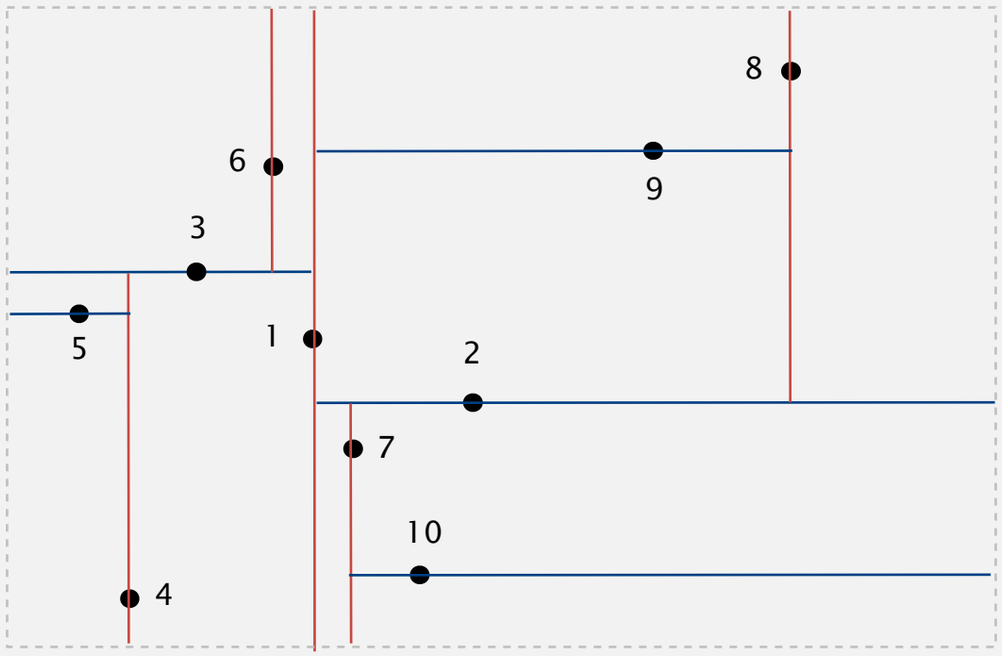
Data structure. BST, but alternate using x- and y-coordinates as key.

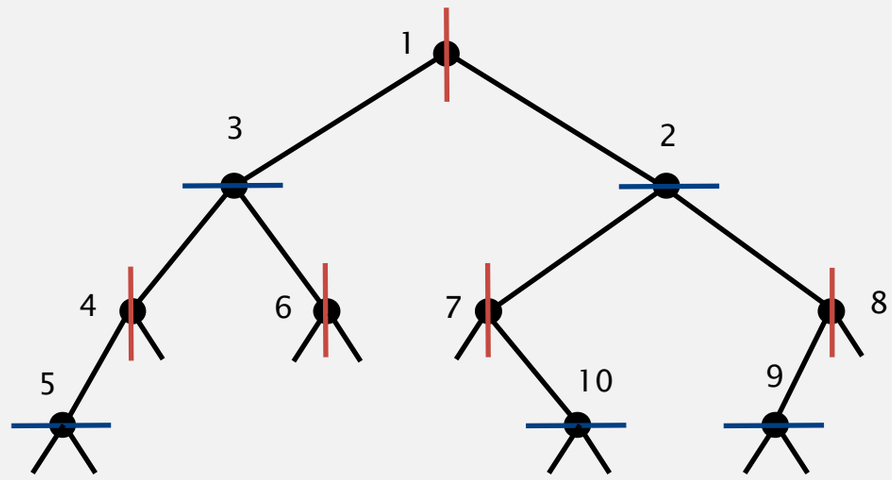
- Search gives rectangle containing point.
- Insert further subdivides the plane.







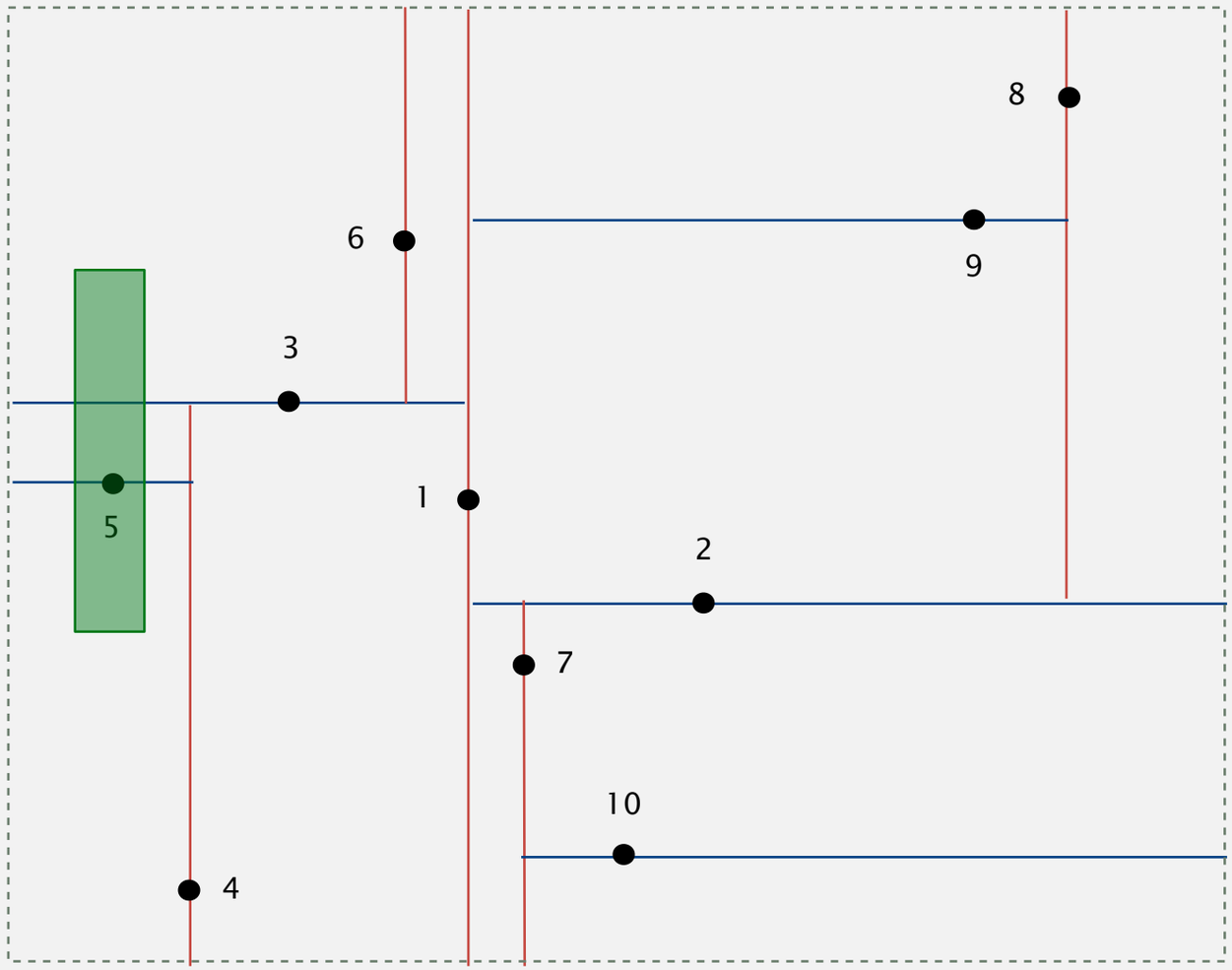


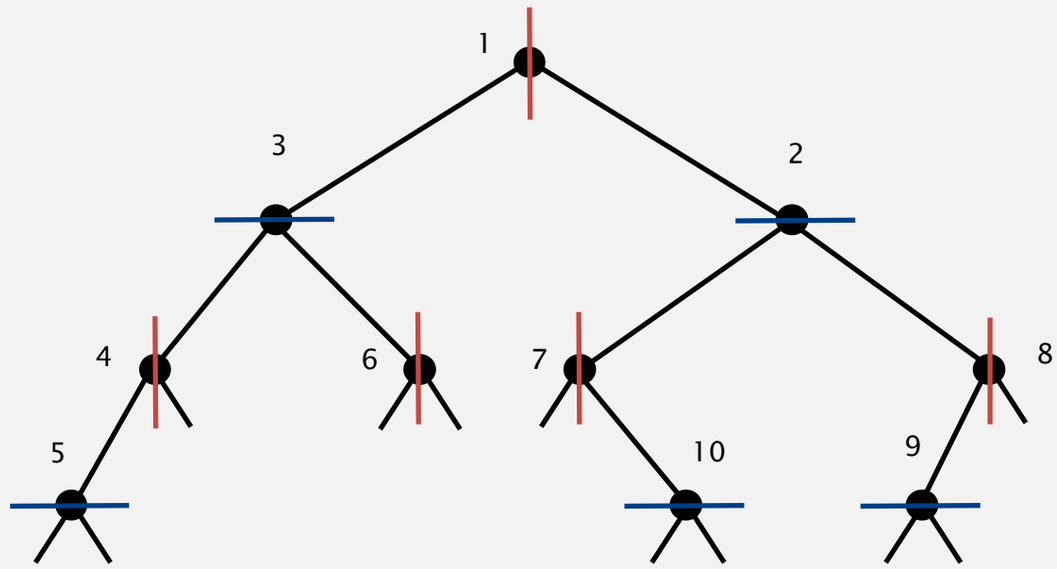


2d tree demo: range search

Goal. Find all points in a query axis-aligned rectangle.

- Check if point in node lies in given rectangle.
- Recursively search left/bottom (if any could fall in rectangle).
- Recursively search right/top (if any could fall in rectangle).





2d tree demo: range search

Goal. Find all points in a query axis-aligned rectangle.

