A stability transfer theorem in d-Tame Metric Abstract Elementary Classes

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Outline

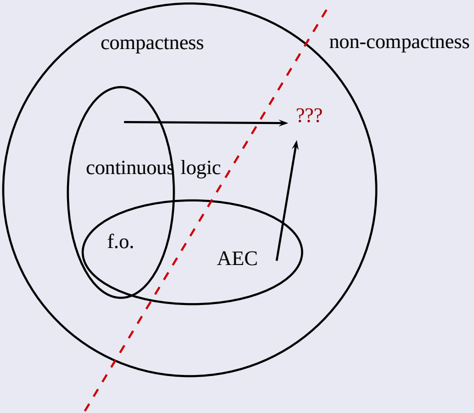
1. Motivation
2. Definition of MAEC
3. d-Tameness and independence in MAEC
4. Stability transfer theorems

Motivation

Abstract Elementary Classes (shortly AECs) correspond to a generalization of $(Mod(T), \prec)$ (T a first order theory), due to B. Jónsson and S. Shelah, towards doing a model-theoretical study of classes of structures which are not first order axiomatizable (e.g., torsion abelian groups, elementary classes in infinitary logics, etc).

Discrete *tame* AEC is a special kind of AEC which have a categoricity transfer theorem (due to Grossberg and VanDieren) and a nice stability transfer theorem (due to Baldwin, Kueker and VanDieren), by using ω-locality.

A map of discrete and metric structures classes



Motivation

MAEC corresponds to a kind of amalgam between *AEC* and *Continuous Logic Elementary Classes*. In this work, we study a version of tameness in this setting and prove a stability transfer theorem, removing the ω-locality assumption and assuming local character of a suitable well-behaved notion of independence.

Definition (MAEC)

Definition

Let $\mathcal{K}$ be a class of L-structures (in the setting of Continuous Logic, but the function symbols need not be uniformly continuous). and let $\prec_{\mathcal{K}}$ be a binary relation defined on $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is a *Metric Abstract Elementary Class* (for short, *MAEC*) iff:

(1) $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under $\cong$

(2) $\prec_{\mathcal{K}}$ is a partial order in $\mathcal{K}$.

(3) If $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.

Definition (MAEC)

(4) (Tarski-Vaught chains) If $(M_i : i < \lambda)$ is an increasing and continuous $\prec_{\mathcal{K}}$-chain, then

- the function symbols in L can be uniquely interpreted in the completion of $\underline{\bigcup_{i<\lambda} M_i}$ such that $\overline{\bigcup_{i<\lambda} M_i} \in \mathcal{K}$
- for all $j < \lambda$, $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\lambda} M_i}$
- if for every $M_i \in \mathcal{K} \prec_{\mathcal{K}} N$, then $\overline{\bigcup_{i<\lambda} M_i} \prec_{\mathcal{K}} N$.

(5) (coherence) If $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, then $M_1 \prec_{\mathcal{K}} M_2$.

(6) (DLS) There is a cardinality $LS(K)$ (called *Löwenheim-Skolem number of* $\mathcal{K}$) such that if $M \in \mathcal{K}$ and $A \subseteq M$, then there exists $N \in \mathcal{K}$ such that $dc(N) \leq dc(A) + LS(K)$ and $A \subseteq N \prec_{\mathcal{K}} M$.

Examples of MAECs

1. The class of Banach spaces
2. The subclass of complete models in an elementary class of a positive bounded theory (W. Henson - J. Iovino).
3. Compact Abstract Theories (I. Ben Yaacov).
4. $(Mod(T), \prec)$, T a (first order) theory in Continuous Logic:
    1. Hilbert spaces with unitary operators (C. Argoty - A. Berenstein)
    2. Nakano spaces with compact essential rank (P. Poitevin).
5. Hilbert spaces with unbounded operators (C. Argoty)
6. Gelfand triplets (Z.)
7. AECs (with the discrete metric)

$\mathcal{K}$-embedding

Let $M, N \in \mathcal{K}$. A $\mathcal{K}$-embedding is an embedding $f : M \to N$ such that $f[M] \prec_{\mathcal{K}} N$.

Remark

Under AP+JEP+existence of large enough models, we can work in a homogeneous monster model $\mathbb{M} \in \mathcal{K}$.

Definition

$ga - tp(a/M) := orb_{Aut(\mathbb{M}/M)}(a)$.

