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NOTES ON OPTIMAL CONTROL THEORY FOR THE NEOCLASSICAL MODEL OF GROWTH

by

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Abstract

The notes derive the salient features of optimal control theory for an infinite horizon objective function. For concreteness, the necessary and sufficient conditions are derived with reference to the neoclassical model of growth. However, the ideas easily extend to other economic problems. The need for the exercise arises from the lack of a single source which students can access for a development of the subject from first principles.

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1 Introduction

For a little over a decade now, endogenous growth theorists have been applying the techniques of optimal control to continuous time dynamic models. These models typically assume, like their neoclassical predecessors Cass (1965), Koopmans (1965) and Ramsey (1928), a dynastic macro household, or a social planner, choosing a consumption (hence, saving) plan by maximizing a welfare integral over an infinite time domain. The mathematical theory (as well as the economic interpretations) underlying optimal control methods (as applied to neoclassical growth models of the pre-endogenous genre) have been discussed by authors such as Arrow & Kurz (1970), Aghion & Howitt (1998), Barro & Sala-i-Martin (1995/1999), Cass (1965), Chiang (1992), Dixit (1990), Dorfman (1968), Intrilligator (1971), Kamien & Schwartz (2000) and others. Since the same principles carry over *mutatis mutandis* to the new growth models, students who wish to embark upon a study of modern growth theory have a large and readily available literature to fall back upon for an introduction to optimal control techniques.

[1]The pages that follow represent the outcome of my attempts to come to grips with the basic ideas underlying Optimal Control Theory. Like most teachers, I learnt through teaching and, to that extent, owe a note of thanks to all students who suffered willy-nilly through my lectures. I am partcularly indebted, however, to Ranajoy Basu, Sayan Datta, Kaushik Gangopadhyay, Raman Khaddaria and Shubhashis Modak Chaudhury, five wonderful students who kept me going through a difficult phase of the learning process. Pradip Maiti's suggestions on certain parts of these notes helped me improve the exposition and clinch an important argument in Appendix 1.

The advantage, however, is weighed down by the fact that while many of these references contain lucid presentations of the basic tenets of control theory, none of them provide a complete treatment of the necessary *and* sufficient conditions characterizing *infinite* horizon optimal paths for the standard neoclassical model. To the best of our knowledge, Cass (1965) contains the most satisfactory *sufficiency* proof for the optimality of the infinite horizon Solow (1956) path. A comparable development of the *necessary* conditions, however, is hard to come by. Arrow & Kurz (1970) and Kamien & Schwartz (2000) provide alternative necessity proofs for the finite horizon problem. By comparison, their discussions of the infinite horizon case are sketchy. The same observation holds for the other references quoted above.[2] A little investigation reveals on the other hand, that the full set of results are scattered about in the literature like pieces of a jigsaw puzzle, though no single source presents them in a comprehensible manner. Consequently, students uninitiated to the subject often find it difficult to put these together on their own. In this connection, matters are complicated further by a famous counter-example due to Halkin (1974), which demonstrated that the infinite horizon necessary condition describing the value of a co-state variable in the limit need not constitute a straightforward generalization of its finite horizon counterpart.

A second lacuna in the literature lies in the economic interpretation accorded to the differential equation describing the evolution of the co-state variable(s) over time when investment is assumed to be irreversible. The general form of this equation (covering both reversible and irreversible investment) undergoes a change to account for the possibility of corner solutions. Cass (1965), once again, contains a clear statement of this condition (for a model without technical progress). The intuitive arguments underlying the Cass equation, however, are not readily available in the literature. Yet, this equation constitutes one of the main pillars of optimal control theory and it is important to clarify its economic content.

The purpose of these notes is to present the control theoretic necessary and sufficient conditions for the optimal path associated with the Solow

[2]The seminal reference on the subject is Pontryagin *et al*, which contains a rigorous treatment of necessary conditions. However, the treatment there is not ideal as a first exposure to the problem. Also, the proofs in that work are not directly motivated by economic arguments.

model in the presence of exogenous technical progress.[3] This is done partly by collecting proofs and explanations from existing sources and partly by filling out the lacunas where they exist. The mathematics used to develop the conditions is mostly rigorous, though it does not go beyond the use of elementary Calculus. Hopefully, this will enable students to obtain a quick grasp of control theory techniques without having to consult several references involving varying degrees of mathematical complexity. By deliberate choice, the treatment is restricted to the one sector Solow model. This is mainly to ensure easy communication. While the approach should be readily extendable to simple two sector versions of the Solow model, thereby ensuring applicability to standard endogenous growth theory, a completely general treatment would call for deeper mathematical methods.

Section 1 is a description of the economic model. Section 2 derives the necessary conditions for the existence of an optimal path. This section draws heavily upon Arrow & Kurz (1970) and Dorfman (1969). Section 3 modifies Cass' treatment of sufficiency to allow for technical progress. Appendix 1 proves two results, used respectively in Sections 2 and 3. The first of these follows from a proposition in Koopmans (1965), though an independent argument is supplied here for completeness. The second result can be found in Koopmans (1965) as well as elsewhere. Appendix 2 discusses the counter-example of Halkin (1974) and indicates why it is inapplicable to the neoclassical growth model.

2 The Model

The description of the economic model relies on Arrow & Kurz (1970), David Cass (1965, 1966), Tjalling Koopmans ((1965), Frank Ramsey (1928) and Robert Solow (1956). The economy evolves over time through the interaction of the households and firms constituting it. There is a single produced commodity (Y), which can be consumed or accumulated as capital (K). The households interact with producers and engage in an intertemporal allocation exercise. We shall abstract completely from interactions between households

[3] We allow here for exogenous technical change alone, but it serves to highlight the alterations in earlier growth exercises that endogeneity will cause.

alone or those between producers alone at any point of time. Consequently, we will pretend that there is a single aggregative or representative household (H) and a single representative business firm (B) in the economy. The household increases in size over time and its rate of growth is captured by the equation

$$L(t) = L_0 e^{nt}, \tag{1}$$

where n is the exponential growth rate of the household size over time and $L(t)$ is the size of H at t. In what follows, $L(t)$ will be referred to as population at t.

The economy being infinitely lived in principle, H too will be assumed to have infinite life. This does not literally mean that individuals live forever. It means rather that members of any generation, even though finitely lived, care for their children, grandchildren and so on. In the present exercise, the generation at $t = 0$ is assumed to plan for its descendants across time forever. In other words, it decides about an optimal consumption path $C(t),\ t \in [0, \infty)$, where $C(t)$ is the aggregate consumption enjoyed by $L(t)$. Optimality of the path is judged with reference to an intertemporal welfare function of the form

$$\begin{aligned} U &= \int_o^\infty u(c(t)).\ e^{nt}.\ e^{-\rho t} dt \\ &= \int_o^\infty u(c(t))\ e^{-(\rho - n)t} dt, \end{aligned} \tag{2}$$

which is a weighted sum of instantaneous utilities derived from $c(t)$, the per capita consumption $C(t)/L(t)$ at t. The weights reflect two facts. First, e^{nt} shows that utilities from per capita consumption receive exponentially higher weights with time to take account of the fact that the household size increases at the rate n. Secondly, $\rho > 0$ stands for the rate of time preference of H. Utilities further down in time are valued less than utilities enjoyed earlier on. This leads to an exponentially decaying weight $e^{-\rho t}$ with the passage of time. We shall generally refer to ρ as the discount rate. It will be assumed all through that $\rho > n$. As will be evident presently, the assumption ensures that U is well defined. The welfare function will be assumed to satisfy the following assumptions:

Assumption u1 $u'(c) > 0,\ u''(c) < 0$,

Assumption u2 $u'(c) \to \infty$ as $c \to 0$ and $u'(c) \to 0$ as $c \to \infty$,

Assumption u3 $\rho > n$.

In other words, utility is a strictly concave function of c, the marginal utility is unboundedly high for small values of c, while it is as close to zero as possible for large c. The first part of **Assumption u2** rules out zero consumption at each point of time. **Assumption u3** ensures that U is well defined.

The entire stock of capital is owned by H. The services of the capital stock are used in production. This causes a fraction of K to depreciate. The rate of depreciation is δ per instant of time. Thus, K is partially durable. The household H receives compensation for capital services at the rate of $r(t)$, the competitive rate of interest prevailing at t. It supplies labour services also and the competitive wage rate at t is $w(t)$. Denoting the capital stock at t by $K(t)$, the household faces an instantaneous constraint on consumption and asset accumulation or investment $(Z(t))$ given by

$$C(t) + Z(t) = w(t)L(t) + r(t)K(t). \tag{3}$$

In other words, H balances its expenditures and incomes at each point of time. It cannot borrow against future to enhance current expenditure.[4] Equation (3) presumes that H supplies all labour and capital inelastically at each t. The household is supposed to be endowed with perfect foresight about the paths of $w(t)$ and $r(t)$ over time. It is standard practice to refer to

[4] H being a macro household, the micro households constituting it may nonetheless borrow from each other. These cancel out in the aggregate and give rise to (3).