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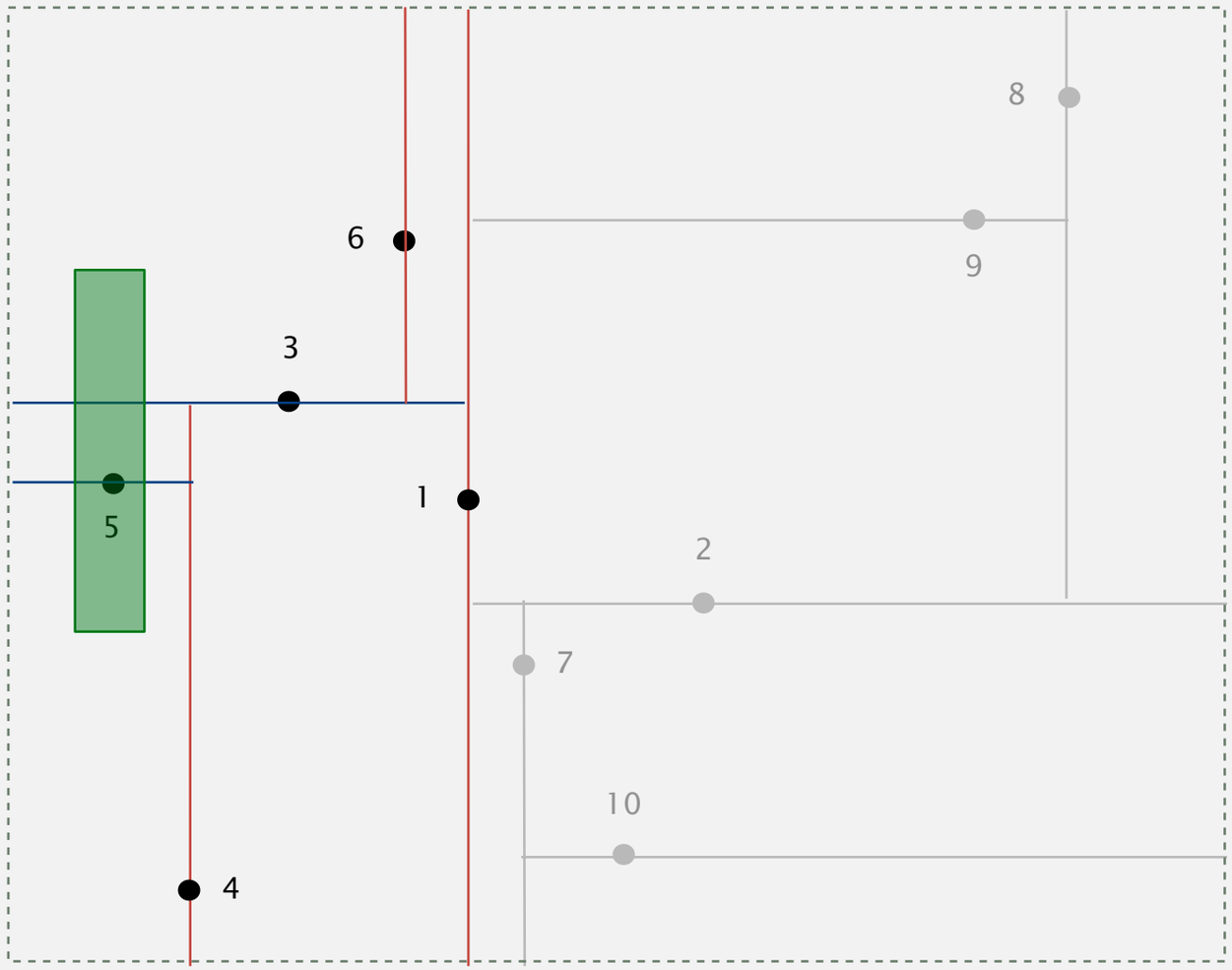
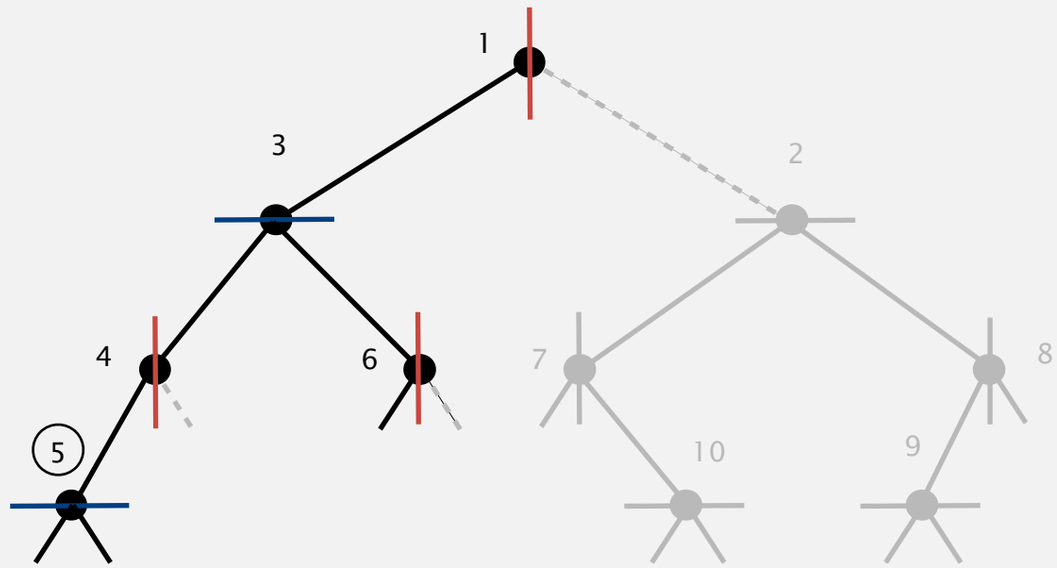


