Aperiodic Cantor and Borel dynamics

Survey and new results

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Conference on Descriptive Set Theory and Model Theory
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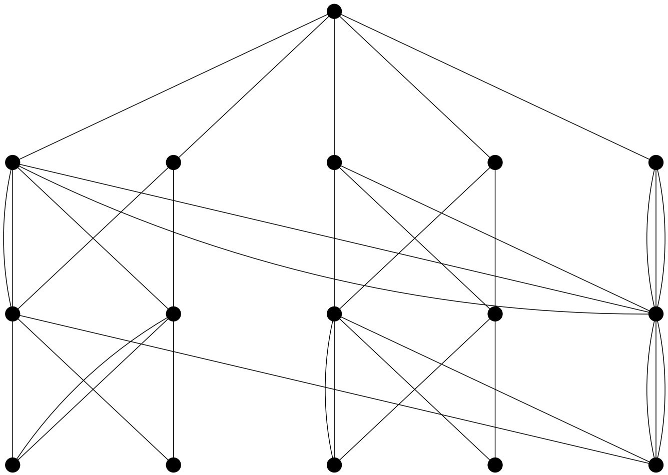
Main definitions and notation

- A Cantor set Ω is a 0-dimensional compact metric space without isolated points; $(X, \mathcal{B})$ denotes a standard Borel space.
- (Ω, T) is called a Cantor dynamical system (d.s.) where T is a self-homeomorphism of Ω; similarly, $(X, \mathcal{B}, T)$ is a Borel d. s. where $T : X \to X$ is a Borel automorphism of X.
- $H(\Omega)$ denotes the group of all homeomorphism of Ω; $Aut(X, \mathcal{B})$ denotes the group of all Borel automorphisms of $(X, \mathcal{B})$.
- $Orb_T(x) = \{T^n x : n \in \mathbb{Z}\}$ is called the T-orbit of x. If $|\{T^n x : n \in \mathbb{Z}\}| < \infty$, then T is periodic at x. If any T-orbit is infinite, then T is called aperiodic (= non-periodic).
- If any T-orbit is dense in Ω, then T is called minimal.
- Homeomorphisms (Borel automorphisms) T, S (acting on the same space Y) are called orbit equivalent if there exists a homeomorphism (Borel automorphism) $\varphi : Y \to Y$ such that $\varphi(Orb_T(x)) = Orb_S(\varphi x)\ \forall x \in Y$. (Here Y is either Ω or X).

Example of a non-simple Bratteli diagram



Goal and motivation

Goal:

Classify aperiodic homeomorphisms of a Cantor set up to orbit equivalence. Study aperiodic transformations and sets formed by them in the context of Cantor and Borel dynamics.

Motivation:

1. Progress in the theory of Cantor minimal systems
2. Bratteli diagrams and aperiodic Cantor and Borel dynamics
3. Full groups and orbit equivalence
4. Invariant finite and infinite ergodic measures
5. Dimension groups and dynamical systems

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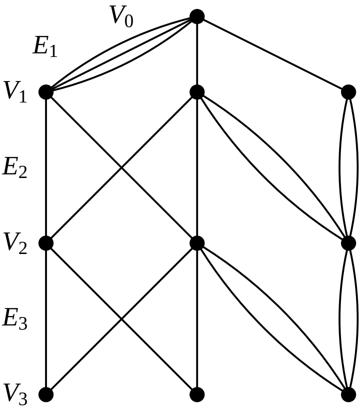
Minimal homeomorphisms of a Cantor set

- Bratteli (1972): AF-algebras and Bratteli diagrams
- Elliott (1976), Effros, Handelman, Shen (1979-1981): dimension groups
- Vershik (1981-1982): Adic transformation model theorem in ergodic theory
- Herman, Putnam, Skau (1992): Bratteli-Vershik model in Cantor dynamics for minimal homeomorphisms
- Giordano, Putnam, Skau (1995), Glasner, Weiss (1995): Orbit equivalence of minimal homeomorphisms; full groups
- Forrest (1997), Durand, Host, Skau (1999): Substitution dynamical systems and simple stationary Bratteli diagrams
- Downarowicz, Maass (2008): Finite rank simple Bratteli diagrams
- Giordano, Matui, Putnam, Skau (2010), Orbit equivalence of minimal $\mathbb{Z}^n$-actions

