Algebraic dynamics

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Some acknowledgments

Some the work I will describe today, I have done on my own, but mostly, it is joint with such people as R. Benedetto, D. Ghioca, B. Hutz, P. Kurlberg, C. G. Lee, A. Medvedev, T. Tucker, and Y. Yasufuku supported by an NSF Focused Research Group grant on algebraic dynamics.

A dynamical system is a pair (X, f) consisting of a set X and a self-map $f : X \to X$.

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For a dynamical system (X, f) we define the following associated sets.

- The orbit of a point: For $a \in X$ we write $\mathscr{O}_f(a) := \{f^{\circ n}(a) \ : \ n \in \mathbb{N}\}$.
- Return sets: For $a \in X$ and $Y \subseteq X$

$$E(a, f, Y) := \{n \in \mathbb{N} \ : \ f^{\circ n}(a) \in Y\}$$

- (Pre-)periodic points:

$$\mathrm{Per}(f) := \{a \in X \ : \ f^{\circ n}(a) = a \text{ for some } n \in \mathbb{Z}_+\}$$

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- For purposes of this lecture, it suffices to think of such a dynamical system as $g : K^n \to K^n$ where K is some field and g is given by an n-tuple of polynomials.
- Generalizing somewhat, it may have been better to define a dynamical system in a category to be an arrow $f : X \to X$ whose source and target are the same object. An algebraic dynamical systems is then a dynamical system in the category of algebraic varieties.

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Special points interpreted dynamically

- If (X, f) is an algebraic variety defined over a field K and $a \in X(K)$ is a K-rational point, then $\mathscr{O}_f(a) \subseteq X(K)$ consists entirely of K-rational points.
- If X is a semi-abelian variety (for example, if $X(\mathbb{C}) = (\mathbb{C}^\times)^g$ is some Cartesian power of the multiplicative group), and $f : X \to X$ is given by multiplication by $n > 1$ (in the sense of the group; $(x_1, \ldots, x_g) \mapsto (x_1^n, \ldots, x_g^n)$ in the example), then $\mathrm{PrePer}(f)$ is the set of torsion points on X and $\mathrm{Per}(f)$ is the set of torsion points of order prime to n.

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- Dynamical Mordell-Lang: If $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$, $a \in X(\mathbb{C})$ is any point, and $Y \subseteq X$ is a closed subvariety, then the return set $E(a, f, Y)$ is a finite union of points and arithmetic progressions.
- Dynamical Manin-Mumford: If $f : X \to X$ is an algebraic dynamical system over $\mathbb{C}$, $Y \subseteq X$ is a closed irreducible subvariety and $Y(\mathbb{C}) \cap \text{PrePer}$ is Zariski dense in Y, then Y is a preperiodic subvariety.
- Dense orbit conjecture: If $f : X \to X$ is an (irreducible) algebraic dynamical system defined over the algebraic numbers, then there is an algebraic point $a \in X(\mathbb{Q}^{alg})$ for which $\mathscr{O}_f(a)$ is Zariski dense in X.

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- Zhang works with a stronger notion of an algebraic dynamical system. A *polarized algebraic dynamical system* is a triple $(X, f, \mathscr{L})$ where $\mathscr{L}$ is an ample line bundle on X, $f : X \to X$ is a self-morphism, and $f^*\mathscr{L} \approx \mathscr{L}^{\otimes q}$ for some $q > 1$.
- The dynamical Mordell-Lang conjecture may still hold with our weaker notion of algebraic dynamical system.
- The dynamical Manin-Mumford conjecture (even with Zhang's stronger hypotheses) is known to be false by work of Ghioca and Tucker.
- In general, for the dense orbit conjecture one needs the hypotheses that $f : X \to X$ is dominant and that there is no dominant rational map $g : X \dashrightarrow Y$ with $\dim(Y) > 0$ and $g \circ f = g$. This follows from polarizability.

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From geometry to arithmetic, a logical caution

While the data defining an algebraic dynamical system are finitary and computable (assuming that the ring operations on our underlying field are computable), it is not obvious that the sets of special points we are considering (eg $\mathscr{O}_f(a)$, $\mathrm{Per}(f)$, $\mathscr{O}_f(a) \cap Y(\mathbb{C})$, *etc.*) are computable.

The dynamical Mordell-Lang conjecture implies, in particular, that the sets of the form $\mathscr{O}_f(a) \cap Y(\mathbb{C})$ are computable (though it is not a formal consequence of this conjecture that one may compute the set from a presentation of the problem).