$Z(t)$ as gross investment and distinguish it from net investment $\dot{K}(t)$. The latter is defined as

$$Z(t) - \delta K(t) = \dot{K}(t), \tag{4}$$

Thus, (3) is rewritten as

$$C(t) + \dot{K}(t) = w(t)L(t) + r(t)K(t) - \delta K(t). \tag{5}$$

A representative firm B has access to a technology for producing Y. This is represented by an aggregate production function

$$Y(t) = F(K(t), A(t)L(t)), \tag{6}$$

where $Y(t)$ stands for the flow of output at t and $K(t)$ and $L(t)$ for the flows of capital and labour services entering the production process at t. Note that the use of the same notations for capital stock and services as well as for population size and labour services implies that the stock-flow ratios for both factors are assumed to be constants (normalized to unity). The coefficient $A(t)$ of $L(t)$ represents (Harrod-neutral) technological progress and satisfies

Assumption T $\dot{A}(t)/A(t) = \mu > 0$.

In view of **Assumption T**, the convergence of (2) will generally call for a strengthening of **Assumption u3**.[5]

Henceforth, particular values attained by variables at a time point such as t_0 will be represented by c_{t_0}, K_{t_0} etc. Also, unless essential for the argument, the time index will be dropped to achieve notational simplicity. The function $F(.,.)$ satisfies the standard neoclassical properties, viz.

[5] See the discussion preceding Proposition 1 in Appendix 1 for further insights.

Assumption F1 $F(0, AL) = F(K, 0) = 0$ and F displays constant returns to scale in K and AL.

Assumption F2 $F_1 > 0, F_2 > 0, F_{11} < 0, F_{22} < 0,$ and F is strictly concave .

As per normal practice, it is useful to rewrite (6) in terms of quantities per unit of *effective labour* (i.e. AL). Thus, denoting Y/AL and K/AL by $\hat{y}$ and $\hat{k}$ respectively and using **Assumption F1**, we obtain

$$\hat{y} = F(\hat{k}, 1)) = f(\hat{k}), \quad f(0) = 0. \tag{7}$$

Assumption F2 implies

Property f1 f is a strictly concave function with $f'(\hat{k}) > 0$.

In addition to **Assumptions F1** and **F2**, we will also impose

Assumption f2 $f'(\hat{k}) \to \infty$ as $\hat{k} \to 0$ and $f'(\hat{k}) \to 0$ as $\hat{k} \to \infty$.

Assumptions u2 and **f2** are referred to as Inada conditions and constitute regularity requirements. They guarantee that the model has mathematically meaningful solutions.

The firm is supposed to treat all prices parametrically, thus implying that the identical firms it subsumes behave competitively in factor and product markets. Choosing Y as the numeraire, the wage rate and the rate of interest are equal to the marginal productivities of labour and capital. Thus,

$$r(t) = \frac{\partial F}{\partial K(t)}$$

$$
\begin{aligned}
&= f'(\hat{k}(t)), \\
w(t) &= \frac{\partial F}{\partial L_t} \\
&= A_t\{f(\hat{k}) - \hat{k}f'(\hat{k})\}. \qquad (8)
\end{aligned}
$$

Assumption F1 and equations (8) may be utilized to rewrite (5) as

$$C(t) + Z(t) = F(K(t), A_tL_t). \qquad (9)$$

This equation may be viewed simultaneously as the household's budget constraint and the transformation frontier between $C(t)$ and $Z(t)$ given $K(t)$. Deflating both sides by A_tL_t, (9) reduces to

$$\frac{c(t)}{A_t} + \hat{z}(t) = f(\hat{k}(t)), \qquad (10)$$

where $\hat{z}(t) = Z(t)/A_tL_t$. An alternative representation of (10) is

$$
\begin{aligned}
\frac{c(t)}{A_t} + \dot{\hat{k}}(t) + (\mu + n + \delta)\hat{k}(t) &= f(\hat{k}(t)), \\
\text{or,} \qquad c(t) + A_t\, \dot{\hat{k}}(t) &= A_t\{f(\hat{k}(t)) - (\mu + n + \delta)\hat{k}(t)\},
\end{aligned}
\qquad (11)
$$

where $\dot{\hat{k}}(t)$ stands for net investment per unit of effective labour and $(\mu + n + \delta)\ \hat{k}(t)$ represents the minimum level of investment per unit of effective labour, (i.e., z), that leaves $\hat{k}(t)$ unchanged.[6]

It is convenient at this stage to consider two cases depending on whether investment is reversible or irreversible.

[6]Since $\hat{k}$ is a constant,

Case 1: Investment is Irreversible. In this case, installed capital cannot be "eaten into" except through depreciation and the smallest possible value $\dot{K}$ can assume is $-\delta K$. At this corner value of net investment, gross investment Z equals zero. This means that $z = 0$ and $\dot{\hat{k}} = -(\mu + n + \delta)\hat{k}$. Consequently, from (11), the maximum possible per capita consumption is $c = Af(\hat{k})$.

Case 2: Investment is Reversible. Reversible investment permits direct capital consumption thus allowing $\dot{K} < -\delta K$ and $Z < 0$. Since δK stands for physical depreciation of capital, the maximum amount of the capital stock that can be directly consumed by the household is $(1 - \delta)K$. Hence, the maximum sustainable net disinvestment equals $\dot{K} = -[(1-\delta)K + \delta K] = -K$. Consequently, $Z = \hat{K} + \delta K = -(1 - \delta)K$. Alternatively, $\hat{z} = -(1 - \delta)\hat{k}$ and $\dot{\hat{k}} = -(1 + \mu + n)\hat{k}$. The corresponding maximum possible level of per capita consumption is $c = A(f(\hat{k}) + (1 - \delta)\hat{k})$. The transformation locus between c and $\dot{\hat{k}}$ is illustrated in Figure 1.

Figure 1 here.

$$\frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{K}}{K} - (\mu + n)$$

$$= 0,$$

or, $(\dot{K}/K) + \delta = (\mu + n) + \delta,$

or, $(\dot{K} + \delta K)/K = \mu + n + \delta,$

or, $\{(\dot{K} + \delta K)/(AL)\}\ (AL)/(K) = \mu + n + \delta,$

or,

$$\hat{z} = \frac{Z}{AL}$$

$$= (\mu + n + \delta)\ \frac{K}{AL}$$

$$= (\mu + n + \delta)\ \hat{k}.$$

Thus, $(\mu + n + \delta)\ \hat{k}$ stands for the gross investment per unit of AL required to maintain $\hat{k}$ constant.

The optimal value of c must necessarily be interior if investment is reversible. If not, suppose

$$c = A(f(\hat{k}) + (1-\delta)\hat{k}). \tag{12}$$

holds for some t. This means that all capital gets exhausted at t and the economy cannot produce positive output beyond t. Hence, $C = 0$ subsequent to t. Given **Assumption u2**, however, a small decrease in consumption at t accompanied by an increase at a later point of time must be welfare improving. Consequently, any optimal consumption path c must satisfy the condition

$$0 < c < A(f(\hat{k}) + (1-\delta)\hat{k}), \tag{13}$$

if investment is reversible.

For the case of *irreversible investment*, it is no longer possible to rule out c hitting the upper bound. At the maximum possible value of c, all output is consumed away, but this does not exhaust the capital stock. The economy bequeaths $(1-\delta)K$ to posterity. Hence, positive production as well as consumption is feasible at subsequent points of time. In other words, an optimal consumption path will satisfy the condition

$$0 < c \leq Af(\hat{k}). \tag{14}$$

Consider the optimum value of $c(t)$ for reversible investment. This being an interior point of the feasible set, a marginal reduction Δ in $\hat{k}$ raises c by $A\ \Delta$ units. (See Figure 1.) But for the wedge represented by A, it is as if $\hat{k}$ and c are not distinguishable as different *economic* goods. As opposed to this, (14) shows that the optimum for the irreversible case could occur at a point where $\hat{k}$ can no longer be sacrificed to yield extra c. Consequently, even in this one sector model, the two variables may have to be treated as distinct *economic* goods. This will show up subsequently in the pricing of c and $\hat{k}$.

The above discussion implies that the statement of the household's optimisation exercise will differ between the cases of reversible and irreversible investment. The two optimisation problems may be stated as follows:

Version 1. Optimisation under Reversible Investment.

Find $\{c^*(t)\}_0^\infty$ to maximize (2)

subject to (10) (alternatively, (11)), (13) and $\hat{k}(0) = \hat{k}_0$.

Version 2. Optimisation under Irreversible Investment.

Find $\{c^*(t)\}_0^\infty$ to maximize (2)

subject to (10) (alternatively, (11)), (14) and $\hat{k}(0) = \hat{k}_0$.