From minimal to aperiodic homeomorphisms

- Bratteli-Vershik model for any aperiodic homeomorphism (B., Dooley, Medynets (2005), Medynets (2006))
- Aperiodic substitution systems and stationary non-simple Bratteli diagrams (B., Kwiatkowski, Medynets (2009))
- Ergodic invariant measures on stationary non-simple Bratteli diagrams (B., Kwiatkowski, Medynets, Solomyak (2010))
- Ergodic invariant measures on finite rank non-simple Bratteli diagrams (B., Kwiatkowski, Medynets, Solomyak (2012))
- Full group is a complete invariant of orbit equivalence for aperiodic homeomorphisms (Medynets (2011))
- Homeomorphic finite and infinite invariant measures on stationary non-simple Bratteli diagrams (B., Karpel (2011))
- Orders on non-simple Bratteli diagrams and the existence of continuous dynamics (Vershik map) on the such diagrams (B., Kwiatkowski, Yassawi (2012))
- Non-simple dimension groups and properties of traces on them (B., Handelman (2012))

Incidence matrix (Example)



The diagram is stationary with incidence matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

In general, the sequence (F_n) of incidence matrices determine the Bratteli diagram.

Topology on the path space X_B: two paths are close if they agree on a large initial segment.

X_B is a Cantor set if it has no isolated points.

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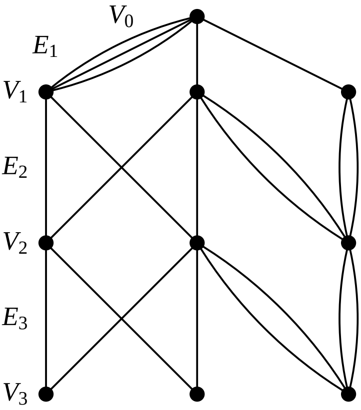

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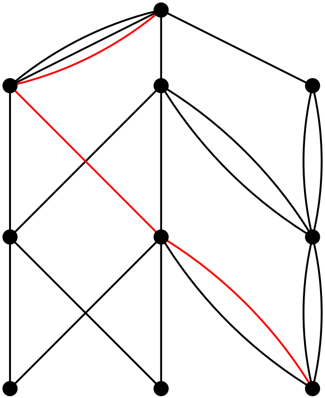
Path space of a Bratteli diagram



- X_B is the set of all infinite paths.
- Consider an infinite path.
- Close paths agree on a large initial segment.
- How can one introduce a dynamics on X_B?

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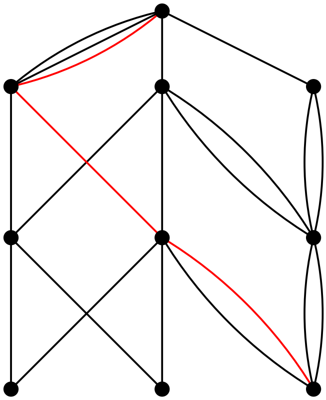

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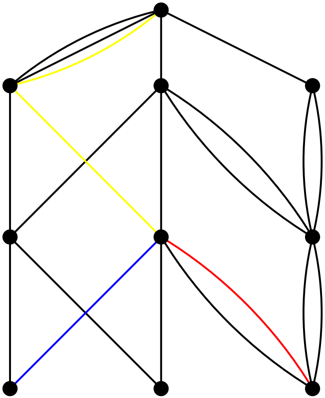

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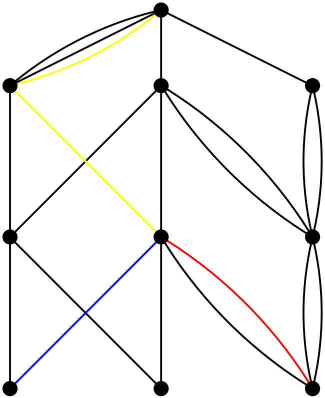

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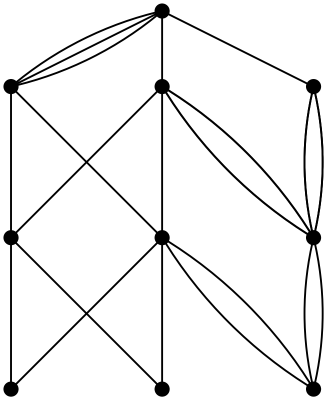
Ordered Bratteli diagrams



- Take a vertex $v \in V \setminus V_0$.
- Consider the set $r^{-1}(v)$ of edges with range at the vertex v.
- Enumerate edges from this set.
- Do the same for every vertex.

