Universally measurable cardinals just beyond $\mathbb{R}/\mathbb{Q}$

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Part I

Introduction

I. Introduction

A broad outline

According to the usual notion of cardinality, one set is smaller than another iff there is an injection of the former into the latter.

Much recent work has involved analogous notions in which definability constraints are placed upon the injection.

Here we explore the effect of this modification.

This is joint work with Clinton Conley.

Notation

Suppose that X and Y are Hausdorff spaces.

With one exception, we will only consider Polish spaces.

Notation

Suppose E and F are Borel equivalence relations on X and Y.

If you wish, you can restrict your attention to equivalence relations generated by free Borel actions of $\mathbb{F}_2$.

Definition

A homomorphism from E to F is a function $\varphi\colon X \to Y$ sending E-equivalent points to F-equivalent points.

Definition

A reduction of E to F is a homomorphism from E to F sending E-inequivalent points to F-inequivalent points.

Disclaimer

For the sake of clarity, we work in $\mathtt{ZFC} + \mathtt{add}(\mathtt{null}) = \mathfrak{c}$.

Definition

A subset of a Polish space X is universally measurable if it is measurable with respect to every Borel probability measure on X.

Definition

A function between Polish spaces is universally measurable if the pre-image of each open set is universally measurable.

Definition

We write $X/E \leq_{\mathrm{UM}} Y/F$ to indicate the existence of a universally measurable reduction of E to F.

Definition

We write $X/E \sim_{\mathrm{UM}} Y/F$ to indicate that both $X/E \leq_{\mathrm{UM}} Y/F$ and $Y/F \leq_{\mathrm{UM}} X/E$.

Definition

We use $|X/E|_{UM}$ to denote the $\sim_{\mathrm{UM}}$-class of X/E.

Definition

We refer to such classes as universally measurable cardinals.

Definition

We write $|X/E|_{UM} \leq |Y/F|_{UM}$ to indicate that $X/E \leq_{\mathrm{UM}} Y/F$.

Proposition 1

The usual cardinals and the universally measurable cardinals agree on countable sets.

Theorem 2 (Silver)

The class $|\mathbb{R}|_{UM}$ is the successor of $|\mathbb{N}|_{UM}$.

Theorem 3 (Harrington-Kechris-Louveau)

The class $|\mathbb{R}/\mathbb{Q}|_{UM}$ is the successor of $|\mathbb{R}|_{UM}$.

Definition

Let $E \times F$ denote the equivalence relation on $X \times Y$ given by

$$(x_1, y_1)\ (E \times F)\ (x_2, y_2) \Leftrightarrow (x_1\ E\ x_2 \text{ and } y_1\ F\ y_2).$$

Definition

Let $|X/E|_{UM} \times |Y/F|_{UM}$ denote $|(X \times Y)/(E \times F)|_{UM}$.

Definition

Let $|X/E|_{UM} \leq |Y/F|_{UM} * |Y'/F'|_{UM}$ denote the existence of a universally measurable homomorphism $\varphi\colon X \to Y'$ from E to F' such that $|\varphi^{-1}([y']_{F'})/E|_{UM} \leq |Y/F|_{UM}$ for all $y' \in Y'$.

Definition

A basis for a quasi-ordered set $(Q, \leq)$ is a set $B \subseteq Q$ with the property that $\forall q \in Q \exists b \in B \; b \leq q$.

Theorem 4

Suppose that $\mathscr{B}$ is a basis for the family of universally measurable cardinals strictly above $\mathbb{R}/\mathbb{Q}$. Then:

1. There exists $\kappa \in \mathscr{B}$ such that $\kappa < \kappa \times \mathfrak{c}$.
2. There exists $\kappa \in \mathscr{B}$ such that $\mathscr{C} = \{\lambda \leq \kappa \mid \lambda < \lambda \times 2\}$ is a basis below κ.
3. There exist $\kappa \in \mathscr{B}$ and $(\kappa_r)_{r \in \mathbb{R}}$ below $\aleph_0 * \kappa$ with the property that $\kappa_r \leq \aleph_0 * \kappa_s$ but $\kappa_r \not\leq \kappa_s \times \aleph_0$ for all distinct $r, s \in \mathbb{R}$.
4. The cardinality of $\mathscr{B}$ is at least $\mathfrak{c}$.
5. There exists $\kappa \in \mathscr{B}$ such that the initial segment below $\aleph_0 * (\kappa \times \mathfrak{c})$ reduces all analytic quasi-orders on Polish spaces.

Part II

A few details

Definition

An equivalence relation is countable if its classes are all countable.

Definition

A Borel equivalence relation E is hyperfinite if $E = \bigcup_{n \in \mathbb{N}} F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ are finite Borel subequivalence relations of E.