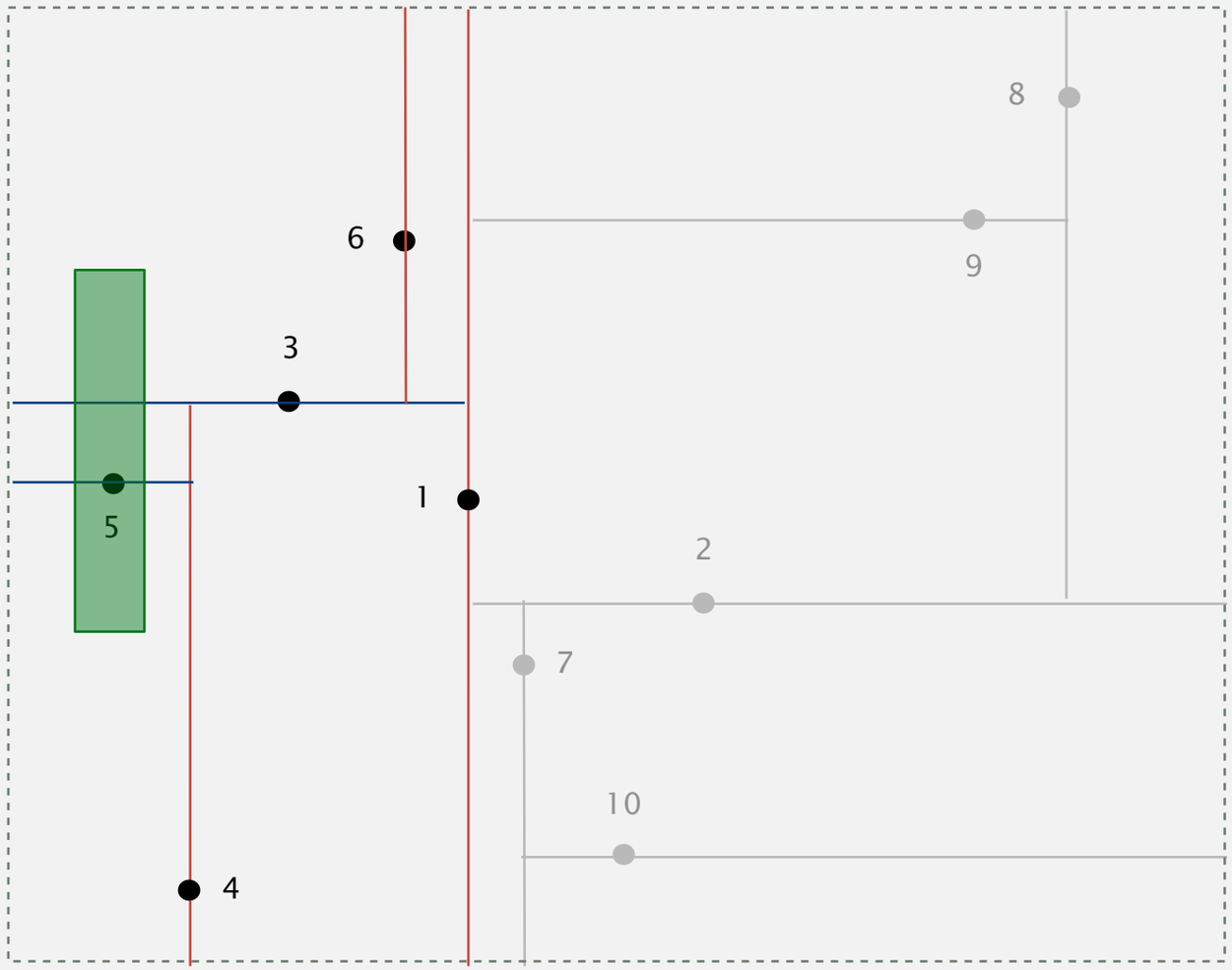


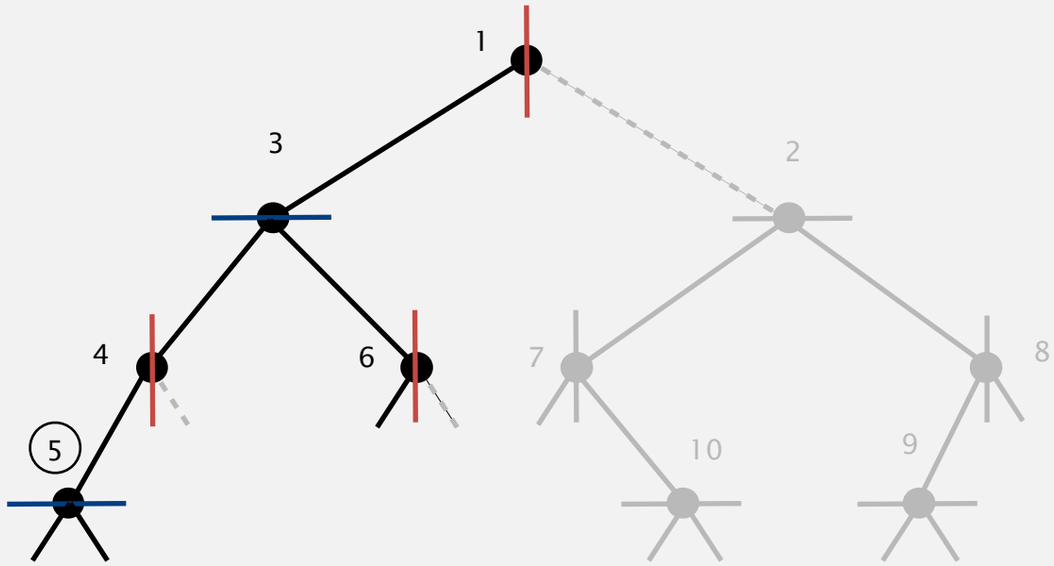
done

Range search in a 2d tree analysis

Typical case. $R + \log N$.

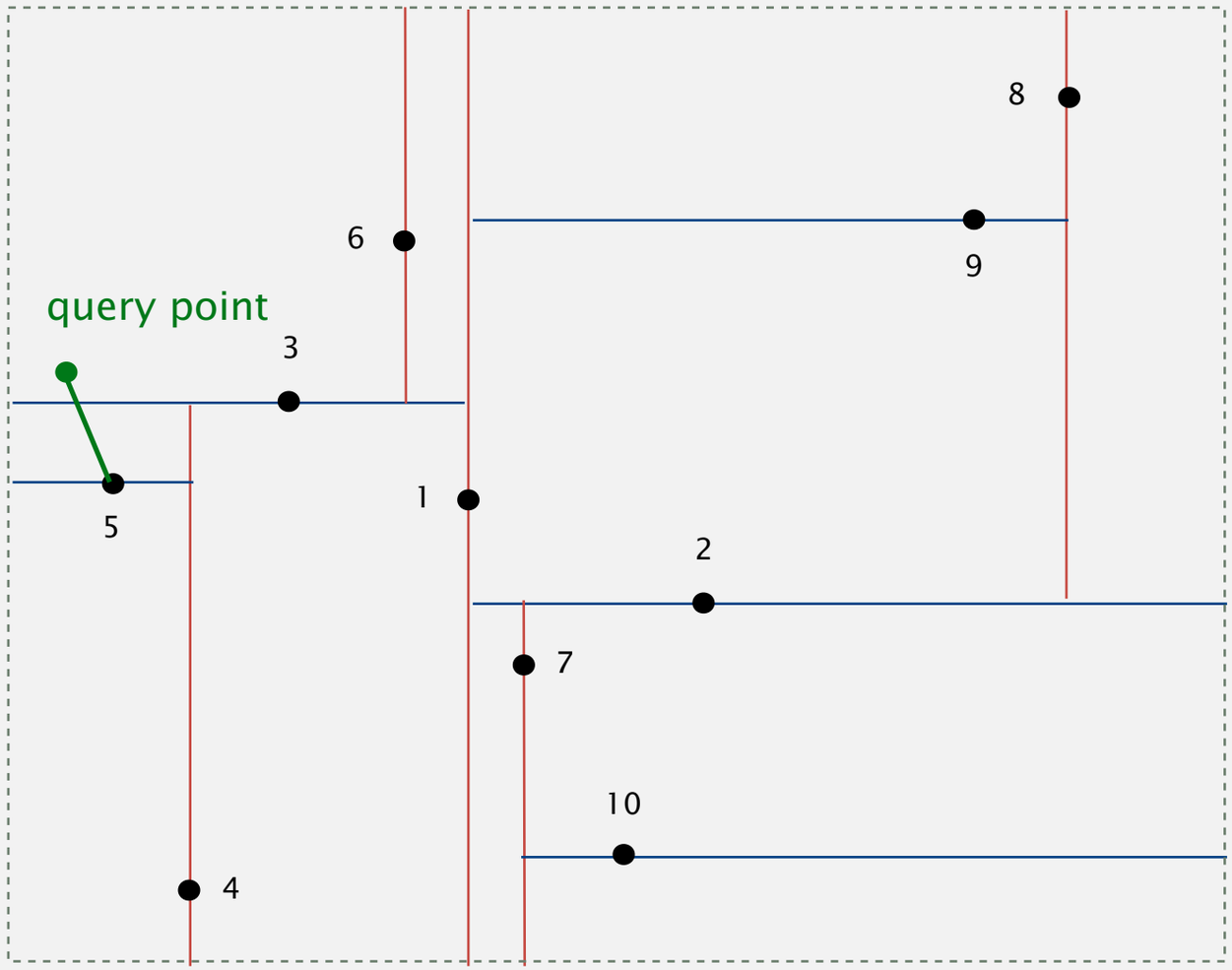
Worst case (assuming tree is balanced). $R + \sqrt{N}$.

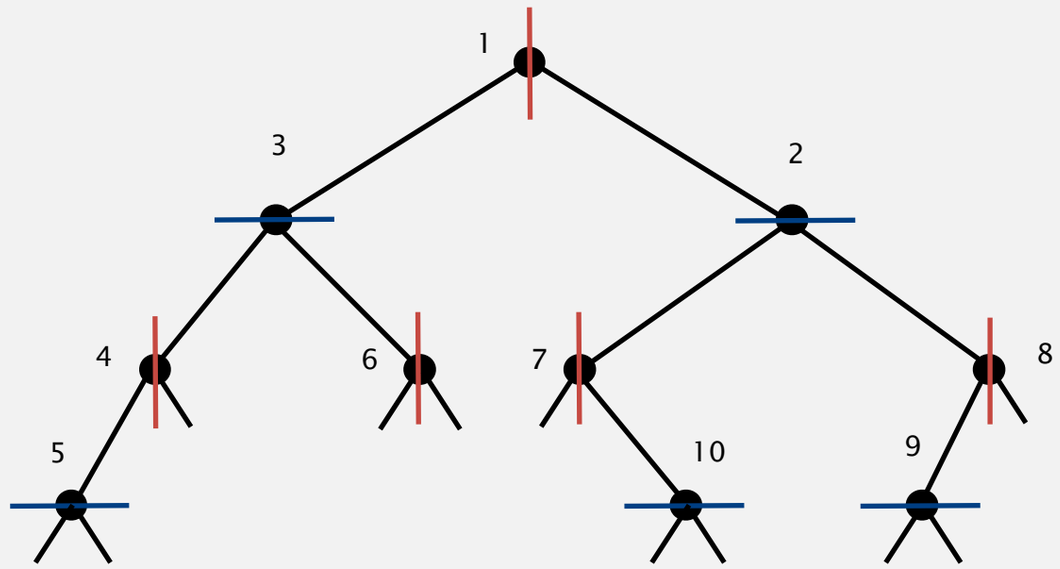




2d tree demo: nearest neighbor

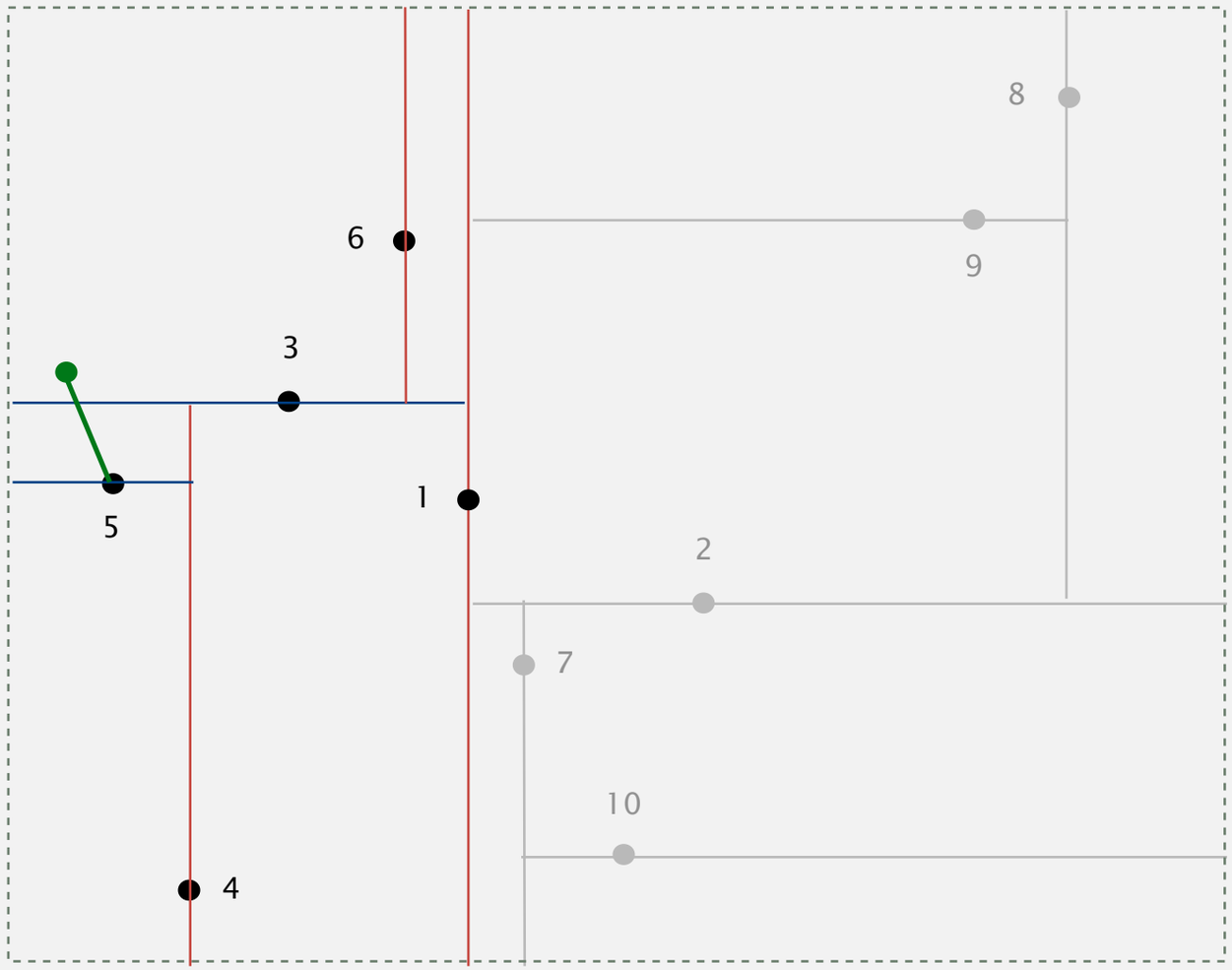
Goal. Find closest point to query point.