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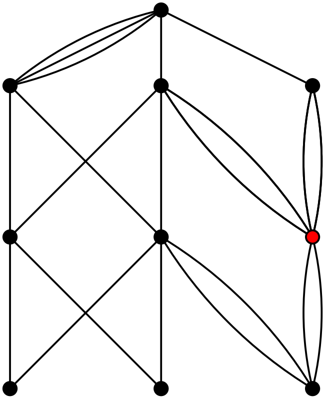

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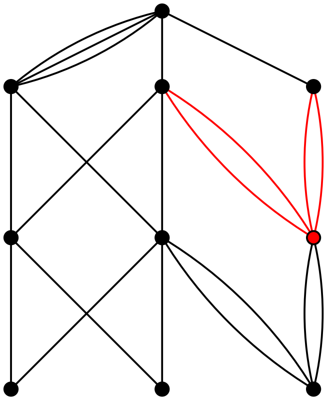

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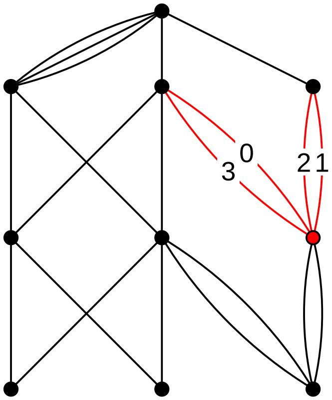

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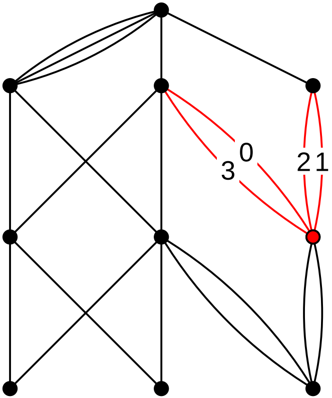

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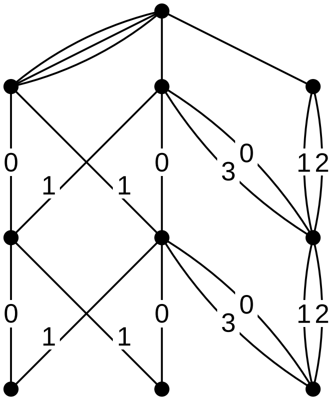
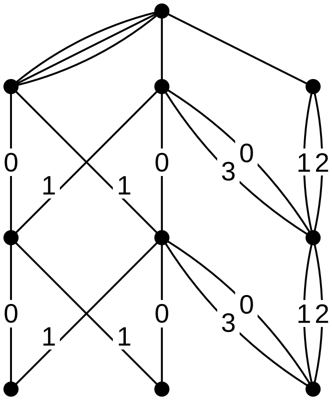
Maximal and minimal paths



- An infinite path $x = (x_n)$ is called maximal if x_n is maximal in $r^{-1}(r(x_n))$. A minimal path is defined similarly.
- The sets $X_{\max}$ and $X_{\min}$ of all maximal and minimal paths are non-empty and closed.
- For simplicity, consider the case of regular diagrams when $X_{\max}$ and $X_{\min}$ have empty interior.

Maximal and minimal paths



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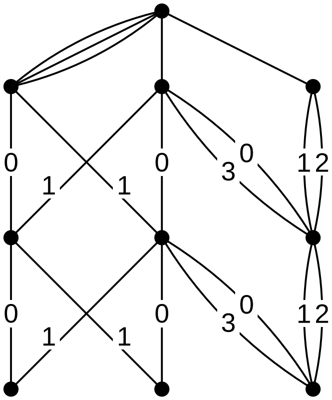

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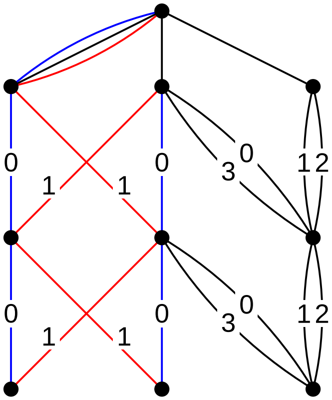