In standard terminology, $c(t)$ is referred to as a control variable and $\hat{k}(t)$ as a state variable.[7]

In what follows, the search for the optimal $c(t)$'s will be restricted to the class of piecewise continuous functions of t. Apiecewise continuous function is defined as follows:

Definition: The function $c : R_+ \rightarrow R$ is piecewise continuous if

(a) $c(\cdot)$ is continuous except over a finite set of points $\{a_1, \cdots, a_n\}$.

(b) At each a_i, lim $c(t)$ exists for $t \uparrow a_i$ as well as for $t \downarrow a_i$, but the two limits are unequal.

In what follows, piecewise continuous functions $c(t)$ satisfying (10) or (11) will be referred to as *feasible paths*.

[7] The constraints (13) and (14) can be alternatively stated as z *unrestricted* and $z \geq 0$. When the problem is stated in this manner, one would normally refer to z, rather than c, as the control variable.

3 Necessary Conditions for Optimum

At $t = 0$, the entire path $\{c(t)\}_0^\infty$ (leading to the associated path $\{\hat{k}(t)\}_{t>0}^\infty$) is the choice variable for H. To this extent, H is engaged in a dynamic exercise. However, this dynamic choice must be consistent with a static optimization exercise at each instant of time. To see this, note that at any t, say t_0, H has an inherited value of $\hat{k_{t_0}}$ from the past. This fixes the RHS of equation (11). Consequently, equation (11) describes the allocation possibilities between $c(t_0)$ and $\dot{\hat{k}}(t_0)$, given $\hat{k}_{t_0}$. An optimal allocation amounts to a static exercise at t_0.

Accordingly, we shall break up the analysis into two parts. The first part will be concerned with *static* optimality conditions, properties that must hold true for a given volume of output at t to be allocated optimally between consumption and investment. The second part will be concerned with *dynamic* conditions of optimal resource allocation across time, i.e., the way in which the optimal choice at a given point of time is linked to choices in the future.

3.1 Static Optimisation

The overall problem being dynamic in nature, even the static optimality conditions need to be derived with reference to a minimal set of dynamic considerations. In this context, we shall begin by developing Bellman's *Principle of Optimality* (Bellman (1957)), a famous mathematical principle underlying multi-stage decision problems.

3.1.1 The Principle of Optimality and the Functional Equation

Starting from any time point t_0, the best possible value of welfare achievable by H depends on $\hat{k}_{t_0}$. Notice that this is a deeper statement than might appear at first sight. If the planning horizon of H were finite, say T, then the best value of welfare would depend on $\hat{k}_{t_0}$ *as well as* t_0, since the residual time horizon shrinks with the passage of time (i.e., $T - t_0$ falls as t_0 rises). The infinite horizon problem does not involve this complication. At any value of t, the residual horizon continues to be infinitely long.

Let $V(\hat{k}_{t_0})$ stand for the optimum welfare starting from $\hat{k}_{t_0}$. The function is normally referred to as the *value function.* Consider the truncated problem

$$\begin{aligned} \text{Maximize} \quad & \int_{t_o}^{\infty} u(c(t))\, e^{-(\rho-n)(t-t_0)}\, dt \\ \text{subject to} \quad & (11),\ (13)\ (\text{alternatively, } (11),\ (14)) \\ \text{and} \quad & \hat{k}(t_0) = \hat{k}_{t_0}. \end{aligned} \tag{15}$$

If $\{\tilde{c}_t\}_{t_0}^{\infty}$ solves this problem, then

$$V(\hat{k}_{t_0}) = \int_{t_o}^{\infty} u(\tilde{c}_t)\, e^{-(\rho-n)(t-t_0)} dt.$$

Bellman's Principle of Optimaility says:

An optimal path has the property that whatever the initial conditions and control variables over some initial period, the control variables over the remaining period must be optimal for the remaining problem, with the state resulting from the early decisions considered as the initial condition.

Let $\{c_t^*\}_0^{\infty}$ solve Version 1 or Version 2 of our problem. Suppose, moreover, that it gives rise to the path $\hat{k}_t^*$. Then, according to the Principle of Optimality,

$$V(\hat{k}_{t_0}^*) = \int_{t_o}^{\infty} u(c_t^*)\, e^{-(\rho-n)(t-t_0)} dt.$$

Proof of the Principle of Optimality: Consider a small interval $0 \leq t \leq h,\ h > 0$. Denoting the truncated path $\{c(t)\}_0^h$ by $c_{0,h}$, it is clear that $\hat{k}_h$ is a function of $c_{0,h}$, given $\hat{k}_0$. Let $\hat{k}(h) = \phi(c_{0,h})$. Then, $V(\hat{k}(h)) = V(\phi(c_{0,h}))$. Suppose then that H chooses $c^*_{0,h}$ over the interval $[0, h]$, but that contrary to the Principle of Optimality, the aggregate utility from $\{c^*_t\}_h^\infty$ falls short of $V(\phi(c^*_{0,h}))$. If possible, let

$$\begin{aligned} V(\phi(c^*_{0,h})) &= \int_h^\infty u(\bar{c}_t) e^{-(\rho-n)\ (t-h)}\ dt \\ &> \int_h^\infty u(c^*_t) e^{-(\rho-n)\ (t-h)}\ dt, \end{aligned}$$

where $\bar{c}_t$ is feasible from $\phi(c^*_{0,h})$ and $\bar{c}_t \neq c^*_t$ except possibly over a set of zero measure. Define

$$c^{**}(t) = \begin{cases} c^*_t, & t \in [0, h] \\ \bar{c}_t & t \in (h, \infty). \end{cases}$$

Clearly, $c^{**}(t)$ is feasible and

$$\int_0^\infty u(c^{**}(t))\ e^{-(\rho-n)t}\ dt > \int_0^\infty u(c^*_t)\ e^{-(\rho-n)t}\ dt,$$

which contradicts the presumed optimality of $\{c^*_t\}_0^\infty$. ■

According to the Principle of Optimality then,

$$\begin{aligned} V(\hat{k}_0) &= \int_o^h u(c^*_t)\ e^{-(\rho-n)t} dt + V(\phi(c^*_{0,h})) \\ &\geq \int_o^h u(c_t)\ e^{-(\rho-n)t} dt + V(\phi(c_{0,h})), \end{aligned} \tag{16}$$

given any feasible path $\{c(t)\}_0^\infty$. Alternatively,

$$V(\hat{k}_0) = max_{c_{0,h}} \; \{\int_o^h \; u(c(t)) \; e^{-(\rho-n)t}dt + V(\phi(c_{0,h}))\}, \tag{17}$$

or, more generally,

$$V(\hat{k}_{t_0}) = max_{c_{t_0,t_{0+h}}} \; \{\int_{t_o}^{t_0+h} \; u(c(t)) \; e^{-(\rho-n)(t-t_0)}dt + V(\phi(c_{t_0,t_{0+h}}))\}, \tag{18}$$

where $max_{c_{a,b}}$ denotes maximization with respect to $c(t),\ a \leq t \leq b$. Equation (17) (alternatively (18)) is referred to as a *functional equation.* This completes our disussion of Bellman's Principle of Optimality.

3.1.2 Necessary Conditions for Static Optimality

In what follows, we shall proceed under

Assumption V $V(\hat{k})$ is continuously differentiable.

Assumption V allows us to make some approximations concerning the RHS of (18). First, for h small,

$$u(c(t)) \cong u(c(t_0)), \quad t_0 \leq t \leq t_0 + h.$$

If t_0 is a point of discontinuity, we choose c_{t_0} as the right hand limit of $c(t)$ at t_0.[8]

Therefore,

[8] Note that t_0 being a point of zero measure, replacing the optimal value of $c(t_0)$ by the right hand limit does not affect the value of the utility integral.

$$\begin{aligned}
\int_{t_o}^{t_0+h} u(c(t))\ e^{-(\rho-n)(t-t_0)}dt &\cong u(c(t_0))\int_{t_o}^{t_0+h} e^{-(\rho-n)(t-t_0)}dt \\
&= u(c(t_0))[-\frac{e^{-(\rho-n)(t-t_0)}}{\rho-n}]_{t_0}^{t_0+h} \\
&= u(c(t_0))[-\frac{e^{-(\rho-n)h}}{\rho-n}+\frac{1}{\rho-n}] \\
&= u(c(t_0))[\frac{1}{\rho-n}\{1-e^{-(\rho-n)h}\}] \\
&\cong u(c(t_0))[\frac{1}{\rho-n}\{1-(1-(\rho-n)h)\}] \\
&\quad \text{by Taylor's approximation ,} \\
&= u(c(t_0))h.
\end{aligned}$$

Thus, (18) can be written as

$$V(\hat{k}_{t_0}) \cong max_{(c_{t_0,t_0+h})}\ \{h\ u(c(t_0)) + V(\hat{k}(t_0+h))\}, \tag{19}$$

where $\hat{k}(t_0+h)$ results from the choice of c_{t_0,t_0+h}. A necessary condition for this optimum is

$$\frac{h\ \partial u(c(t_0))}{\partial c(t_0)} + \frac{\partial V(\hat{k}(t_0+h))}{\partial c(t_0)} \geq 0. \tag{20}$$

The *inequality* is explained by the fact that under irreversible investment, the optimum value of $c(t_0)$ might hit its upper bound given by (14). This being a corner solution, the partial derivative may turn out to be strictly positive.