Definition

The difference set associated with $\varphi\colon X \to Y$ and $\psi\colon X \to Y$, or $D(\varphi, \psi)$, is given by $D(\varphi, \psi) = \{x \in X \mid \varphi(x) \neq \psi(x)\}$.

Definition

Given $\Gamma \curvearrowright Y$, we say that a homomorphism $\varphi\colon X \to Y$ from E to E_Γ^Y is ρ-invariant if $\varphi(x_1) = \rho(x_1, x_2) \cdot \varphi(x_2)$ for all $x_1, x_2 \in X$.

Definition

We say that $\Gamma \curvearrowright Y$ is locally rigid if whenever X is a Polish space, E is a countable Borel equivalence relation on X, $\rho \colon E \to \Gamma$ is a Borel function, and φ, ψ are ρ-invariant countable-to-one Borel homomorphisms from E to E_Γ^Y, the relation $E \upharpoonright D(\varphi, \psi)$ is hyperfinite.

Theorem 5

The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.

Proof (Sketch)

We will just mention a few observations that go into the proof.

Lemma 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho\colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, and φ and ψ are ρ-invariant countable-to-one Borel homomorphisms from E to $E^{\mathbb{R}^2}_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}$. Then there is a countable-to-one Borel homomorphism from $E \restriction D(\varphi, \psi)$ to $E^{\mathbb{T} \times \mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$.

Lemma 7 (Jackson-Kechris-Louveau)

The equivalence relation $E^{\mathbb{T}}_{\mathrm{SL}_2(\mathbb{Z})}$ is hyperfinite.

Lemma 8

All but countably many stabilizers of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial, and the non-trivial ones are infinite cyclic.

Lemma 9

The equivalence relation $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}\times\mathbb{T}^2}$ is hyperfinite.

It follows that $E \restriction D(\varphi, \psi)$ is hyperfinite. ⊠

Definition

Let $L(X, \mu, Y)$ denote the set of μ-measurable functions from X to Y, equipped with the pseudo-metric $d_\mu(\varphi, \psi) = \mu(D(\varphi, \psi))$.

Definition

Let $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ denote the subspace of all $\varphi \in L(X, \mu, Y)$ for which there is a μ-conull set $C \subseteq X$ such that $\varphi \restriction C$ is a countable-to-one homomorphism from $E \restriction C$ to F.

Definition

We say that F has separable homomorphisms if whenever X is a Polish space, E is a countable Borel equivalence relation, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite, the space $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Proposition 10

Suppose X is a Polish space and $\Gamma \curvearrowright X$ is a locally rigid Borel action of a countable group. Then E_Γ^X has separable homomorphisms.

Proposition 11

The countable Borel equivalence relations with separable homomorphisms are closed under countable-to-one Borel homomorphism.

Theorem 12

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and there is a countable-to-one Borel homomorphism from E to $E^{\mathbb{R}^2}_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}$. Then E has separable homomorphisms.

Definition

A Borel equivalence relation E is universally measurable hyperfinite if $E = \bigcup_{n \in \mathbb{N}} F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ are finite universally measurable subequivalence relations of E.

Proposition 13

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X which has separable homomorphisms but is not universally measurable hyperfinite. Then there is no universally measurable reduction of $E \times \Delta(2^{\mathbb{N}})$ to E.

Proposition 14

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a Borel probability measure on X, $\mathcal{I}$ is a σ-ideal on Y, E is μ-nowhere hyperfinite, and F has separable homomorphisms. Then there is an $\mathcal{I}$-conull set $C \subseteq Y$ such that the image of every μ-positive Borel set $B \subseteq X$ under every Borel homomorphism from $E \upharpoonright B$ to $F \upharpoonright C$ is $\mathcal{I}$-positive.

Proposition 15

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X which has separable homomorphisms but is not universally measurable hyperfinite. Then there is a Borel set $B \subseteq X$ for which there is no universally measurable reduction of $(E \upharpoonright B) \times \Delta(2)$ to $E \upharpoonright B$.

Definition

A Borel equivalence relation E is treeable if there is an acyclic Borel graph whose components coincide with the classes of E.

Proposition 16

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not universally measurable hyperfinite. Then there is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E which are pairwise incomparable under universally measurable reducibility.

Proposition 17

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not universally measurable hyperfinite. Then there is no basis of cardinality $< \mathfrak{c}$ for the family of non-universally measurable hyperfinite countable Borel equivalence relations which admit countable-to-one Borel homomorphisms to E.

Proposition 18

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not universally measurable hyperfinite. Then the initial segment of the universally measurable reducibility hierarchy consisting of relations with countable-to-one Borel homomorphisms to $E \times \Delta(2^{\mathbb{N}})$ contains copies of all analytic quasi-orders on Polish spaces.