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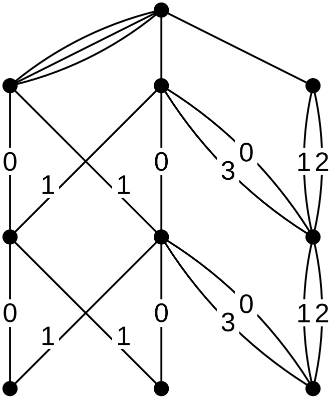
Vershik map



Define the Vershik map
$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$:

Fix $x \in X_B \setminus X_{\max}$.

Find the first k with x_k non-maximal.

Take the successor $\overline{x}_k$ of x_k.

Connect $s(\overline{x}_k)$ (the source of x_k) to the top vertex V_0 by the minimal path.

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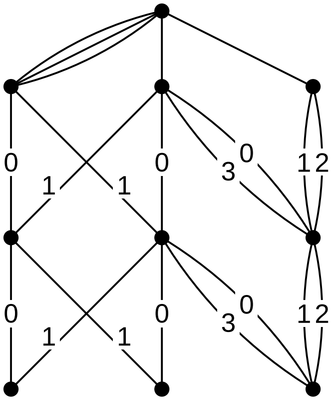

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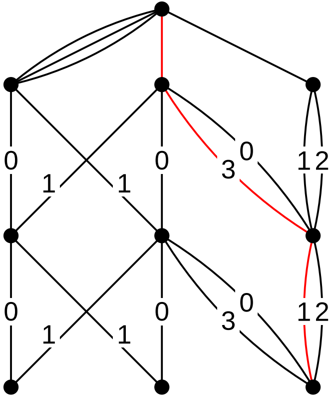

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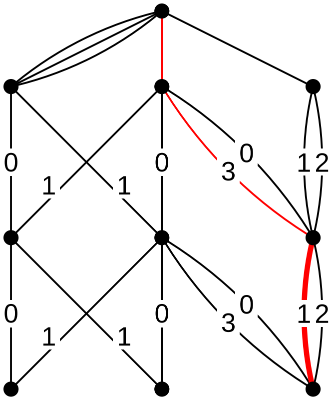

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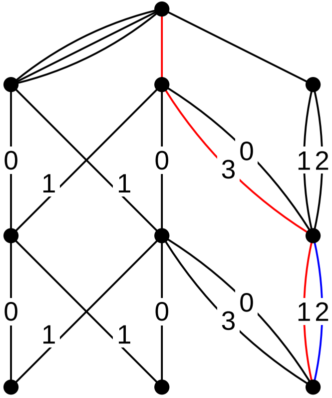

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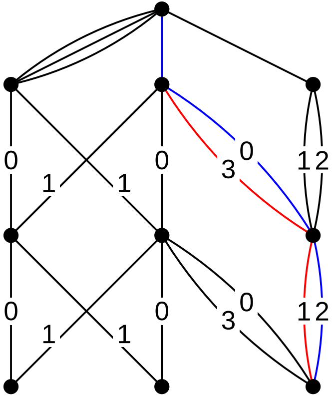

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Vershik map

- φ_B is defined everywhere on $X_B \setminus X_{\max}$
- $\varphi_B(X_B \setminus X_{\max}) = X_B \setminus X_{\min}$

Definition

If the map φ_B can be extended to a homeomorphism of X_B such that $\varphi_B(X_{\max}) = X_{\min}$, then (X_B, φ_B) is called a Bratteli-Vershik system and φ_B is called the Vershik map.

Question

Does any order on a Bratteli diagram define a Vershik map?

Answer

In general, No (even for simple Bratteli diagrams). On the other hand, there is a Bratteli diagram (odometer) such that any order produces a Vershik map.

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Existence of a Vershik map

Open problem: Under what conditions does a (non-simple) Bratteli diagram admit a Vershik map?

If $X_{\max}$ and $X_{\min}$ are singletons, then forcing $\varphi_B(X_{\max}) = X_{\min}$ we get that $\varphi_B : X_B \to X_B$ is a homeomorphism.