Definition

$\text{ga-S}(M) := \{\text{ga-tp}(a/M) : a \in \mathbb{M}\}$

Some preliminary results

Definition

Let $p, q \in \text{ga-S}(M)$ ($M \in \mathcal{K}$). $d(p,q) := \inf\{d(a,b) : a \vDash p, b \vDash q\}$.

Definition

We say that a MAEC satisfies the *continuity* property (for short, *CP*) iff $(a_n) \to b$ and $\text{ga-tp}(a_0/M) = \text{ga-tp}(a_n/M)$ for every $n < \omega$ implies that $\text{ga-tp}(b/M) = \text{ga-tp}(a_0/M)$.

Fact (Hirvonen-Hyttinen)

d is a metric in $\text{ga-S}(M)$ ($M \in \mathcal{K}$) iff $\mathcal{K}$ satisfies the *CP*.

Cofinal stability

Assumption

Assume that d define above is a metric.

Definition

We say that a MAEC $\mathcal{K}$ is μ-**d**-stable iff for every $M \in \mathcal{K}$ of density character μ we have that $dc(\text{ga-S}(M)) \leq \mu$.

Cofinal d-stability

Let $\mathcal{K}$ be an MAEC with AP and JEP and $LS(\mathcal{K}) \leq \lambda < \kappa$. We say that $\mathcal{K}$ is $[\lambda, \kappa)$-*cofinally d-stable* iff given $\theta \in [\lambda, \kappa)$ there exists $\theta' \geq \theta$ in $[\lambda, \kappa)$ such that $\mathcal{K}$ is θ'-d-stable.

Tameness (discrete AEC)

Let $\mathcal{K}$ be an AEC and $\mu \geq LS(\mathcal{K})$. We say that $\mathcal{K}$ is μ-*tame* iff for any $M \in \mathcal{K}$ of cardinality $\geq \mu$, if $p \neq q$ where $p, q \in$ ga-S(M), then there exists $N \prec_{\mathcal{K}} M$ of cardinality μ such that $p \upharpoonright N \neq q \upharpoonright N$.

d-Tameness

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d-Tameness

Let $\mathcal{K}$ be a MAEC and $\mu \geq LS(\mathcal{K})$. We say that $\mathcal{K}$ is μ-**d**-*tame* iff for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if for any $M \in \mathcal{K}$ of dc $\geq \mu$ we have that $\mathbf{d}(p, q) \geq \varepsilon$ where $p, q \in$ ga-S(M), then there exists $N \prec_{\mathcal{K}} M$ of dc μ such that $\mathbf{d}(p \upharpoonright N, q \upharpoonright N) \geq \delta_\varepsilon$.

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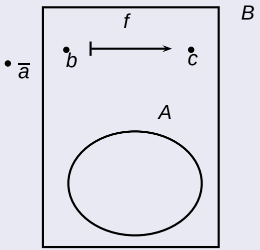
Some assumptions (*)

We assume $\mathcal{K}$ is a μ-d-tame ($\mu < \kappa$) and $[LS(\mathcal{K}), \kappa)$-cofinally-d-stable. Define $\lambda := \min\{\mu < \chi < \kappa : \mathcal{K}$ is χ-d-stable$\}$ $\zeta := \min\{\chi : 2^\chi > \lambda\}$ and $\zeta^* := \max\{\mu^+, \zeta\}$. We also require that $cf(\kappa) \geq \zeta^*$ and $\kappa > \zeta^*$.

Splitting in first order

Definition (f. o. splitting)

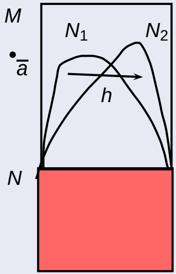
Let $A \subset B$ and $\overline{a} \in \mathbb{C}$. We say that $tp(\overline{a}/B)$ splits over A iff there exist $b, c \in B$ such that $tp(b/A) = tp(c/A)$ (witnessed by $f \in Aut(\mathbb{C}/A)$ in such a way that $f(b) = c$) but $tp(a/Ac) \neq tp(f(a)/Ac)$.



Independence in tame MAEC

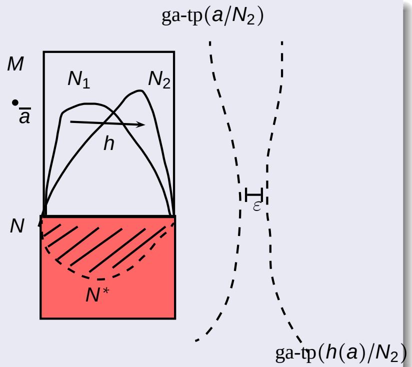
Definition (Splitting, AEC)

Let $N \prec_{\mathcal{K}} M$. We say that ga-tp(a/M) splits over N iff there exist N_1 and N_2 such that $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \cong_N N_2$ such that ga-tp$(a/N_2) \neq h($ga-tp$(a/N_1))$.