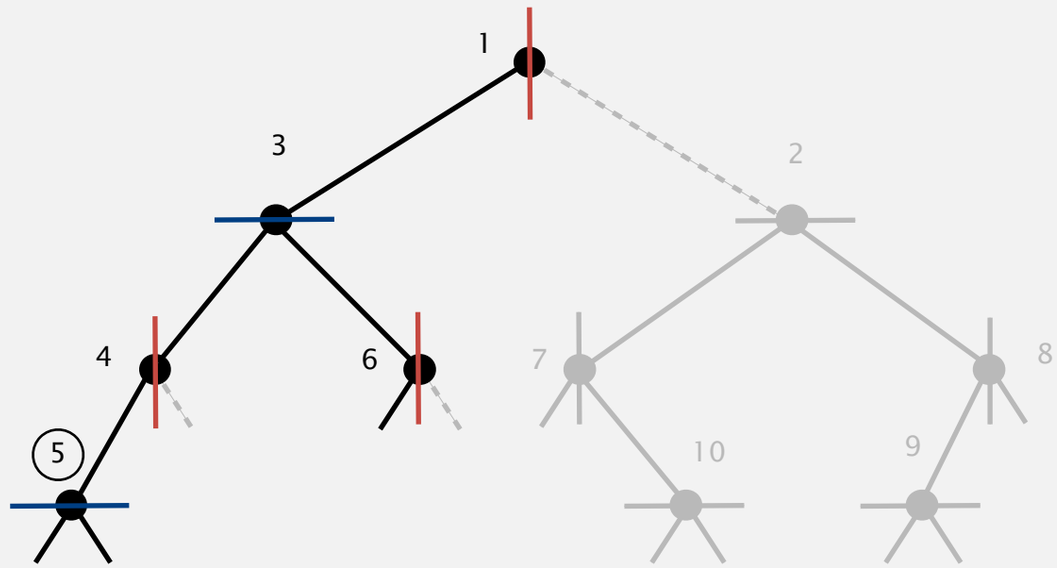




2d tree demo: nearest neighbor

- Check distance from point in node to query point.
- Recursively search left/bottom (if it could contain a closer point).
- Recursively search right/top (if it could contain a closer point).
- Organize method so that it begins by searching for query point.



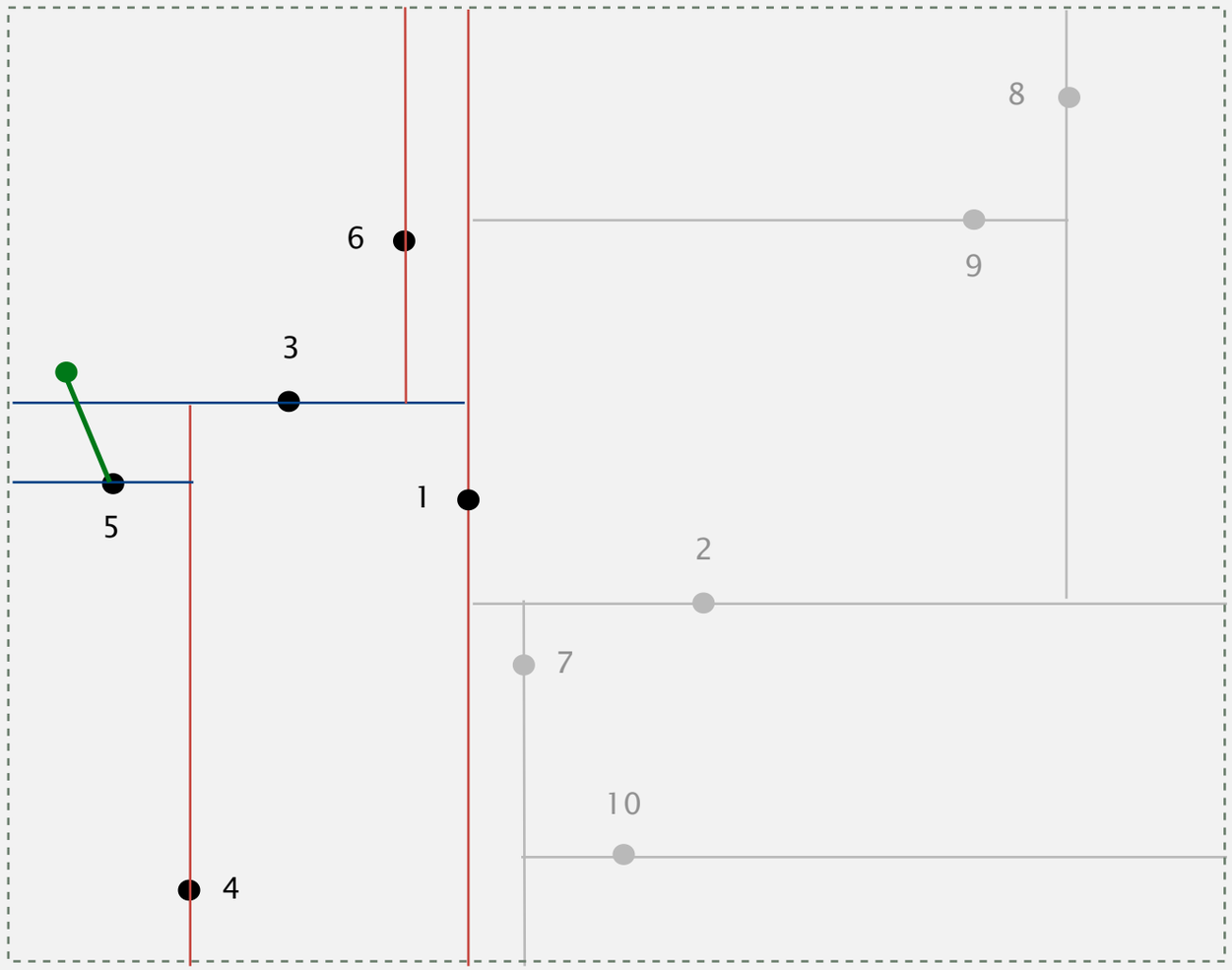


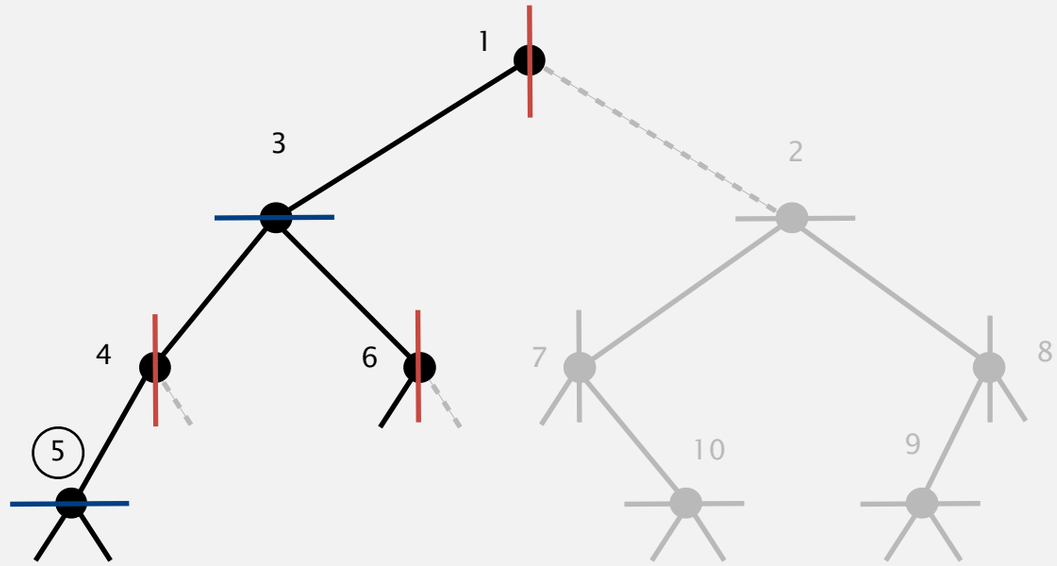
nearest neighbor = 5

Nearest neighbor search in a 2d tree analysis

Typical case. $\log N$.

Worst case (even if tree is balanced). N.