Question:

Can any Bratteli diagram be given an order which defines a Vershik map?

Answer:

No! There are examples of even stationary non-simple diagrams that do not admit a continuous Vershik map.

An example of such a stationary Bratteli diagram (Medynets (2006)):

$$F_n = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

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$$F_n := \begin{pmatrix} A_n^{(1)} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_n^{(k)} & 0 \\ B_n^{(1)} & \dots & B_n^{(k)} & C_n \end{pmatrix}$$

where (1) for $1 \leq i \leq k$ each matrix $A_n^{(i)}$ is of size $d_i \times d_i$; (2) all matrices $A_n^{(i)}$, $B_n^{(i)}$ and C_n are strictly positive; (3) C_n is an $s \times s$ matrix; (4) there exists $j \in \{\sum_{i=1}^{k} d_i + 1, \dots \sum_{i=1}^{k} d_i + s\}$ such that for each n, the j-th row of F_n is strictly positive.

Theorem (Bezuglyi, Kwiatkowski, Yassawi (2012))

Let B be as above, C_n an $s \times s$ matrix where $1 \leq s \leq k-1$. If $k = 2$, there are Vershik maps on B only if $C_n = (1)$ for all n. If $k > 2$, then there is no Vershik map on B.

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Set of orders of a Bratteli diagram

Let $\mathcal{O}_B$ be the set of all orders B. Then

$$\mathcal{O}_B = \prod_{v \in V} \{1, ..., |r^{-1}(v)|!\},$$

$\mathcal{O}_B$ can be viewed as a measure space with product measure $\mu = \prod_{v \in V} \nu_v$ where ν_v is the uniformly distributed measure on $\{1, ..., |r^{-1}(v)|!\}$.

Let $\mathcal{P}_B$ be the set of orders that admit a Vershik map. Then $\mathcal{P}_B = \mathcal{O}_B$ if and only if B is an odometer. In general case,

$$\overline{\mathcal{P}_B} = \overline{\mathcal{O}_B \setminus \mathcal{P}_B} = \mathcal{O}_B, \quad int(\mathcal{P}_B) = int(\mathcal{O}_B \setminus \mathcal{P}_B) = \emptyset$$

Theorem (Bezuglyi, Kwiatkowski, Yassawi (2012))

Let B be a finite rank d aperiodic Bratteli diagram. Then there exists $j \in \{1, ..., d\}$ such that μ-almost all orders have j maximal and j minimal elements.

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Set of orders of a Bratteli diagram

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$$\mathcal{O}_B = \prod_{v \in V} \{1, ..., |r^{-1}(v)|!\},$$

$\mathcal{O}_B$ can be viewed as a measure space with product measure $\mu = \prod_{v \in V} \nu_v$ where ν_v is the uniformly distributed measure on $\{1, ..., |r^{-1}(v)|!\}$.

Let $\mathcal{P}_B$ be the set of orders that admit a Vershik map. Then $\mathcal{P}_B = \mathcal{O}_B$ if and only if B is an odometer. In general case,

$$\overline{\mathcal{P}_B} = \overline{\mathcal{O}_B \setminus \mathcal{P}_B} = \mathcal{O}_B, \quad int(\mathcal{P}_B) = int(\mathcal{O}_B \setminus \mathcal{P}_B) = \emptyset$$

Theorem (Bezuglyi, Kwiatkowski, Yassawi (2012))

Let B be a finite rank d aperiodic Bratteli diagram. Then there exists $j \in \{1, ..., d\}$ such that μ-almost all orders have j maximal and j minimal elements.

From Cantor aperiodic systems to Bratteli diagrams

Theorem (Herman - Putnam - Skau (1992))

Each Cantor minimal system (X, T) is conjugate to a Bratteli-Vershik system (X_B, φ_B) for which $|X_{\max}| = |X_{\min}| = 1$.

A closed subset Y of a Cantor set X is called basic if every clopen neighborhood of Y is a complete T-section and Y meets every T-orbit at most once.

Claim

Every Cantor aperiodic system has a basic set.

Theorem (Medynets 2006)

Let (X, T, Y) be a Cantor aperiodic system with a basic set Y. There exists an ordered Bratteli diagram B such that (X, T) is homeomorphic to a Bratteli-Vershik model $(X_B; \varphi_B)$ and the homeomorphism implementing this conjugacy maps Y onto $X_{\min}$. Regular Bratteli diagrams correspond to nowhere dense basic sets.

