An abstract approach to finite Ramsey theory with applications

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Outline of Topics

Introduction

Ramsey theory has become important in topological dynamics of Polish, not necessarily locally compact, groups, particularly after close connections between Ramsey theory and extreme amenability were found by Kechris–Pestov–Todorcevic.

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I will outline an approach that recovers most of the unstructured Ramsey theory and makes it possible to prove new results.

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We formulate within this approach an **abstract pigeonhole principle** and an **abstract Ramsey theorem**, and prove that the pigeonhole principle implies the Ramsey theorem.

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— a hierarchy of the Ramsey results according to the number of times the abstract Ramsey theorem is applied in their proofs:
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— a possibility of classifying concrete Ramsey theorems.

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$(\mathcal{R}, C)$ is an m-**connection** if $\mathcal{R}$ and C have m elements each and, upon listing $\mathcal{R}$ as $R_1, \ldots, R_m$ with $\min R_i < \min R_{i+1}$ and C as $c_1, \ldots, c_m$ with $c_i < c_{i+1}$

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An l-connection $(\mathcal{Q}, B)$ is an l-**subconnection** of an m-connection $(\mathcal{R}, C)$ if $\mathcal{R}$ is a coarser partition than $\mathcal{Q}$ and $B \subseteq C$.

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Algebraic notions

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find $f_0 \in F$ with $f_0 \,.\, P$ monochromatic.

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then there exists $f_0 \in F$ such that $\{f_0 \circ x \colon x \in P\}$ is monochromatic.

Normed backgrounds

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- o is an element of A (**zero**).

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(iv) there is $t = t_a \in \mathbb{N}$ with $\partial^t a = o$, and $\partial o = o$.

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$a \cdot b$ defined when $[k]$ contains the image of b and

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A with the above defined operations is a normed background.

Lifting multiplication to sets

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For $F, G \subseteq A$, $F \cdot G$ **is defined** if $f \cdot g$ is defined for all $f \in F$ and $g \in G$, and we let

$$F \cdot G = \{f \cdot g \colon f \in F,\, g \in G\}.$$

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We say that $\mathcal{F}$ with this operation is a **family over** A provided that whenever $F \bullet G$ is defined, then so is $F \cdot G$ and

$$F \bullet G = F \cdot G.$$

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For $k, l \in \mathbb{N}$ with $k \leq l$, let

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Clear: $\binom{n}{l} \bullet \binom{l}{k} = \binom{n}{l} \cdot \binom{l}{k}$.

Abstract Ramsey and abstract pigeonhole statements

Ramsey statement

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This is the classical Ramsey theorem.

Pigeonhole statement

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$$F^a = \{f \in F\colon a \text{ is a restriction of } f\}.$$

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Price: make f behave as prescribed by some $a \in A$ on a part of A containing y.

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(P) for $d > 0$, $P \in \mathcal{F}$, and $y \in \partial P$,
there is $F \in \mathcal{F}$ and $a \in A$ such that
$F \bullet P$ is defined, $a \,.\, y$ is defined and
for every d-coloring of $F \,.\, P_y$ there is $f \in F$ such that $f \,.\, P_y$ is monochromatic.

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for every d-coloring of $F \,.\, P_y$ there is $f \in F^a$ such that $f \,.\, P_y$ is monochromatic.

Example.(ctd)

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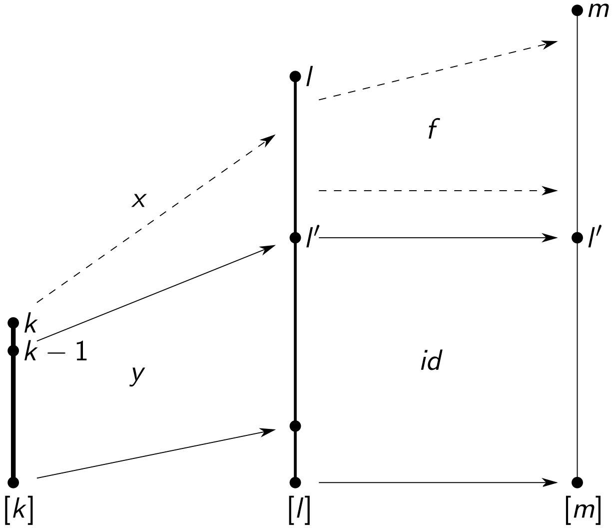

Aim: prove a theorem showing that (P) implies (R).

The theorem

Additional conditions

(A) if $F \in \mathcal{F}$, then $\partial F \in \mathcal{F}$;

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(A) if $F \in \mathcal{F}$, then $\partial F \in \mathcal{F}$;

(B) if $F, G \in \mathcal{F}$ and $F \bullet \partial G$ is defined, then there is $F' \in \mathcal{F}$ such that $F' \bullet G$ is defined and for each $f \in F$ there is $f' \in F'$ extending f.

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Statement of the theorem

Theorem (S.)

Let $\mathcal{F}$ be a family over a normed background fulfilling (A), (B) and ($*$). Assume each set in $\mathcal{F}$ is finite. Then (P) implies (R).

Example.(ctd)

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is defined. Then $m = l - 1$. Take $F' = \binom{n+1}{l}$ to witness (B).
It works since $\binom{n+1}{l} \bullet \binom{l}{k}$ is defined and each element of $\binom{n}{l-1}$ is extended by an element of $\binom{n+1}{l}$.

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is defined, then $m = l$ and $p = n$,

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is defined, then $m = l$ and $p = n$, but then

$$(\binom{q}{p} \bullet \binom{n}{m}) \bullet \binom{l}{k}$$

is defined.

Conclusion:

Conclusion: the classical Ramsey theorem holds.