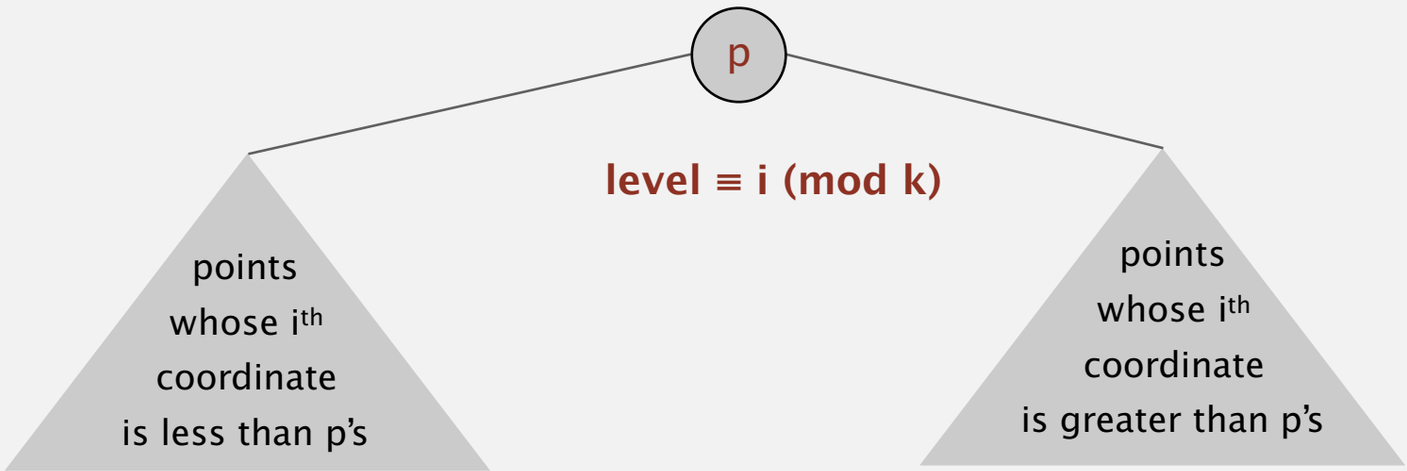


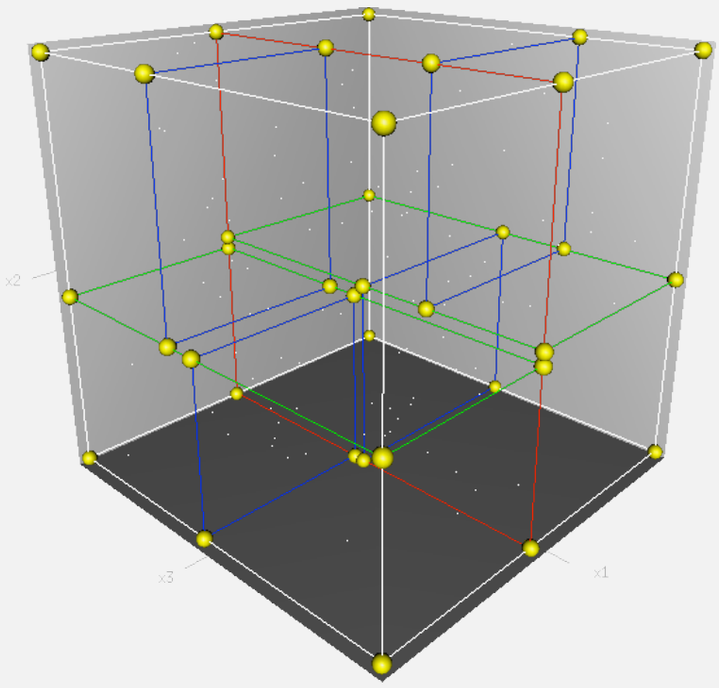
nearest neighbor = 5

Kd tree

Kd tree. Recursively partition k-dimensional space into 2 halfspaces.

Implementation. BST, but cycle through dimensions ala 2d trees.





Efficient, simple data structure for processing k-dimensional data.

- Widely used.
- Adapts well to high-dimensional and clustered data.
- Discovered by an undergrad in an algorithms class!

Jon Bentley

http://algs4.cs.princeton.edu

Geometric Applications of BSTs

- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

1d interval search

1d interval search. Data structure to hold set of (overlapping) intervals.

- Insert an interval (lo, hi).
- Search for an interval (lo, hi).
- Delete an interval (lo, hi).
- Interval intersection query: given an interval (lo, hi), find all intervals (or one interval) in data structure that intersects (lo, hi).

Q. Which intervals intersect $(9, 16)$?

A. $(7, 10)$ and $(15, 18)$.

1d interval search API

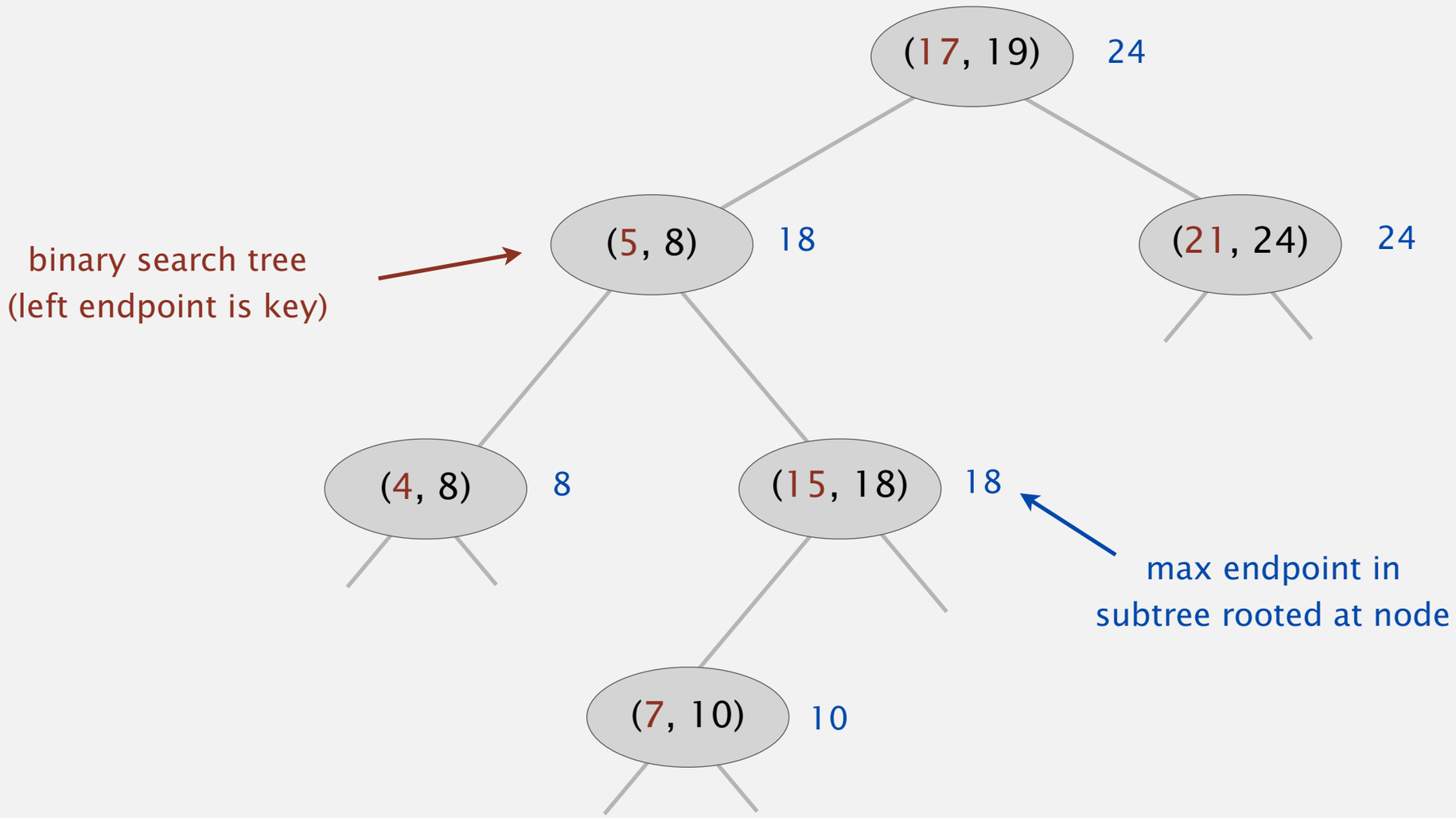
public class IntervalST<Key extends Comparable<Key>, Value>	
IntervalST()	*create interval search tree*
void put(Key lo, Key hi, Value val)	*put interval-value pair into ST*
Value get(Key lo, Key hi)	*value paired with given interval*
void delete(Key lo, Key hi)	*delete the given interval*
Iterable<Value> intersects(Key lo, Key hi)	*all intervals that intersect (lo, hi)*

Nondegeneracy assumption. No two intervals have the same left endpoint.

Interval search trees

Create BST, where each node stores an interval (lo, hi).

- Use left endpoint as BST key.
- Store max endpoint in subtree rooted at node.