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Borel-Bratteli diagrams

Definition (Borel-Bratteli diagram)

A Borel-Bratteli diagram is an infinite graph $B = (V, E)$ such that the vertex set V and edge set E are partitioned into sets $V = \bigcup_{i \geq 0} V_i$ and $E = \bigcup_{i \geq 1} E_i$ having the following properties:
(i) $V_0 = \{v_0\}$ is a singleton, and every V_i and E_i are at most countable sets;
(ii) there exist a range map r and a source map s from E to V such that $r(E_i) \subset V_i$, $s(E_i) \subset V_{i-1}$, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
(iii) for every $v \in V \setminus V_0$, the set $r^{-1}(v)$ is finite.

Definition (Ordered Borel-Bratteli diagram)

Enumerate all edges from $r^{-1}(v)$ for all $v \neq v_0$. A Borel-Bratteli diagram B is called an ordered Borel-Bratteli diagram if the path space Y_B has no cofinal minimal and maximal paths.

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Borel-Bratteli diagrams (continuation)

A few simple facts

- Path space Y_B is a 0-dimensional Polish space.
- Incidence matrices $F_n = (f_{ik}^{(n)})$ have only finitely many non-zero entries at each row and $\sum_{k=1}^{\infty} f_{ik}^{(n)} = |r^{-1}(v_i(n))|$.
- Every order on B defines a Vershik map (homeomorphism) $\varphi_B : Y_B \to Y_B$.
- Given an ordered Borel-Bratteli diagram B, (Y_B, φ_B) is called Borel-Bratteli dynamical system.

Kakutani-Rokhlin partition

Let T be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$, and let A be a complete T-section whose points are T-recurrent. Then $\forall x \in A\ \exists n_A(x) > 0$ such that $T^{n(x)}x \in A$ and $T^i x \notin A,\ 0 < i < n(x)$. Let $C_k = \{x \in A \mid n_A(x) = k\},\ k \in \mathbb{N}$, then $T^k C_k \subset A$ and $\{T^i C_k \mid i = 0, ..., k-1\}$ are pairwise disjoint. Thus,

$$X = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k$$

and X is partitioned into T-towers $\xi_k = \{T^i C_k \mid\ = 0, ..., k-1\},\ k \in \mathbb{N}$, where C_k is the base and $T^{k-1}C_k$ is the top of ξ_k. This partition of X is called Kakutani-Rokhlin partition.

Vanishing sequence of markers

Given an aperiodic automorphism T, there exists a sequence (A_n) of Borel sets such that
(i) $X = A_0 \supset A_1 \supset A_2 \supset \cdots$,
(ii) $\bigcap_n A_n = \emptyset$,
(iii) A_n and $X \setminus A_n$ are complete T-sections, $n \in \mathbb{N}$,
(iv) for $n \in \mathbb{N}$, every point in A_n is recurrent,
(v) for $n \in \mathbb{N}$, $A_n \cap T^i(A_n) = \emptyset, \ i = 1, ..., n-1$,
(vi) for $n \in \mathbb{N}$, the base $C_k(n)$ of every non-empty T-tower is an uncountable Borel set, $k \in \mathbb{N}$.

Definition

A sequence of Borel sets satisfying conditions (i) - (vi) is called a vanishing sequence of markers.

From Borel automorphisms to Borel-Bratteli diagrams

Given an aperiodic automorphism T of $(X, \mathcal{B})$ and a vanishing sequence of markers $X = A_0 \supset A_1 \supset A_2 \supset \cdots$, construct a Borel-Bratteli diagram coming from (A_n) and T.
Let $(\xi_n = \{\xi_n(v) : v\})$ be the sequence of refining Kakutani-Rokhlin partitions constructed by $(A_n), T$. Towers of ξ_n correspond to vertices of V_n, and the i-th row of the incidence matrix F_n is determined by the intersection of $\xi_{n+1}(i)$ with towers of ξ_n. This automatically defines an order on $r^{-1}(v)$ for each v.