Next, note that

$$\frac{\partial V(\hat{k}(t_0+h))}{\partial c(t_0)} = \frac{\partial V(\hat{k}(t_0+h))}{\partial \hat{k}(t_0+h)}.\frac{\partial \hat{k}(t_0+h)}{\partial c(t_0)}.$$

Linearizing again

$$\hat{k}(t_0 + h) \cong \hat{k}(t_0) + h\ \dot{\hat{k}}(t_0),$$

where, according to (11),

$$\begin{aligned}\dot{\hat{k}}(t_0) &= \dot{\hat{k}}(t_0)(c(t_0)) \\ &= f(\hat{k}(t_0)) - (\mu + n + \delta)\hat{k}(t_0) - \frac{c(t_0)}{A_{t_0}}.\end{aligned}$$

Thus,

$$\frac{\partial \hat{k}(t_0 + h)}{\partial c(t_0)} \cong -\frac{h}{A_{t_0}}.$$

Denote $\partial V(\hat{k}_t^*)/\partial \hat{k}(t)$ by q_t^*. The variable $q(t)$ stands for the maximum possible change in the social welfare from t onwards on account of a marginal change in $\hat{k}(t)$. In other words, it is the marginal value or shadow price of $\hat{k}$ at t along the optimal path. The assumption that V is differentiable implies that at any given value of $\hat{k}(t)$, the value of $q(t)$ is uniquely defined. Using these facts, the optimality of $\{c_t^*\}_{t_0}^{\infty}$ and the definition of q_t^*, (20) reduces to

$$\begin{aligned}\frac{h\ \partial u(c_{t_0}^*)}{\partial c(t_0)} - q_{t_0+h}^*\ \frac{h}{A_{t_0}} &= \frac{h\ \partial u(c_{t_0}^*)}{\partial c(t_0)} - \frac{h\ q_{t_0}^*}{A_{t_0}} - \frac{h\ (q_{t_0+h}^* - q_{t_0}^*)}{A_{t_0}} \\ &\geq 0,\end{aligned}$$

or,

$$\frac{\partial u(c_{t_0}^*)}{\partial c(t_0)} - \frac{q_{t_0}^*}{A_{t_0}} - \frac{(q_{t_0+h}^* - q_{t_0}^*)}{A_{t_0}} \geq 0.$$

Allowing $h \to 0$, replacing t_0 by t and using **Assumption V**, we see that for $\{c_t^*\}_0^\infty$ to be optimal,

$$\frac{\partial u(c_t^*)}{\partial c(t)} - \frac{q_t^*}{A_t} \geq 0, \text{ with equality if } c_t^* \text{ is interior }, \tag{21}$$

$$\text{and} \quad \left(\frac{\partial u(c_t^*)}{\partial c(t)} - \frac{q_t^*}{A_t}\right) z^*(t) = 0 \tag{22}$$

must hold for all t.

3.1.3 The Maximal Principle

Instead of moving from (19) to (20) as we did in the last section, we could have followed an alternative route. To see this, note that

$$\begin{aligned} V(\hat{k}(t_0+h)) &\cong V(\hat{k}(t_0) + h\ \dot{\hat{k}}(t_0)), \\ &\quad \text{(linearizing } \hat{k}(t_0+h)) \text{ }, \\ &\cong V(\hat{k}(t_0)) + h\ \dot{\hat{k}}(t_0))\ V'(\hat{k}(t_0), \\ &\quad \text{(linearizing again) }. \end{aligned} \tag{23}$$

Thus, (19) reduces to

$$V(\hat{k}_{t_0}) \cong max_{(c_{t_0, t_0+h})}\ \{h\ (u(c(t_0)) + V'(\hat{k}(t_0))\ \dot{\hat{k}}(t_0))\} + V(\hat{k}(t_0)). \tag{24}$$

Cancelling out $V(\hat{k}_{t_0})$ from both sides,

$$h\ max_{(c_{t_0,t_0+h})}\ \{(u(c(t_0)) + V'(\hat{k}(t_0))\ \dot{\hat{k}}(t_0))\} \cong 0,$$

which, despite its trivial nature, seems to indicate a relationship between the optimisation result discussed in the last section and the maximum of the function $\{(u(c(t_0)) + V'(\hat{k}(t_0))\ \dot{\hat{k}}(t_0))\}$ at t.[9] Accordingly, we proceed to construct an auxiliary function, usually referred to as the Hamiltonian, as follows:

$$\mathcal{H}(c(t), \hat{k}(t), q(t)) = u(c(t)) + q(t)\dot{\hat{k}}(t), \tag{25}$$

where the arguments of $\mathcal{H}$ follow from (11).

Recalling that $q(t)$ stands for the shadow price of a unit of $\hat{k}(t)$, the Hamiltonian $\mathcal{H}$ can be interpreted approximately as the value in utils imputed to per capita net national product at t.[10] For given values of $q(t)$ and $\hat{k}(t)$, we may treat $\mathcal{H}$ as a static welfare function at t, defined over $c(t)$ and $\dot{\hat{k}}(t)$ alone. To establish the shape of a level curve generated by $\mathcal{H}$ in the $(c(t), \dot{\hat{k}}(t))$ plane, differentiate totally to get

$$\frac{d\ c(t)}{d\ \dot{\hat{k}}(t)} = -\frac{q(t)}{u'(c(t))} < 0$$

and

$$\frac{d^2\ c(t)}{d\ \dot{\hat{k}}(t)^2} = q(t)\ \frac{1}{u'(c(t))^2}\ u''(c(t))\ \frac{d\ c(t)}{d\ \dot{\hat{k}}(t)} > 0.$$

[9] Note that h being small, $h\ \ \{(u(c(t_0)) + V'(\hat{k}(t_0))\ \dot{\hat{k}}(t_0))\} \cong 0$ whatever may be the value of $max_{(c_{t_0,t_0+h})}\ \{(u(c(t_0)) + V'(\hat{k}(t_0))\ \dot{\hat{k}}(t_0))\}$.

[10] The qualification "approximate" is needed since $\hat{k}$ is investment per unit of effective labour AL, rather than per capita investment.

Thus, the level curves corresponding to $\mathcal{H}$ are downward falling and strictly convex to the origin.

Given these preliminaries, we can restate (21) in terms of the Hamiltonian. Consider first the reversible investment case. We shall argue that in this case, c_t^* maximizes $\mathcal{H}$ subject to (11), $\hat{k}(t) = \hat{k}_t^*$ and $q(t) = q_t^*$ for each t. Moreover, the *FOC* characterising such a solution is identically the same as the equality version of (21). To see this, use (11) to get

$$\dot{\hat{k}}(t) = \{f(\hat{k}^*(t)) - (\mu + n + \delta)\hat{k}^*(t)\} - \frac{c(t)}{A_t}.$$

Substituting in (25), $\mathcal{H}$ reduces to

$$\mathcal{H}(c(t), \hat{k}^*(t), q^*(t)) = u(c(t)) + q^*(t)[\{f(\hat{k}^*(t)) - (\mu + n + \delta)\hat{k}^*(t)\} - \frac{c(t)}{A_t}],$$

which is a function of $c(t)$ alone. Differentiating $\mathcal{H}$ with respect to $c(t)$, we obtain $\partial u(c(t))/\partial c(t) - q^*(t)/A_t$. By assumption, $\exists$ a value of c_t^* satisfying (11) and $\hat{k}(t) = \hat{k}_t^*$ such that

$$\frac{\partial u(c_t^*)}{\partial c(t)} - \frac{q_t^*}{A_t} = 0.$$

The shape of the level curves of $\mathcal{H}$ tell us further that c_t^* is an unique solution to the problem

$$\text{Maximize } \mathcal{H}(c(t), \hat{k}_t^*, q_t^*) \text{ subject to (11).} \tag{26}$$

Next, consider a corner solution corresponding to irreversible investment. To relate it to the Hamiltonian, let us reformulate the relevant constraints in the Kuhn-Tucker form. Rewrite (11) as the inequality constraint

$$A_t\{f(\hat{k}(t)) - (\mu + n + \delta)\hat{k}(t)\} - c(t) - A_t\ \dot{\hat{k}}(t) \geq 0 \tag{27}$$

Similarly, note that

$$(\mu + n + \delta)\hat{k}(t) + \dot{\hat{k}}(t) \geq 0 \tag{28}$$

must hold. The inequality (21) may now be viewed as the *FOC* satisfying a corner solution to the problem

$$\text{Maximize } \mathcal{H}(c(t), \hat{k}^*(t), q_t^*) \text{ subject to (27) and (28) .} \tag{29}$$

Figure 1 shows that at the corner solution, both constraints are binding. The gradients to these constraints at the optimum point are $(-1, -A_t)$ and $(0, 1)$ respectively and the gradient to the objective function is $(u'(c_t^*), q_t^*)$. As per the Kuhn-Tucker conditions then, ∃ nonnegative Lagrange multipliers λ_1 and λ_2 such that

$$(-u'(c_t^*), -q_t^*) = \lambda_1\ (-1, -A_t) + \lambda_2\ (0, 1).$$

Moreover, it is easy to read from Figure 1 that (21) must hold as a strict inequality.