Definition (ε-splitting)

Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) ζ^*-ε-splits over N iff for every $N^* \prec_{\mathcal{K}} N$ of dc $< \zeta^*$ there exist N_1 and N_2 of dc $< \zeta^*$ such that $N^* \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \cong_{N^*} N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1)) \geq \varepsilon$.



Independence in tame MAEC

Notation

If ga-tp(a/M) does not tame-ε-split over N, we denote that by $a \downarrow_N^\varepsilon M$.

Definition

Let $N \prec_{\mathcal{K}} M$. We say that a is ζ^*-independent from M over N iff for every $\varepsilon > 0$ $a \downarrow_N^\varepsilon M$.

Some properties of ε-independence

Some properties

1. (Local character I) For every M, a and every $\varepsilon > 0$ there exists $N \prec_{\mathcal{K}} M$ of density character $< \zeta^*$ such that $a \downarrow^{\varepsilon}_{N} M$.
2. (Weak stationarity) For every $\varepsilon > 0$ there exists δ such that for every $N_0 \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ and every a, b, if N_1 is universal over N_0, $a, b \downarrow^{\delta}_{N_0} N_2$ and $\mathbf{d}(\text{ga-tp}(a/N_1), \text{ga-tp}(b/N_1)) < \delta$, therefore $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) < \varepsilon$.

More assumptions (**) -local character II-

For every tuple $\overline{a}$, every $\varepsilon > 0$ and every increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \sigma \rangle$, there exists $j < \sigma$ such that $\overline{a} \downarrow^{\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.

Stability transfer theorems

Assumption (*)

We assume $\mathcal{K}$ is μ-d-tame ($\mu < \kappa$) and $[LS(\mathcal{K}), \kappa)$-cofinally-d-stable. Define $\lambda := \min\{\mu < \chi < \kappa : \mathcal{K} \text{ is } \chi\text{-d-stable}\}$ $\zeta := \min\{\chi : 2^\chi > \lambda\}$ and $\zeta^* := \max\{\mu^+, \zeta\}$. We also require that $cf(\kappa) \geq \zeta^*$ and $\kappa > \zeta^*$.

Theorem ($\sharp$)

Let $\mathcal{K}$ be an MAEC satisfying assumption $(*)$. Then $\mathcal{K}$ is κ-**d**-stable.

Idea of the proof.

By RA, using local character I, $cf(\kappa) \geq \zeta^*$ and pigeon-hole principle, contradicting cofinal stability. □

Remark

If $\mu = \aleph_0$ and $\lambda = \aleph_1$, then $\zeta := \min\{\chi : 2^\chi > \lambda\} \leq \aleph_1$ and $< \zeta^* = \aleph_0$.

Corollary

Let $\mathcal{K}$ be an $\aleph_0$-**d**-tame MAEC. Suppose that $\mathcal{K}$ is $\aleph_0$-*d*-stable and $\aleph_1$-d-stable. Then $\mathcal{K}$ is $\aleph_n$-d-stable for all $n < \omega$

Corollary (♯ ♯)

Let $\mathcal{K}$ be an $\aleph_0$-**d**-tame MAEC. Suppose that $\mathcal{K}$ is $\aleph_0$-*d*-stable and $\aleph_1$-d-stable. Then $\mathcal{K}$ is $\aleph_\omega$-d-stable.

Idea of the proof

By RA, use local character II, $\aleph_0$-d-tameness and pigeon-hole principle to contradict $\aleph_n$-d-stability. □

Proposition

Let $\mathcal{K}$ be an $\aleph_0$-d-tame, $\aleph_0$-d-stable and $\aleph_1$-d-stable MAEC, which also satisfies assumption (**). Then $\mathcal{K}$ is κ-*d*-stable for every cardinality κ.

Idea of the proof.

If $cf(\kappa) \geq \zeta^* = \omega_1$, use theorem $\sharp$. If $cf(\kappa) = \omega$, use a similar argument as in corollary $(\sharp\,\sharp)$

Question

What of the done work can we do if *d* is not a metric?

Adknowledgment

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THANKS!

View from Colpatria tower - Bogotá.