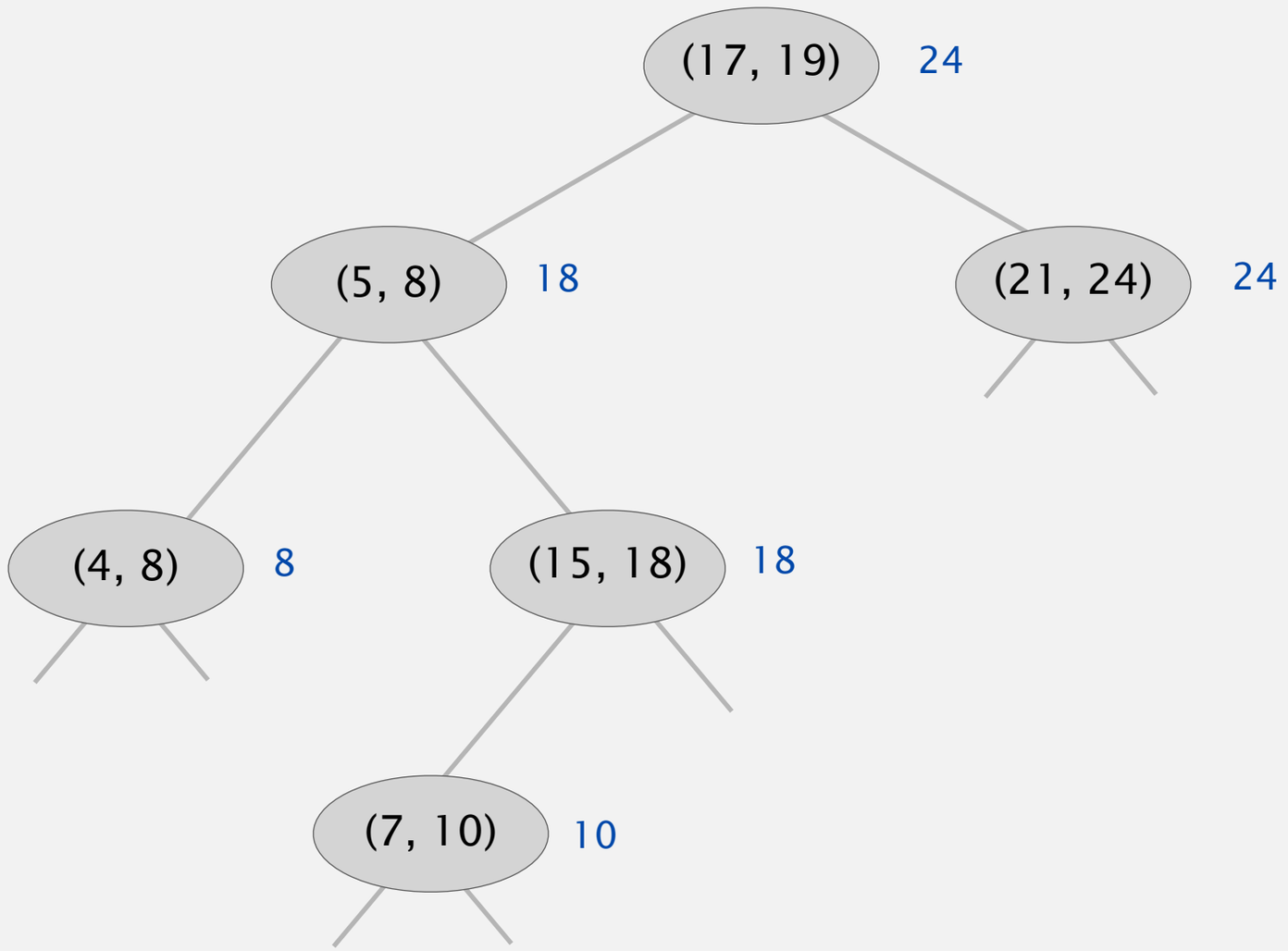


Interval search tree demo: insertion

To insert an interval (lo, hi) :

- Insert into BST, using lo as the key.
- Update max in each node on search path.

insert interval (16, 22)



Interval search tree demo: insertion

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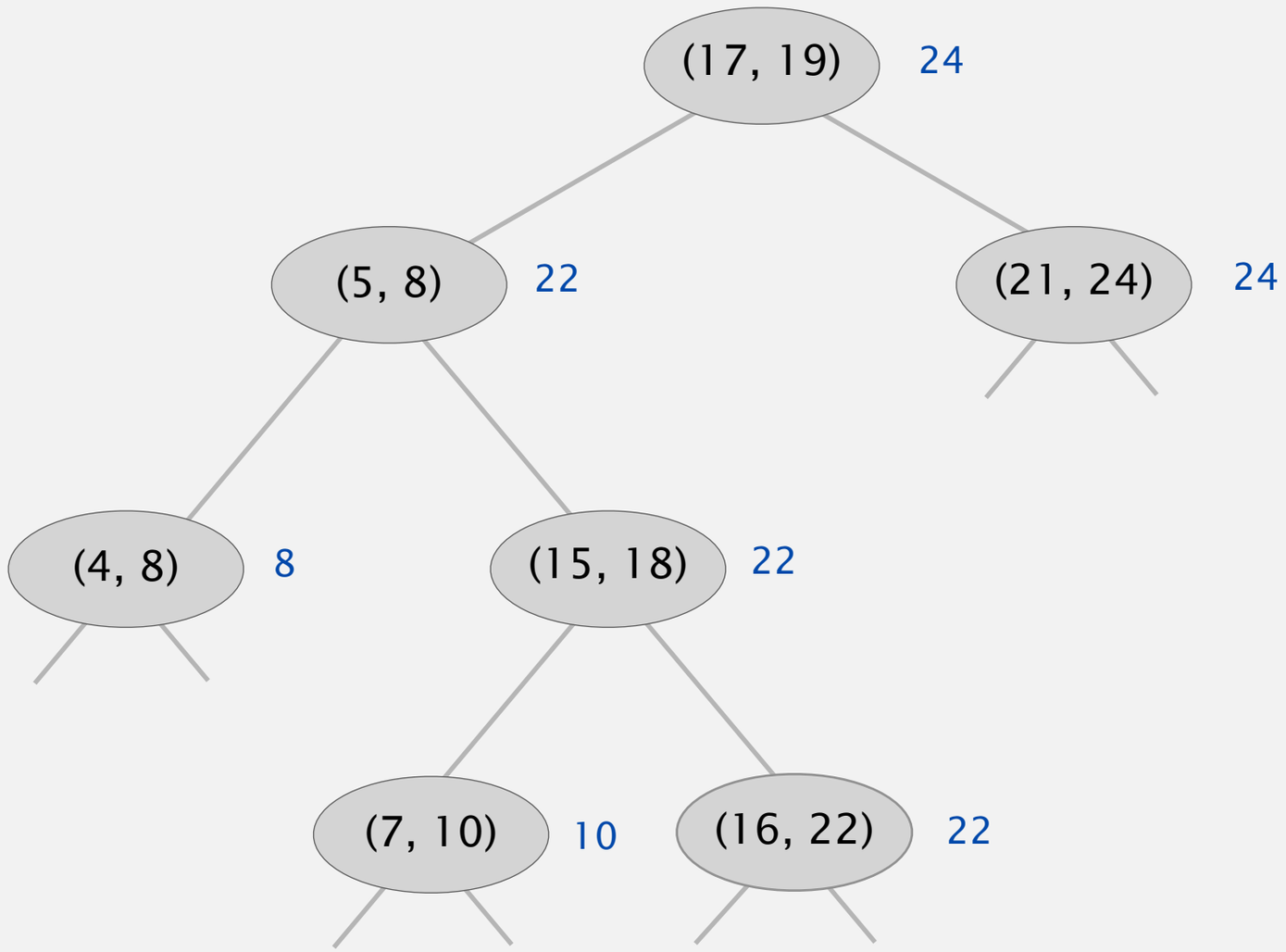
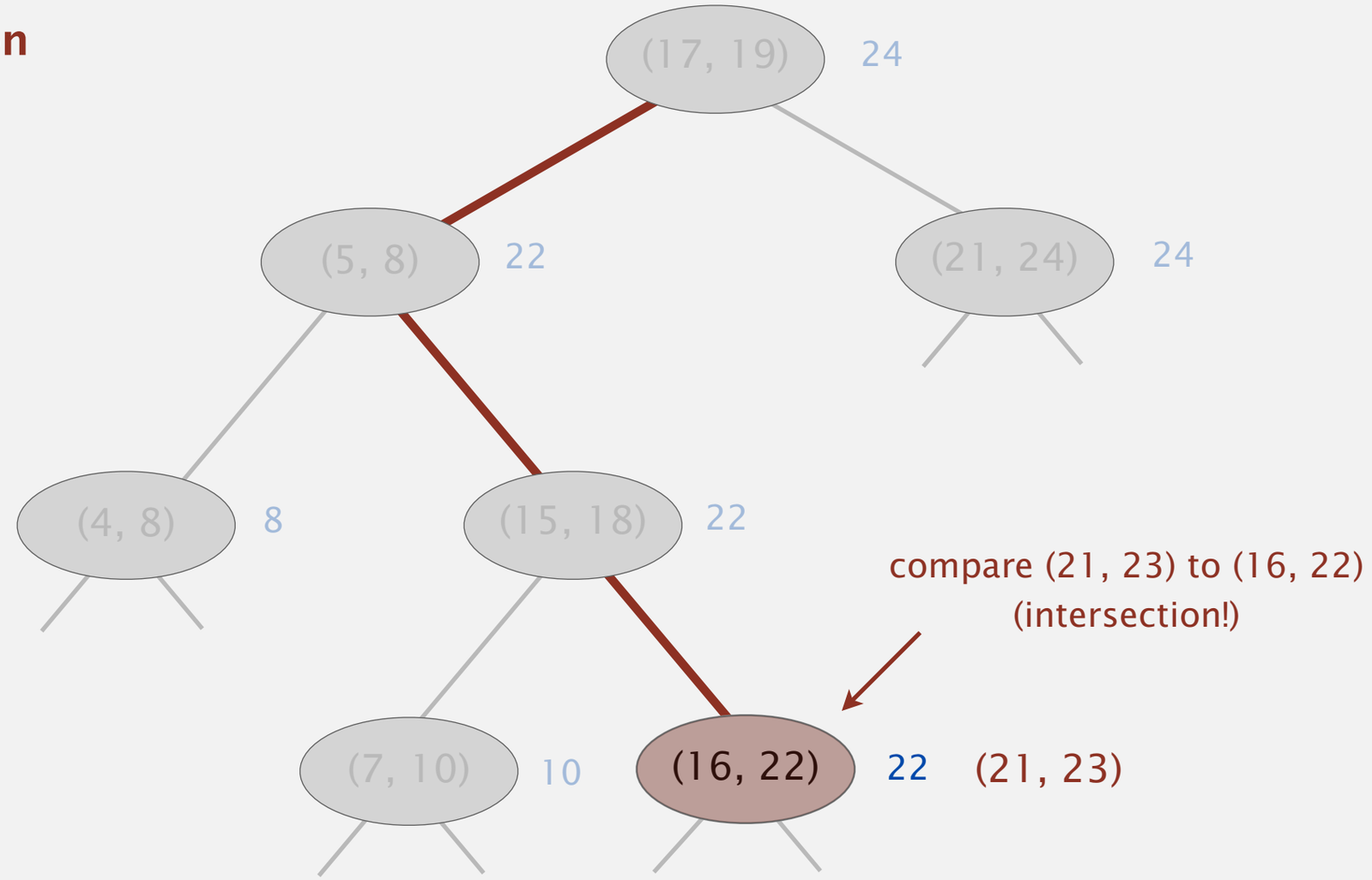