Theorem (Bezuglyi, Dooley, Kwiatkowski (2006))

Let T be an aperiodic Borel automorphism of $(X, \mathcal{B})$. Then there exists an ordered Borel-Bratteli diagram $B = (V, E, \geq)$ and a Vershik map $\varphi_B : Y_B \to Y_B$ such that (X, T) is isomorphic to (Y_B, φ).

If T is an aperiodic homeomorphism of a locally compact 0-dimensional Polish space, then the arising Borel-Bratteli diagram has only finitely many non-trivial towers at each level.

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Subsets of transformations

- $\mathcal{A}p(Y)$ = the set of all aperiodic transformations of Y;
- $\mathcal{P}er(Y)$ = the set of all periodic transformations Y;
- $\mathcal{M}in$ = the set of all minimal homeomorphisms from $H(\Omega)$;
- $\mathcal{M}in \supset \mathcal{O}d$ = the set of all odometers from $H(\Omega)$ (Example: $\Omega = \{0,1\}^{\mathbb{N}}$, $T \in \mathcal{O}d \Longleftrightarrow$ $T(\underbrace{1...1}_{n}0x_i) = \underbrace{0...0}_{n}1x_i, T(1^\infty) = (0^\infty)$);
- $\mathcal{M}ov = \{T \in H(\Omega) : TF \setminus F \neq \emptyset,\ F \setminus TF \neq \emptyset,\ \forall$ clopen $F\}$ is the set of moving homeomorphisms;
- $[T]_C = \{R \in H(\Omega) : Rx \in Orb_T(x)\ \forall x \in \Omega\}$, i.e. $Rx = T^{n_R(x)}x$ (full group generated by T). If $T \in Aut(X, \mathcal{B})$, the (Borel) full group $[T]_B$ is defined similarly.
- $[[T]]_C = \{R \in [T] : n_R : \Omega \to \mathbb{Z}$ is continuos$\}$ (topological full group).

Topologies on $Aut(X, \mathcal{B})$

- The uniform topology τ on $Aut(X, \mathcal{B})$ is defined by the base of neighborhoods

$$U(T; \mu_1, ..., \mu_n; \varepsilon) := \{S \in Aut(X, \mathcal{B}) : \\ \mu_i(\{x : Tx \neq Sx\} \cup \{x \in X : T^{-1}x \neq S^{-1}x\}) < \varepsilon, \ i = 1, ..., n\}$$

  where $T \in Aut(X, \mathcal{B})$, $\mu_1, ..., \mu_n$ are probability Borel measures, and $\varepsilon > 0$.

- The p-topology on $Aut(X, \mathcal{B})$ is defined by the base of neighborhoods

$$W(T; F_1, ..., F_k) = \{S \in Aut(X, \mathcal{B}) \mid SF_i = TF_i, \ i = 1, ..., k\}$$

  where $T \in Aut(X, \mathcal{B})$ and $F_1, ..., F_k$ are any Borel sets.

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Topologies on $H(\Omega)$

- For a Cantor set Ω, the topology τ on $H(\Omega)$ is induced from $Aut(\Omega, \mathcal{B})$.
- The topology d of uniform convergence on (Ω, ρ):

$$d(S,T) = \sup_{x\in\Omega} \rho(Sx, Tx) + \sup_{x\in\Omega} \rho(S^{-1}x, T^{-1}x).$$

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- (1) The p-topology and the topology defined by the metric d coincide on $H(\Omega)$.
  (2) $(H(\Omega), d)$ is a Polish 0-dimensional group.

Question: *Find topological properties of $\mathcal{A}p, \mathcal{P}er, \mathcal{M}in, \mathcal{O}d, [T]_C$, $[[T]]_C, [T]_B$ in $Aut(X, \mathcal{B})$ and $H(\Omega)$ with respect to the defined topologies.*

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Results

Theorem (Bezuglyi, Dooley, Kwiatkowski (2006); Bezuglyi, Medynets (2004))

- *(i)* $\mathcal{A}p(X)$ *is a nowhere dense closed set and* $\mathcal{P}er(X)$ *is a dense set* $Aut(X, \mathcal{B})$ *w.r.t.* τ*;*
  (ii) $\overline{\mathcal{A}p}^{d} = H(\Omega)$.
- *(1)* $\mathcal{O}d$ *is a dense* G_δ *subset of* $(\mathcal{M}in, d)$*;*
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  (3) $\overline{\mathcal{O}d}^{d} = \overline{\mathcal{M}in}^{d} = \mathcal{M}ov = \mathcal{C}h$
  where $\mathcal{C}h$ *is the set of chain transitive homeomorphisms.*
- *(i) Periodic transformations are dense in* $Aut(X, \mathcal{B})$ *and* $H(\Omega)$ *w.r.t.* τ*;*
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Results (continuation)