Collecting the observations of this section, it follows that an optimal path $\{c_t^*, \hat{k}_t^*\}_0^\infty$ has associated with it a path of $\{\hat{q}^*(t)\}_0^\infty$ such that the corresponding imputed value of per capita net national product is maximized with respect to $c(t)$ at each point of time. In view of this implication, (26) (alternatively (29)) is often referred to as the *Maximal Principle*.

3.2 Dynamic Optimality

The discussion of static optimality was concerned with the allocation of output between consumption and investment at any given point of time. This led to a relationship between the marginal utility from per capita consumption and the shadow price of $\hat{k}$ at each t. As opposed to this, dynamic optimality links the shadow price of $\hat{k}$ at any t with the shadow prices at subsequent t's. As we shall see, finding this link is tantamount to stating a differential equation describing the evolution of q_t^* over time. In order to describe that equation, we shall use the principle that a small perturbation around the optimal path $\{c_t^*, \hat{k}_t^*\}_0^\infty$ leaves the aggregate utility of the household unchanged. In particular, we shall consider the following perturbation:

(a) at t_0, consumption is lowered and investment increased marginally so as to raise $\hat{k}_{t_0}^*$ to $\hat{k}_{t_0}'$, where $\hat{k}_{t_0}' - \hat{k}_{t_0}^* = \Delta$;

(b) $\hat{k}_s' = \hat{k}_s^* + \Delta \;\forall\; s > t_0$, or, as shown in Figure 2, $\hat{k}_s'$ is merely a parallel upward shift in $\hat{k}^*(s)$ for $s > t_0$;

(c) $\forall\; s > t_0$, the extra *per capita* output realized by the higher $\hat{k}_s'$ *after maintaining the additional* Δ *for all time* is consumed away.[11]

Figure 2 here.

The definition of $q(\cdot)$ implies that the price of a unit of $\hat{k}(t_0)$ in units of $c(t_0)$ is $q_{t_0}^*/u'(c_{t_0}^*)$ along the optimal path. Thus, the sacrifice of $c(t_0)$ required to raise $\hat{k}_{t_0}^*$ by Δ equals $(q_{t_0}^*/u'(c_{t_0}^*))\,\Delta$. This entails a loss of *utility* equal to $u'(c^*(t_0)) \times \Delta\ (q_{t_0}^*/u'(c_{t_0}^*) = \Delta\ q_{t_0}^*$.

Let us now compute the extra utility provided by the new path $\forall\; s > t_0$. The extra per capita output brought forth by Δ at s equals $A_s f'(\hat{k}_s^*)\,\Delta$. This extra output is partly invested to maintain $\hat{k}(s)$ at the higher level. In units of z, the required investment is $(\mu + n + \delta)\,\Delta$, which equals $(q_s^*/u'(c_s^*))\,(\mu + n + \delta)\,\Delta$ in units of c. The extra per capita consumption permitted by the extra output after subtracting the investment is $A_s f'(\hat{k}_s^*)\,\Delta - (q_s^*/u'(c_s^*))\,(\mu +$

[11]The construction of the perturbed path follows Solow (2000).

$n + \delta)\ \Delta$. Multiplying out by $u'(c_s^*)$, the extra utility from the extra consumption at each s is given by $[A_s\ u'(c_s^*)\ f'(\hat{k}(s)) - q_s^*\ (\mu + n + \delta)]\ \Delta$. Thus, the total discounted gain in utility at t_0 from the perturbation equals $\Delta\ \int_{t_0}^{\infty} e^{-(\rho-n)(s-t_0)}\{A_s u'(c_s^*) f'(\hat{k}_s^*) - q_s^*\ (\mu + n + \delta)\}ds$.

Optimality, as noted, requires that the gain and the loss be equal. Hence,

$$\Delta\ q_{t_0}^* = \Delta\ \int_{t_0}^{\infty} e^{-(\rho-n)(s-t_0)}\{A_s u'(c_s^*)\ f'(\hat{k}_s^*) - q_s^*\ (\mu + n + \delta)\}ds,$$

or,

$$q_{t_0}^* = \int_{t_0}^{\infty} e^{-(\rho-n)(s-t_0)}\{A_s u'(c_s^*)\ f'(\hat{k}_s^*) - q_s^*\ (\mu + n + \delta)\}ds. \qquad (30)$$

Replacing t_0 by t for notational ease, consistency between (21) and (30) implies

$$q_t^* \geq \int_{t}^{\infty} e^{-(\rho-n)(s-t)} q_s^*\ \{f'(\hat{k}_s^*) - (\mu + n + \delta)\}ds. \qquad (31)$$

For q_t^* to be well-defined, the integral on the RHS must exist for each t. We shall demonstrate in Appendix 1 (Proposition 1) that the optimality of $\{c_t^*, \hat{k}_t^*\}_0^{\infty}$ implies $f'(\hat{k}_s^*) - (\mu + n + \delta)$ is bounded strictly away from zero for s sufficiently large.[12] Anticipating this result, the integral can exist $\forall\ t$ only if

$$e^{-(\rho-n)t} q(t)^* \to 0 \text{ as } t \to \infty. \qquad (32)$$

Equation (32) is called the transversality condition and constitutes a restriction on an optimal path. Intuitively, (32) implies that efficient paths

[12] The result follows directly from Proposition (J) of Koopmans (1965). For the sake of completeness, however, Appendix 1 constructs an independent proof.

view capital stocks far out in the future to be increasingly useless relative to present stocks.[13]

The balance between cost and benefits captured by equation (30) can be expressed in another manner. This is done by differentiating[14] (30) with respect to t to get

$$\begin{aligned} \dot{q}_t^* &= -\{A_t u'(c_t^*) f'(\hat{k}_t^*) - q_t^*(\mu + n + \delta)\} \\ &\quad +(\rho - n) \int_t^\infty e^{-(\rho-n)(s-t)} \{A_s u'(c_s^*) f'(\hat{k}_s^*) \\ &\quad -q_s^*(\mu + n + \delta)\} ds. \end{aligned}$$

Using (30), the last equation reduces to

$$\dot{q}_t^* = -\{A_t u'(c_t^*) f'(\hat{k}_t^*) - q_t^*(\mu + n + \delta)\} + (\rho - n) q_t^*. \tag{33}$$

In order to interpret (33), consider a different scenario where p is the money price of a unit of the commodity that acts both as a consumption and a

[13]It is worth noting here that the transversality condition follows from the fact that $f'(\hat{k}_s^*) - (\mu + n + \delta)$ is bounded strictly away from zero, rather than the other way round. Certain sources liken the transversality condition to a no-Ponzi game restriction. The latter is a vacuous restriction for the present model, since it assumes a single macro household which can neither be a net borrower nor a net lender within the household sector. See footnote 3 above. In any case, as we have demonstrated, optimality implies this condition. It is *not* an exogenous stipulation on the model. As we shall see in Appendix 2, Halkin's counter-example to the necessity of the transversality condition depends crucially on the fact that the objective function of his problem assigns a disproportionately high weight on capital in the distant future.