Interval search tree demo: intersection

To search for any one interval that intersects query interval (lo, hi):

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo, go right.
- Else go left.

interval intersection search for (21, 23)



Search for an intersecting interval: implementation

To search for any one interval that intersects query interval (lo, hi) :

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo, go right.
- Else go left.

```
Node x = root;
while (x != null)
{
   if      (x.interval.intersects(lo, hi)) return x.interval;
   else if (x.left == null)                x = x.right;
   else if (x.left.max < lo)               x = x.right;
   else                                    x = x.left;
}
return null;
```

Search for an intersecting interval: analysis

To search for any one interval that intersects query interval $(\ lo,\ hi\ )$:

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo, go right.
- Else go left.

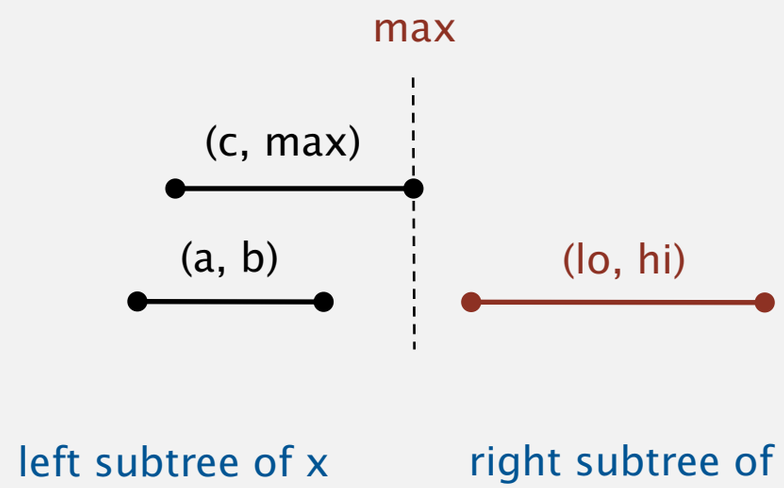
Case 1. If search goes right, then no intersection in left.

Pf. Suppose search goes right and left subtree is non empty.

- Since went right, we have $max < lo$.
- For any interval (a, b) in left subtree of x, we have $b \leq max < lo$.

definition of max — reason for going right

- Thus, (a, b) will not intersect $(\ lo,\ hi\ )$.



Search for an intersecting interval: analysis

To search for any one interval that intersects query interval $(\ lo,\ hi\ )$:

- If interval in node intersects query interval, return it.
- Else if left subtree is null, go right.
- Else if max endpoint in left subtree is less than lo, go right.
- Else go left.

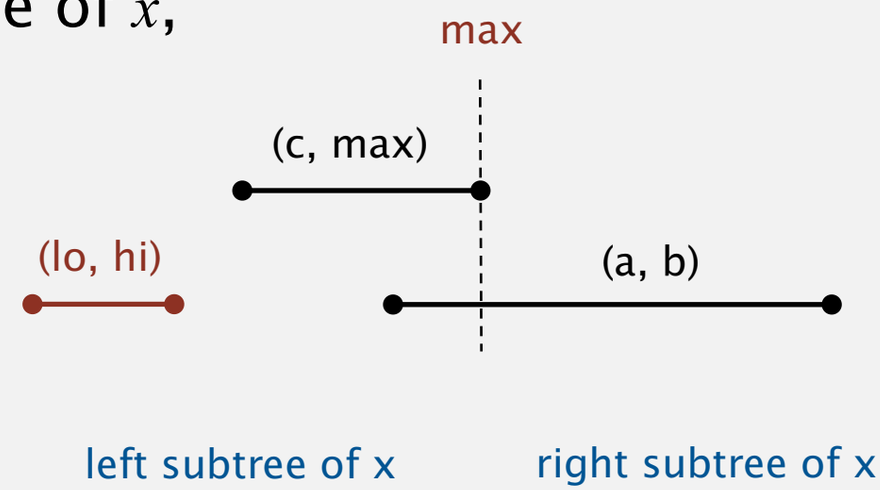
Case 2. If search goes left, then there is either an intersection in left subtree or no intersections in either.

Pf. Suppose no intersection in left.

- Since went left, we have $lo \leq max$.
- Then for any interval (a, b) in right subtree of x, $hi < c \leq a \Rightarrow$ no intersection in right.

no intersections in left subtree

intervals sorted by left endpoint



Interval search tree: analysis

Implementation. Use a red-black BST to guarantee performance.

easy to maintain auxiliary information
(log N extra work per operation)

operation	brute	interval search tree	best in theory
insert interval	1	$\log N$	$\log N$
find interval	N	$\log N$	$\log N$
delete interval	N	$\log N$	$\log N$
find any one interval that intersects (lo, hi)	N	$\log N$	$\log N$
find all intervals that intersects (lo, hi)	N	$R \log N$	$R + \log N$

order of growth of running time for N intervals

http://algs4.cs.princeton.edu

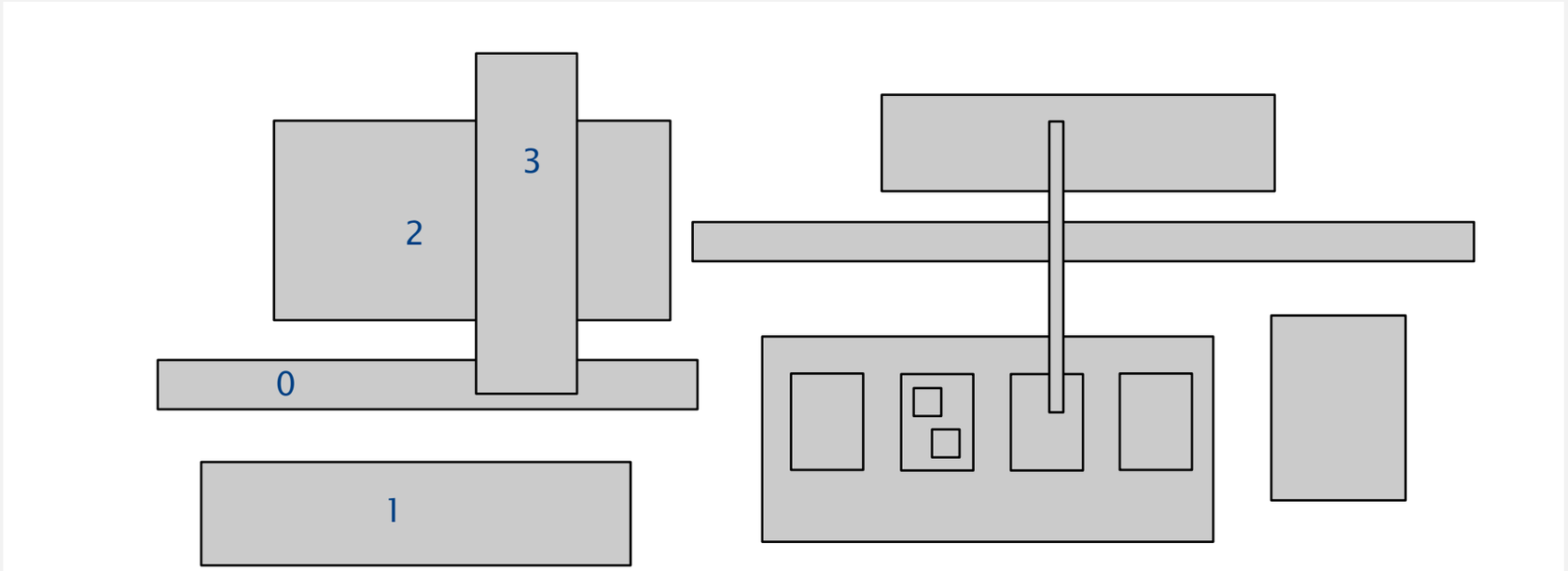
Geometric Applications of BSTs

- *1d range search*
- *line segment intersection*
- *kd trees*
- *interval search trees*
- *rectangle intersection*

Orthogonal rectangle intersection

Goal. Find all intersections among a set of N orthogonal rectangles.

Quadratic algorithm. Check all pairs of rectangles for intersection.



Non-degeneracy assumption. All x- and y-coordinates are distinct.