Let $Ctbl = \{T \in Aut(X, \mathcal{B}) : |\{x \in X : Tx \neq x\}| \leq \aleph_0\}$, then $Ctbl$ is a normal subgroup of $Aut(X, \mathcal{B})$ closed w.r.t. τ and p. Set $Aut_0(X, \mathcal{B}) = Aut(X, \mathcal{B})/Ctbl$ and let τ_0, p_0 denote the quotient topologies.

A Borel automorphism S is called smooth if S has a complete section that meets every S-orbit at most once. The set of all smooth automorphisms is denoted by $\mathcal{S}m$.

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  (ii) $\mathcal{S}m \cap \mathcal{A}p$ is not dense in $Aut(X, \mathcal{B})$ w.r.t. p;
  (iii) $(\mathcal{S}m \cap \mathcal{A}p)_0$ is dense in $Aut_0(X, \mathcal{B})$ w.r.t. p_0.
- *(the Rokhlin property): (i) The group $(Aut_0(X, \mathcal{B}), p_0)$ has the Rokhlin property, that is $Aut_0(X, \mathcal{B}) = \overline{\{TST^{-1} : T \in Aut_0(X, \mathcal{B})\}}^{p_0}$ for any $S \in (\mathcal{S}m \cap \mathcal{A}p)_0$;*
  (ii) The group $(H(\Omega), d)$ has the Rokhlin property (Glasner, Weiss (2001))

Results (continuation)

Let $Ctbl = \{T \in Aut(X, \mathcal{B}) : |\{x \in X : Tx \neq x\}| \leq \aleph_0\}$, then $Ctbl$ is a normal subgroup of $Aut(X, \mathcal{B})$ closed w.r.t. τ and p. Set $Aut_0(X, \mathcal{B}) = Aut(X, \mathcal{B})/Ctbl$ and let τ_0, p_0 denote the quotient topologies.

A Borel automorphism S is called smooth if S has a complete section that meets every S-orbit at most once. The set of all smooth automorphisms is denoted by $\mathcal{S}m$.

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Theorem (continuation)

- $\overline{H(\Omega)}^{\tau} = Aut(\Omega, \mathcal{B})$ *(Luzin theorem)*
- *(i)* $\forall T \in \mathcal{A}p(\Omega),\ \overline{\{STS^{-1} : S \in H(\Omega)\}}^{\tau} = \mathcal{A}p(\Omega)$;
  (ii) $\forall T \in \mathcal{A}p(X),\ \overline{\{STS^{-1} : S \in Aut(X, \mathcal{B})\}}^{\tau} = \mathcal{A}p(X)$.
- *The group* $(Aut_0(X, \mathcal{B}), p_0)$ *is path-connected; the group* $(Aut(X, \mathcal{B}), \tau)$ *is not path-connected.*
- *Let* T *be an aperiodic homeomorphism of* Ω. *Then* $\overline{[[T]]_C}^{\tau} = [T]_B$ *and* $\overline{[[T]]_C}^{\tau} \bigcap H(\Omega) = [T]_C$ *where* $[T]_B$ *is formed by all Borel automorphisms of* Ω *preserving all* T*-orbits.*

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Full groups and orbit equivalence

Theorem (Giordano, Putnam, Skau (1999))

Two minimal homeomorphisms T and S from $H(\Omega)$ are orbit equivalent if and only if their full groups $[T]_C$ and $[S]_C$ are isomorphic as abstract groups.

The next theorem is not formulated in full generality.

Theorem (Miller, Rosendal (2007))

Let T and S be two aperiodic Borel automorphisms of $(X, \mathcal{B})$. Then they are orbit equivalent if and only if their full groups $[T]_B$ and $[S]_B$ are isomorphic as abstract groups.

Medynets (2011) also proved that the above result of Giordano-Putnam-Skau can be generalized, in particular, to transitive aperiodic actions of more general groups.

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