[14]The formula for differentiating a definite integral of the form

$$K(x) = \int_a^{b(x)} F(t, x) dt$$

is

$$\frac{dK(x)}{dx} = \int_a^{b(x)} F_x(t, x) dt + F(b(x), x) b'(x).$$

See Chiang (1992), (2.11), p.31.

capital good.[15] Suppose further that there exists, alongside the capital good, an alternative monetary asset, a long term bond, yielding a nominal rate of interest $i(t)$ for each t. An infinitely lived agent is engaged in evaluating a chosen path of capital accumulation $\{k(t)\}_0^\infty$. When p units of money are invested in a unit of the capital good at time t, the marginal product is $f'(k(s)) \ \ \forall \ s \geq t$, assuming as before that the agent maintains the extra unit of capital for all $s \geq t$ and consumes any residual output brought forth by the extra capital. Then, the agent's return from maintaining an extra unit of k forever from s onwards is $f'(k(s)) \ \ \forall \ s \geq t$. In nominal terms, the return equals $p(s)f'(k(s))$ at each s. For the agent to be indifferent between investing in the physical capital and investing in the bond, the two investments must yield the same rate of return per instant of time. The rate of return from the physical capital investment, $r(s)$, is given by

$$p(t) = \int_t^\infty p(s)f'(k(s))e^{-\int_t^s r(x)dx}ds.$$

If the two rates of return are equal, then $r(s) = i(s) \ \ \forall \ \ s$. Hence,

$$p(t) = \int_t^\infty p(s)f'(k(s))e^{-\int_t^s i(x)dx}ds. \tag{34}$$

Differentiation of (34) with respect to t gives

$$\dot{p}(t) = -p(t)f'(k(t)) + i(t)p(t),$$

or,

$$p(t)f'(k(t)) + \dot{p}(t) = i(t)p(t). \tag{35}$$

The first term on the LHS of (35) stands for the value of the instantaneous marginal product of investment in k in nominal terms, while the second

[15] This interpretation is based on Solow (1956). Note that under reversible investment, (33) reduces to the simpler condition $\dot{q}_t^* = -q_t^*\{f'(\hat{k}_t^*) - (\mu + \rho + \delta)\}$.

term represents capital gain (or loss) on account of price change. The LHS then gives the *net* instantaneous nominal return from investing in k. The RHS, on the other hand, is the nominal return at t from holding the bond. The equality implies that the agent is indifferent between the two ways of investing.[16]

The logic underlying (35) may be applied to (33), which we rewrite as

$$\{A_t u'(c_t^*) f'(\hat{k}_t^*) - q_t^*(\mu + n + \delta)\} + \dot{q}_t^* = (\rho - n)\, q_t^*. \tag{36}$$

The LHS now gives the instantaneous *net* return in utils of a unit of investment in $\hat{k}(t)$, where $\dot{q}_t^*$ is the capital gain or loss measured in utils. On the RHS, the term $\rho - n$ is the rate at which utils *ought to grow* in the household's judgement. It is the counterpart of $i(t)$ in (35). This rate applied to the shadow price of capital yields the instantaneous return that H finds acceptable, the imputed opportunity cost of investment in physical capital. Hence, (36) says that the rate of return from investment along the optimal path equals the household's minimum acceptable return. When (33) or (36) holds therefore, H has no incentive to divert away from the chosen path of asset accumulation.

The necessary conditions for static and dynamic optimality are then captured by (11), (22), (36) and (32). For ease of reference, we renumber and rewrite these below as

PROPOSITION **1** *If* $\{c_t^*, \hat{k}_t^*\}_0^\infty$ *is optimal, then there exists a path of co-state variables* $\{q_t^*\}_0^\infty$ *such that*

$$u'(c_t^*) \geq \frac{q_t^*}{A_t}, \text{ with equality for an interior } c_t^*; \tag{37}$$

$$\dot{\hat{k}}_t^* = f(\hat{k}_t^*) - \frac{c_t^*}{A_t} - (\mu + n + \delta)\hat{k}_t^*; \tag{38}$$

[16] A more common way of writing (35) is $f'(k(t)) + \dot{p}(t)/p(t) = i(t)$, commonly called the Fisher equation, after Irving Fisher.

$$\begin{aligned}
\dot{q}_t^* &= -\{A_t u'(c_t^*) f'(\hat{k}_t^*) - q_t^*(\mu + n + \delta)\} \\
&\quad +(\rho - n) q_t^* \\
&= -A_t u'(c_t^*) f'(\hat{k}_t^*) + (\mu + \rho + \delta) q_t^*; \qquad (39)
\end{aligned}$$

$$\text{and} \qquad e^{-(\rho-n)t}\ q_t^* \to 0\ \ as\ t \to \infty. \qquad (40)$$

3.2.1 The Hamiltonian Function Again

The link between (37) of Proposition 1 and the Hamiltonian was already indicated by (26) and (29). The remaining parts of this proposition can also be stated in terms of the same Hamiltonian function. Equation (38) is derived by maximizing $\mathcal{H}$ with respect to $q(t)$:

$$\frac{\partial \mathcal{H}(c_t^*, \hat{k}_t^*, q(t))}{\partial q(t))} = \dot{\hat{k}}^*(t). \qquad (41)$$

We may also claim that

$$\begin{aligned}
\dot{q}_t^* &= \frac{d(\int_t^\infty (\partial \mathcal{H}(c_t^*, \hat{k}_t^*, q_t^*)/\partial \hat{k}(s))\ e^{-(\rho-n)(s-t)} ds)}{dt} \\
&= -A_t u'(c_t^*) f'(\hat{k}_t^*) + (\mu + \rho + \delta) q_t^*. \qquad (42)
\end{aligned}$$

establishes the connection between $\mathcal{H}$ and (39). The derivation of equation (42) is based on the perturbation discussed in subsection 3.2. Property (b) of the comparison path $\hat{k}_t'$ implies that $d\dot{\hat{k}}^*(s)/d\hat{k}(s) = 0\ \forall\ s > t$. Using (11) and (14),

$$\begin{aligned}
\mathcal{H} &= u(Af(\hat{k})) - q(\mu + n + \delta)\hat{k},\ \text{when } c \text{ has a corner solution },\\
&= u(A\{f(\hat{k}) - (\mu + n + \delta)\hat{k} - \dot{\hat{k}}\}) + q\dot{\hat{k}},\ \text{otherwise}.
\end{aligned}$$

Differentiating $\mathcal{H}$ with respect to $\hat{k}$, holding $\dot{\hat{k}}(t)$ fixed, and utilizing (37), we see that

$$\frac{\partial \mathcal{H}(c_s^*, \hat{k}_s^*, q_s^*)}{\partial \hat{k}(s)} = u'(c_s^*)\ A_s\ f'(\hat{k}_s^*) - q_s^*(\mu + n + \delta), \tag{43}$$

irrespective of whether c_s^* has an interior or boundary value. Thus, referring back to Section 3.2, the derivative in (43) stands for the increment in utility at each point of time along the perturbed path $\{\hat{k}_t'\}$. Using (30) now, $q_t^* = \int_t^\infty (\partial \mathcal{H}/\partial \hat{k}(s))\ e^{-(\rho-n)(s-t)} ds$. Consequently, (42) is a restatement of (33) or (36). Let us collect the necessary conditions stated in terms of the Hamiltonian function as[17]

PROPOSITION **2** *Suppose $\{c_t^*, \hat{k}_t^*\}_0^\infty$ solves Version 1 or Version 2 of the problem. Then, there exists a path of co-state variables q_t^* such that (26) (alternatively (29)), (41), (42) and (40) are satisfied.*

4 Sufficient Conditions for an Optimum

We proceed now to prove that under **Assumptions u1** and **f1**, any path $\{c_t^*, \hat{k}_t^*, q_t^*\}_0^\infty$ satisfying (37)-(40) constitutes a unique solution to the problems stated as Version 1 and Version 2 above. Assume then that $\{c(t), \hat{k}(t)\}_0^\infty$ ia any feasible path. Then, equation (10) gives

$$A(f(\hat{k}) - \hat{z}) - c = 0, \tag{44}$$

[17]The conditions resemble standard representations of the first three necessary conditions, except for (42). Equation (42) is borrowed from Cass (1965, 1966). The advantage of choosing the form (42) is that it makes direct reference to the economic interpretation of a co-state variable. Moreover, it uses a single differential equation to describe the evolution of the co-state variable for both reversible and irreversible investment.

where the time index t has been dropped for convenience. In what follows, we shall also use the fact that u1 and f1 imply

$$\begin{aligned} u(c^*) - u(c) - u'(c^*)(c^* - c) &> 0 \\ f(\hat{k}^*) - f(\hat{k}) - f'(\hat{k}^*)(\hat{k}^* - \hat{k}) &> 0. \end{aligned} \tag{45}$$

Our claim is established if we can show that

$$\begin{aligned} D &= \int_0^\infty \{u(c^*) - u(c)\}\, e^{-(\rho-n)t} dt \\ &> 0. \end{aligned}$$

By adding and subtracting terms, we may use (44) and the identity $\hat{z} = \dot{\hat{k}} + (\mu + n + \delta)\hat{k}$ to write

$$\begin{aligned} D = \int_0^\infty & [\{u(c^*) - u(c)\} + u'(c^*)\{(A(f(\hat{k}^*) - \hat{z}^*) - c^*) - (A(f(\hat{k}) - \hat{z}) - c)\} \\ & q^*\{(\hat{z}^* - \lambda\, \hat{k}^* - \dot{\hat{k}}^*) - ((\hat{z} - \lambda\, \hat{k} - \dot{\hat{k}})\}]\, e^{-(\rho-n)t}\, dt, \end{aligned}$$

where $\lambda = \mu + n + \delta$. Collecting terms,

$$\begin{aligned} D = \int_0^\infty & [\{u(c^*) - u(c) - u'(c^*)(c^* - c)\} + \{q^*(\hat{z}^* - \hat{z}) \\ & - Au'(c^*)(\hat{z}^* - \hat{z})\} - q^*\{\lambda(\hat{k}^* - \hat{k}) + (\dot{\hat{k}}^* - \dot{\hat{k}})\} \\ & + Au'(c^*)\{f(\hat{k}^*) - f(\hat{k})\}]\, e^{-(\rho-n)t} dt, \end{aligned} \tag{46}$$