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Dense orbit conjecture and geometry

Recall that the dense orbit conjecture says that if (X, f) is an (irreducible, dominant) algebraic dynamical system over an algebraically closed field K which does not dominate a constant dynamical system, then there is a point $a \in X(K)$ whose orbit is Zariski dense.

Considering the contrapositive, if the conjecture were false witnessed by (X, f), then for each algebraic point $a \in X(K)$, the Zariski closure of $\mathcal{O}_f(a)$ would be a proper subvariety of X. Clearly, $f(\mathcal{O}_f(a)) \subseteq \mathcal{O}_f(a)$ so that $f\overline{\mathcal{O}_f(a)} \subseteq \overline{\mathcal{O}_f(a)}$.

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The real problem for which the dense orbit conjecture is merely a test question for its solution is:

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Qualitatively describe the invariant varieties for algebraic dynamical systems.

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The class of algebraic dynamical systems and their invariant subvarieties may be analyzed as special cases of definable sets in difference closed fields. As such, they admit a tame geometry.

Being definable in general has no consequences, but if we require that X be definable in a fairly inexpressive language, then we may deduce that X enjoys certain tameness properties and avoids the pathologies of general mathematical objects.

For the dynamical problems, we consider sets definable in difference closed fields.

A difference field is a pair (K, σ) consisting of a field K and an endomorphism $\sigma : K \to K$. It is difference closed if every finite system of polynomial difference equations and inequations with coefficients from K which has a solution in some difference field extension already has a solution in K.

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If (X, f) is an algebraic dynamical system defined over the field K, then by regarding K as a difference field with the identity map $\sigma = \mathrm{id} : K \to K$, the definable set $(X, f)^\sharp$ defined by $a \in X$ and $\sigma(a) = f(a)$ encodes the dynamical system.

When evaluated on (K, id), the definable set $(X, f)^\sharp$ contains only the fixed points of f, but when evaluated on a difference closed field $(\mathbb{U}, \sigma)$, the set is Zariski dense in X (assuming that f is dominant and X is absolutely irreducible) and the invariant subvarieties of f correspond to definable subset of $(X, f)^\sharp$.

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- The definable sets in difference closed fields admit dimension theories more refined than the usual algebraic dimensions. Generally, if (X, f) is a dynamical system, then $\dim(X, f)^\sharp \leq \dim(X)$ with strict inequality possible. [Here $\dim(X, f)^\sharp$ could be taken to be Lascar rank and $\dim(X)$ is the usual dimension from algebraic geometry.]
- The one dimensional sets fall into three classes: field-like (essentially algebraic curves over the fixed field), group-like (in correspondence with definable groups have the property that every definable subset of any Cartesian power is essentially a translate of a group), and disintegrated (all definable relations on the set are essentially binary).
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Structure theory for definable sets in difference closed fields

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Theorem (Medvedev)

If $f(x)$ is a one-variable nonconstant rational function, then either the set defined by $\sigma(x) = f(x)$ is disintegrated or there is a one-dimensional algebraic group G, isogeny $\phi : G \to G^\sigma$ and nonconstant rational function $\pi : G \to \mathbb{P}^1$ for which $f \circ \pi = \pi \circ \phi$.

Theorem (Medvedev, S.)

If $f_1, \ldots, f_n$ is a sequence of one variable nonlinear polynomials over $\mathbb{C}$ none of which is special, then every invariant subvariety of the algebraic dynamical system given by $(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n))$ comes from invariant curves for $(x_i, x_j) \mapsto (f_i(x_i), f_j(x_j))$.

- Here, "special" means linearly conjugate to a monomial or a scalar multiple of a Chebyshev polynomial.
- The invariant curves for (f_i, f_j) have a very simple form. In the special case of (f, f), they are horizontal, vertical or of the form $y = g(x)$ for some g commuting with f.

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Corollaries for algebraic dynamics

- Zhang's dense orbit conjecture is true for dynamical systems of the form $(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n))$ where each f_i is a nonconstant polynomial and at most one f_i is linear.
- The dynamical Manin-Mumford conjecture is true for such dynamical systems (for periodic points) provided that each f_i lifts the Frobenius in the sense that each is defined over $\mathbb{Z}$ and $f_i(x) \equiv x^p \pmod{p}$.
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Skolem's method

Skolem introduced a p-adic analytic method for proving some instances of the Mordell conjecture in the 1930s.