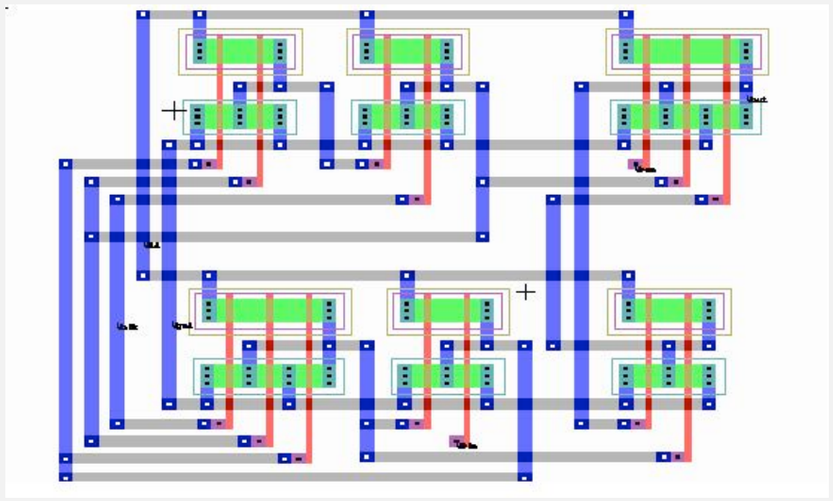
Microprocessors and geometry

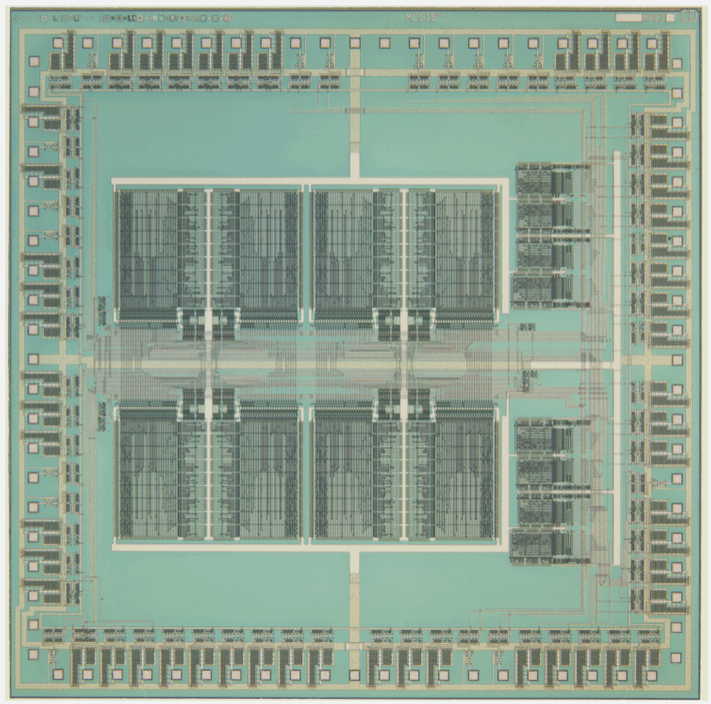
Early 1970s. microprocessor design became a geometric problem.

- Very Large Scale Integration (VLSI).
- Computer-Aided Design (CAD).

Design-rule checking.

- Certain wires cannot intersect.
- Certain spacing needed between different types of wires.
- Debugging = orthogonal rectangle intersection search.

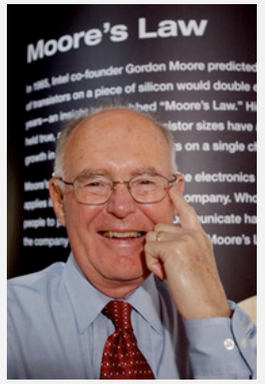




Algorithms and Moore's law

"Moore's law." Processing power doubles every 18 months.

- 197x: check N rectangles.
- 197(x+1.5): check $2N$ rectangles on a 2x-faster computer.



Gordon Moore

Bootstrapping. We get to use the faster computer for bigger circuits.

But bootstrapping is not enough if using a quadratic algorithm:

- 197x: takes M days.
- 197(x+1.5): takes $(4M)/2 = 2M$ days. (!)

quadratic algorithm

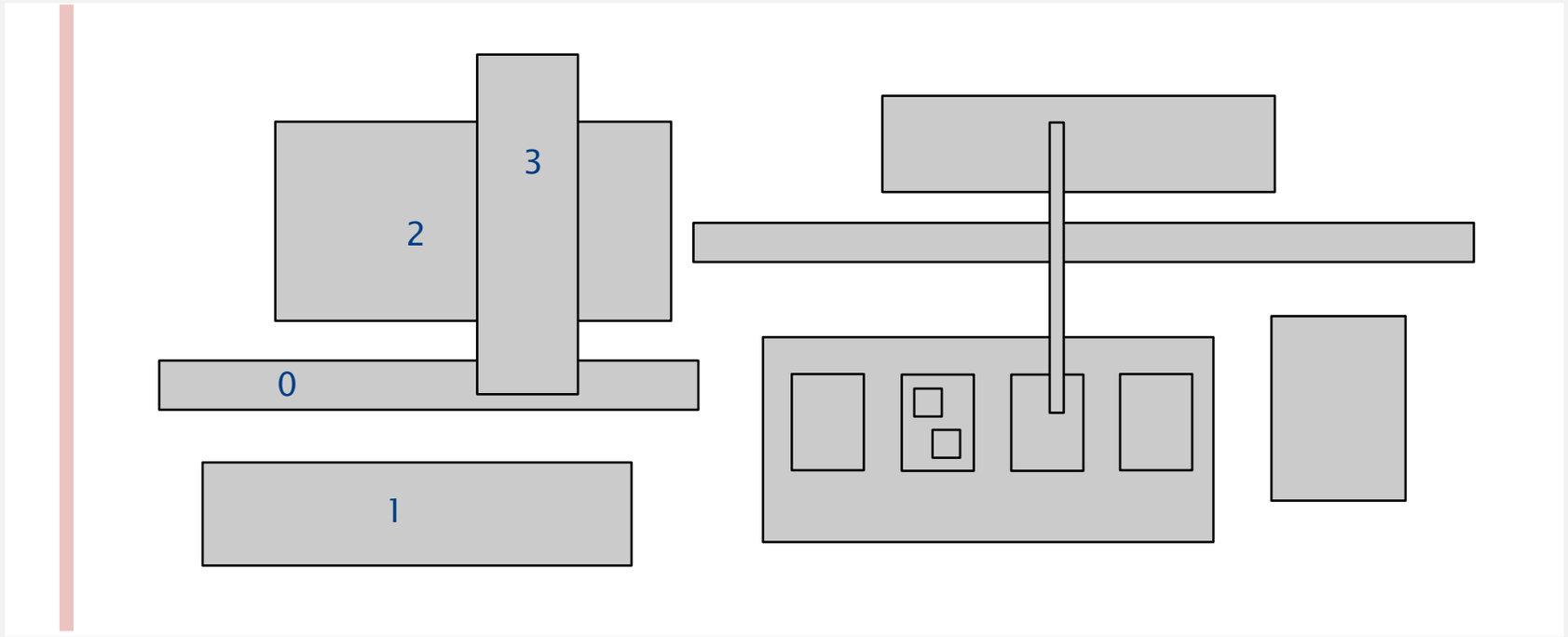
2x-faster computer

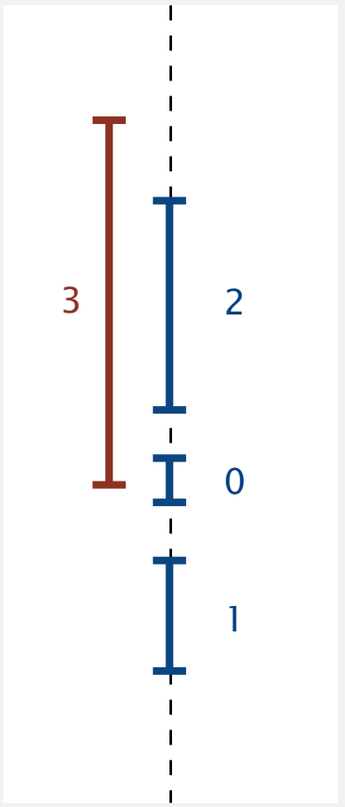
Bottom line. Linearithmic algorithm is necessary to sustain Moore's Law.

Orthogonal rectangle intersection: sweep-line algorithm

Sweep vertical line from left to right.

- x-coordinates of left and right endpoints define events.
- Maintain set of rectangles that intersect the sweep line in an interval search tree (using y-intervals of rectangle).
- Left endpoint: interval search for y-interval of rectangle; insert y-interval.
- Right endpoint: remove y-interval.





y-coordinates

Orthogonal rectangle intersection: sweep-line analysis

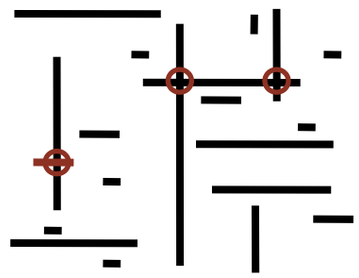
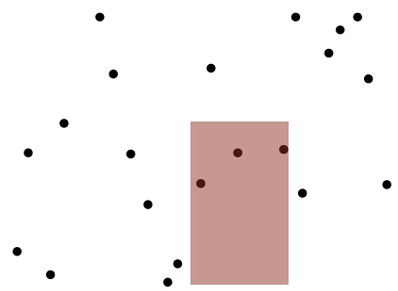
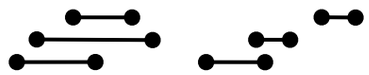
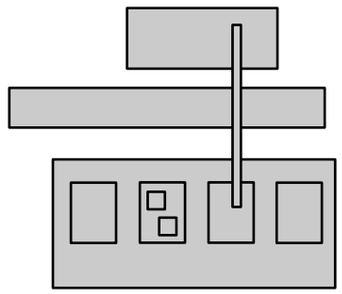
Proposition. Sweep line algorithm takes time proportional to $N \log N + R \log N$ to find R intersections among a set of N rectangles.

Pf.

- Put x-coordinates on a PQ (or sort). ← N log N
- Insert y-intervals into ST. ← N log N
- Delete y-intervals from ST. ← N log N
- Interval searches for y-intervals. ← N log N + R log N

Bottom line. Sweep line reduces 2d orthogonal rectangle intersection search to 1d interval search.

Geometric applications of BSTs

problem	example	solution
1d range search		BST
2d orthogonal line segment intersection		sweep line reduces to 1d range search
kd range search		kd tree
1d interval search		interval search tree
2d orthogonal rectangle intersection		sweep line reduces to 1d interval search