Equation (46) may be reduced further by integrating $\int_0^\infty q^*(\dot{\hat{k}}^* - \dot{\hat{k}})e^{-(\rho-n)t}dt$ by parts. Thus,

$$\begin{aligned}\int_0^\infty q^*(\dot{\hat{k}}^* - \dot{\hat{k}})e^{-(\rho-n)t}dt &= e^{-(\rho-n)t}q^*(\hat{k}^* - \hat{k}) \mid_0^\infty \\ &\quad - \int_0^\infty (\hat{k}^* - \hat{k})\{\dot{q}^* e^{-(\rho-n)t} - (\rho-n)q^* e^{-(\rho-n)t}\}dt.\end{aligned}$$

We shall demonstrate in Appendix 2 that any feasible path $\{\hat{k}\}$ is bounded above. Assuming this result for the time being and using (32), the last equation reduces to

$$\int_0^\infty q^*(\dot{\hat{k}}^* - \dot{\hat{k}})e^{-(\rho-n)t}dt = -\int_0^\infty (\hat{k}^* - \hat{k})\{\dot{q}^* e^{-(\rho-n)t} - (\rho-n)q^* e^{-(\rho-n)t}\}dt. \tag{47}$$

Plugging (47) into (46), we get

$$\begin{aligned} D &= \int_0^\infty [\{u(c^*) - u(c) - u'(c^*)(c^* - c)\} + \{q^* - Au'(c^*)\}(\hat{z}^* - \hat{z}) \\ &\quad - q^*\lambda(\hat{k}^* - \hat{k}) + (\dot{q}^* - (\rho - n)q^*)(\hat{k}^* - \hat{k}) \\ &\quad + Au'(c^*)\{f(\hat{k}^*) - f(\hat{k})\}]\ e^{-(\rho-n)t}dt \\ &= \int_0^\infty [\{u(c^*) - u(c) - u'(c^*)(c^* - c)\} + \{q^* - Au'(c^*)\}(\hat{z}^* - \hat{z}) \\ &\quad + (\hat{k}^* - \hat{k})\{\dot{q}^* - (\rho + \lambda - n)q^* + Au'(c^*)f'(\hat{k}^*)\} \\ &\quad + Au'(c^*)\{f(\hat{k}^*) - f(\hat{k}) - f'(\hat{k}^*)(\hat{k}^* - \hat{k})\}]\ e^{-(\rho-n)t}, \end{aligned} \tag{48}$$

adding and subtracting $Au'(c^*)f'(\hat{k}^*)(\hat{k}^* - \hat{k})$. Note that $(q^* - Au'(c^*))\hat{z} = 0$ according to (22). Further, $(q^* - Au'(c^*))\hat{z} = 0$ for reversible investment.

In case of irreversible investment, $q^* - Au'(c^*) \leq 0$ and $\hat{z} \geq 0$. Hence, $(q^* - Au'(c^*))(\hat{z}^* - \hat{z}) = 0$ in all cases. Appealing to this fact along with (45), the definition of λ and (39), equation (48) implies

$$
\begin{aligned}
D &> \int_0^\infty [(\hat{k}^* - \hat{k})\{\dot{q}^* - (\rho + \lambda - n)q^* \\
&\quad + Au'(c^*)f'(\hat{k}^*)\}]\ e^{-(\rho - n)t} \\
&= \int_0^\infty [(\hat{k}^* - \hat{k})\{\dot{q}^* - (\rho + \delta + \mu)q^* \\
&\quad + Au'(c^*)f'(\hat{k}^*)\}]\ e^{-(\rho - n)t} \\
&= \int_0^\infty (\hat{k}^* - \hat{k})\{\dot{q}^* - \dot{q}^*\} \\
&= 0.
\end{aligned}
$$

This establishes that $\{c_t^*, \hat{k}_t^*\}_0^\infty$ is a unique optimum path. We may note in passing that the inequality in (46) will be weak if either u or f is weakly concave. Thus, we have proved the following result[18]:

Proposition **3** *The conditions enumerated in Proposition 2, along with the strict concavity of u and f, are sufficient for the existence of a unique solution to Version 1 or Version 2 of the problem.*

Appendix 1

A steady state path of capital accumulation is defined to be one along which Y/AL, $\hat{k}$ and $\hat{c} = c/A$ are constants. It follows from (11) that along any steady state path

[18] Mangasarian (1966) proved the corresponding result for the finite horizon problem.

$$\hat{c} = f(\hat{k}) - \lambda \hat{k}, \tag{A.1}$$

where $\lambda = \mu + n + \delta$ as before. The value of $\hat{k}$ maximizing $\hat{c}$ in steady state is referred to as the Golden Rule (GR) value of the effective capital-labour ratio. It is the solution $\hat{k}^*$ to the equation

$$f'(\hat{k}) = \lambda. \tag{A.2}$$

The corresponding value of $\hat{c}$ satisfying (A.1) is denoted by $\hat{c}^*$. The per capita consumption corresponding to $\hat{c}^*$ is

$$c_t^* = A_t\ \hat{c}^* = A_t\ (f(\hat{k}^*) - \lambda \hat{k}^*)$$

The path of c_t^* will be referred to as the GR path of per capita consumption. Given any steady state pair, $(\hat{c}, \hat{k}) \neq (\hat{c}^*, \hat{k}^*)$, it follows from definition that $c_t^* > c(t)\ \forall\ t$. The welfare associated with any steady state path is $\int_0^\infty u(A_t\ \hat{c})\ e^{-(\rho-n)t} dt$. As already noted in connection with **Assumption T** in Section 2, , the integral may not exist unless the assumptions on the model are strengthened further. For example, we may assume that the instantaneous utility function has the form $u(c) = (c^{1-\theta} - 1)/(1-\theta),\ \theta > 0$. Such a utility function has a constant elasticity of marginal utility from consumption ($= \theta$). Alternatively, it admits a constant elasticity of substitution between consumption at different points of time. When u has this form, convergence of $\int_0^\infty u(A_t\ \hat{c}(t))\ e^{-(\rho-n)t} dt$ is guaranteed if $\rho - n > (1-\theta)\ \mu$.[19] We are now ready to prove the following

PROPOSITION **A.1** *For any optimal path* $\{c_t^*, \hat{k}_t^*\}_0^\infty$, $\exists$ *a* t_0 *such that* $f'(\hat{k}_t^*) - \lambda$ *is bounded strictly away from zero* $\forall\ t > t_0$. *In other words,*

[19]This condition turns out to be problematic for endogenous growth theory, since μ is not exogenously given. It is frequently encountered in the literature, however. See, for example, Barro (1990). Dasgupta (2001) discusses further details.

the optimal path of capital accumulation stays away from the GR in the long run.[20]

Proof: The result will be derived in two steps. The first will demonstrate that the GR per capita consumption path $\{A_t\hat{c}^*\}$ associated with indefinite maintenance of $\hat{k}^*$ is a suboptimal policy. The second step will then show that a path for which $f'(\hat{k}^*_t) \to \lambda$ is suboptimal.

<u>Step 1</u>.

Equation (A.1) implies that along the GR path

$$\hat{c}^* + \lambda\hat{k}^* = f(\hat{k}^*), \tag{A.3}$$

As an alternative to the path $\{A_t\hat{c}^*, \hat{k}^*\}$, consider a path which raises per capita consumption at $t = 0$ above $A_0\ \hat{c}^*$ by reducing $\hat{k}^*$ to $\hat{k}' = \hat{k}^* - \Delta$. It is possible to achieve this by reducing $\hat{z}$ below $\lambda\ \hat{k}^*$.[21] Thus, we have

$$\begin{aligned} f(\hat{k}^*) &= \hat{c}^* + \lambda\hat{k}' + \lambda(\hat{k}^* - \hat{k}') \\ &= \hat{c}^* + \lambda\Delta + \lambda\hat{k}'. \end{aligned}$$

Thus, the change in $\hat{c}^*$ is $\lambda\Delta$ and the rise in per capita consumption at $t = 0$ is $A_0\ \lambda\ \Delta$.

The alternative path is constructed to maintain $\hat{k}$ at this constant value $\hat{k}'\ \forall\ t > 0$. Per capita consumption for all $t > 0$ is $A_t\hat{c}'$ along the alternative

[20] Proposition (J) in Koopmans (1965) proves the stronger statement that $f'(k^*(t)) \to (\mu + \rho + \delta)$. Since, $\rho > n$, Koopmans' result implies ours. The control theory methods developed here can be used to prove Koopmans' result also in a straightforward manner. We shall not undertake that exercise here and stay satisfied with the weaker statement, given that our objective is to establish the transversality condition (40).