In some cases, his method generalizes to dynamical problems: given an algebraic dynamical system (X, f) over a field K of characteristic zero, a point $a \in X(K)$, and a subvariety $Y \subseteq X$ one may choose a prime p and regard (X, f) and Y as being defined over $\mathbb{Q}_p$.

Sometimes, one can find a p-adic analytic function $\Phi : \mathbb{Z}_p \to X(\mathbb{Q}_p)$ which interpolates the map $n \mapsto f^{\circ n}(a)$. In this case, the set $\{\alpha \in \mathbb{Z}_p : \Phi(\alpha) \in Y(\mathbb{Q}_p)\}$ is the zero set of a one variable p-adic analytic function which by the identity principle must be a finite union of points and sets of the form $a + p^n\mathbb{Z}_p$. The dynamical Mordell-Lang conjecture follows in this case.

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If instead of a p-adic analytic function one had a real analytic function $\Phi : (0, \infty) \to X(\mathbb{C})$ satisfying $\Phi(n) = f^{\circ n}(a)$, then it would seem that we could deduce nothing about the set $\{n \in \mathbb{N} \ : \ f^{\circ n}(a) \in Y(\mathbb{C})\}$ from the fact that it is the intersection of $\mathbb{N}$ with the zero set of a real analytic function.

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However, if we could arrange for Φ to be o-minimally definable, then the dynamical Mordell-Lang conjecture would follow.

A structure $(R, <, \ldots)$ is o-minimal if $<$ is a total order and every definable subset of R is a finite union of points and intervals.

It follows from a theorem of Tarski that $(\mathbb{R}, <, +, \cdot, 0, 1)$ is o-minimal. A deep theorem of Wilkie shows that $(\mathbb{R}, <, +, \cdot, \exp, 0, 1)$ is also o-minimal. By work of van den Dries and Miller, the real numbers given together with all polynomials, the exponential function, and all real analytic functions (in any number of variables) restricted to boxes $[0, 1]^n$ is o-minimal.

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O-minimal Skolem's method

- There are algebraic dynamical systems to which the o-minimal Skolem method applies though the p-adic method does not. (Alas, not every algebraic dynamical system is susceptible to this method.)
- The definable sets in higher dimension in o-minimal structures are also tame. If one considers a dynamical system (X, f) and point a for which there is an o-minimally definable interpolating map $\Phi : (0, \infty) \to X(\mathbb{C})$ for the orbit of a, then for any subvariety $Y \subseteq X^n$, one may interpret $\{(\ell_1, \ldots, \ell_n) \in \mathbb{N}^n \ : \ (f^{\circ \ell_1}(a), \ldots, f^{\circ \ell_n}(a)) \in Y(\mathbb{C})\}$ as the integer points on a definable set. A counting theorem of Pila and Wilkie limits the number of such points not coming from "obvious" relations to being subpolynomial in the size of the integer.

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Higher rank problem

Let X be an algebraic variety over some field K, $f_1, \ldots, f_n$ a sequence of self-maps $f_i : X \to X$ which commute with each other, $a \in X(K)$ a point, and $Y \subseteq X$ a subvariety. What can one say about
$E := \{(\ell_1, \ldots, \ell_n) \in \mathbb{N}^n \ : \ f_1^{\circ \ell_1} \circ \cdots \circ f_n^{\circ \ell_n}(a) \in Y(K)\}$?

In general, not much. Consider $X(\mathbb{C}) = \mathbb{C}^n$, $f_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$, Y defined over $\mathbb{Z}$, and $a = (0, \ldots, 0)$. Then $E = Y(\mathbb{Z})$ which is notoriously complicated.

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With Y. Yasufuku, I showed that, in fact, every exponential-polynomial set, a set of n-tuples of natural numbers defined by the vanishing of a function of the form $(\ell_1, \ldots, \ell_n) \mapsto P(\ell_1, \ldots, \ell_n; \{\alpha_i^{\ell_j}\})$ where P is a polynomial with coefficients from the ring of all algebraic integers and each α_i is an algebraic integer, may be realized as a return set for a sequence of n commuting endomorphisms of an algebraic torus.

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The model theoretic methods suggest that by averting to properties of definable sets we may tame algebraic dynamics.

However, in joint work with several collaborators (Benedetto, Ghioca, Hutz, Kurlberg and Tucker), we showed that while the initial step of Skolem's method may be made to work in low dimension, probabilistically, it should almost always fail in dimensions four and higher.

Thus, I would expect that the chaos inherent in Gödel's incompleteness theorem will show itself even in the algebraic dynamics of a single map $f : X \to X$.

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