[21] One may consume part of the capital also in the reversible investment case, but we do not follow up this possibility. The proof we construct instead works for both reversible as well as irreversible investment.

path, where $(\hat{c}', \hat{k}')$ solves (A.1). Linearizing around $A_0\hat{c}^*$, the gain in utility at $t = 0$ from the change is

$$\begin{aligned} \mathcal{G}(0) &= u(A_0\hat{c}^* + A_0\ \lambda\ \Delta) - u(A_0\hat{c}^*) \\ &\cong u(A_0\hat{c}^*) + A_0\ \lambda\ \Delta\ u'(A_0\hat{c}^*) - u(A_0\hat{c}^*) \\ &= A_0\ \lambda\ \Delta\ u'(A_0\hat{c}^*). \end{aligned} \tag{A.4}$$

Denote $u'(A_0\hat{c}^*)$ by u'^*.

We proceed now to compare the initial gain $A_0\ \lambda\ \Delta\ u'^*$ with subsequent losses. The loss in utility from the change at each $t > 0$ is

$$\begin{aligned} \mathcal{L}(t) &= u(A_t\hat{c}^*) - u(A_t\hat{c}') \\ &= u(A_t\hat{c}^*) - u(A_t(f(\hat{k}') - \lambda\hat{k}')). \end{aligned}$$

Linearizing around $\hat{k}^*$,

$$\begin{aligned} \mathcal{L}(t) &\cong u(A_t\hat{c}^*) - u[A_t(f(\hat{k}^*) - \Delta f'(\hat{k}^*) + \frac{\Delta^2}{2} f''(\hat{k}^*) \\ &\quad -\lambda\hat{k}^* + \lambda\Delta)] \\ &= u(A_t\hat{c}^*) - u[A_t((f(\hat{k}^*) - \lambda\hat{k}^*) - \Delta(f'(\hat{k}^*) \\ &\quad -\lambda) + \frac{\Delta^2}{2} f''(\hat{k}^*))] \\ &= u(A_t\hat{c}^*) - u(A_t(\hat{c}^* + \frac{\Delta^2}{2} f''(\hat{k}^*))),\ \text{using } (A.2), \\ &\cong u(A_t\hat{c}^*) - (u(A_t\hat{c}^*) + \frac{\Delta^2}{2} A_t f''(\hat{k}^*) u'(A_t\hat{c}^*)), \end{aligned}$$

$$\begin{aligned} & \text{(linearizing around } A_t \hat{c}^*\text{)}, \\ = & -\frac{\Delta^2}{2} A_t f''^* u'(A_t \hat{c}^*), \end{aligned} \tag{A.5}$$

where $f''^* = f''(\hat{k}^*)$. The discounted stream of losses incurred during $(0, \infty)$ is

$$\begin{aligned} \int_0^\infty \mathcal{L}(t)\ e^{-(\rho-n)t} dt &= \frac{\Delta^2}{2} \int_0^\infty (-f''^* A_t\ u'(A_t\ \hat{c}^*)\ e^{-(\rho-n)t} dt \\ &> 0, \end{aligned} \tag{A.6}$$

since $f'' < 0$ by **Assumption f1**. The net change in welfare to the household from the perturbation is

$$\begin{aligned} \omega &= \mathcal{G}(0) - \int_0^\infty \mathcal{L}(t)\ e^{-(\rho-n)t} dt \\ &= A_0\ \lambda\ \Delta\ u'^* - (\Delta^2/2) \int_0^\infty (-f''^* A_t\ u'(A_t\ \hat{c}^*) e^{-(\rho-n)t} dt. \end{aligned}$$

Let

$$\xi^* = \int_0^\infty (-f''^* A_t u'(A_t\ \hat{c}^*) e^{-(\rho-n)t} dt,$$

so that

$$\begin{aligned} \omega &= A_0\ \lambda\ \Delta u'^* - (\Delta^2/2)\ \xi^* \\ &= \Delta\ \xi^* (A_0\ \lambda\ u'^* / \xi^* - \Delta/2). \end{aligned}$$

Since A_0, λ, u'^* and ξ^* are fixed, $\exists$ an ϵ such that $\Delta < \epsilon \Rightarrow \omega > 0$. So long as the reduction in $\hat{k}$ falls short of ϵ, the perturbation from the path $\{A_t\ \hat{c}^*, \hat{k}^*\}$ constructed above is welfare improving.

<u>Step 2</u>.

Suppose now that the proposition is false. Then, $\exists$ an optimal path $\{\hat{c}_t^*, \hat{k}_t^*\}$ such that $|\hat{c}_t^* - \hat{c}^*|$ and $|\hat{k}_t^* - \hat{k}^*|$ are arbitrarily small for t large enough. Consider the following perturbation. At a large enough t_0, disinvest down to $\hat{k}'$ (defined in Step 1) and maintain $\{\hat{c}', \hat{k}'\}$ then onwards. The extra consumption generated is Δ_{t_0}, where $|\Delta_{t_0} - A_{t_0}\ \lambda\ \Delta|$ is arbitrarily small for t_0 large enough (given the definition of Δ in Step 1).

The per capita consumption at t_0 changes to $A_{t_0}\hat{c}_{t_0}^* + \Delta_{t_0}$ and the gain in utility from the increased consumption is $u(A_{t_0}\hat{c}_{t_0}^* + \Delta_{t_0}) - u(A_{t_0}\hat{c}_{t_0}^*) = \nu$ (say). For large enough t_0, the value of $\nu \cong (u(A_{t_0}\hat{c}^* + A_{t_0}\ \lambda\ \Delta) - u(A_{t_0}\hat{c}^*)$. Thus, using Step 1 again, we may assume $\nu \cong A_{t_0}\lambda\ \Delta u'(A_{t_0}\ \hat{c}^*)$.

Since $\hat{k}_t^* \rightarrow \hat{k}^*$, it is possible to assume *wlog* that $u(c_t^*) - u(A_t\hat{c}') > 0\ \forall\ t > t_0$. Thus, utility falls by $u(c_t^*) - u(A_t\hat{c}')$ at each $t > t_0$. The discounted present value of the stream of losses is $\int_{t_0}^{\infty}(u(c_t^*) - u(A_t\hat{c}'))e^{-(\rho-n)(t-t_0)}dt$. We have, by definition GR,

$$\int_{t_0}^{\infty}(u(A_t\hat{c}^*) - u(A_t\hat{c}'))\ e^{-(\rho-n)(t-t_0)}dt$$

$$> \int_{t_0}^{\infty}(u(c_t^*) - u(A_t\hat{c}'))e^{-(\rho-n)(t-t_0)}dt,$$

or,

$$-\int_{t_0}^{\infty}(u(A_t\hat{c}^*) - u(A_t\hat{c}'))\ e^{-(\rho-n)(t-t_0)}dt$$

$$< -\int_{t_0}^{\infty}(u(c_t^*) - u(A_t\hat{c}'))e^{-(\rho-n)(t-t_0)}dt,$$

or,

$$-(\Delta^2/2)\ \xi^*_{t_0} < -\int_{t_0}^{\infty} (u(c^*_t) - u(A_t\hat{c}'))\ e^{-(\rho-n)(t-t_0)}dt,$$

where $\xi^*_{t_0}$ corresponds to ξ^* of Step 1 with due alteration of details. Thus, the (Gain - Loss) is approximately equal to

$$A_{t_0}\ \lambda\ \Delta\ u'(A_{t_0}\hat{c}^*) - \textstyle\int_{t_0}^{\infty}(u(c^*_t) - u(A_t\hat{c}'))\ e^{-(\rho-n)(t-t_0)}dt$$

$$> A_{t_0}\ \lambda\ \Delta\ u'(A_{t_0}\hat{c}^*) - (\Delta^2/2)\ \xi^*_{t_0} > 0$$

for an appropriately small value of Δ.

This completes the proof. ∎

PROPOSITION **A.2** *Any feasible path $\hat{k}(t)$ satisfying (11) is bounded above.*

Proof: The maximum value of $\hat{k}(t)$ possible at each t is found by equating c/A to zero for all t. This may be referred to as the path of pure capital accumulation. Denote the path of pure capital accumulation by $\bar{\hat{k}}(t)$. It satisfies the equation

$$\dot{\bar{\hat{k}}}(t) = f(\bar{\hat{k}}(t)) - \lambda\bar{\hat{k}}(t).$$

By virtue of **Assumption f2**, $\exists$ a $\bar{\hat{k}}$ such that $\hat{k}(t) > \bar{\hat{k}} \Rightarrow \dot{\bar{\hat{k}}}(t) < 0$. Thus, max $\{\hat{k}(0), \bar{\hat{k}}\}$ is the claimed upper bound on $\hat{k}(t)$. ∎

Appendix 2

Halkin's Example.

This section discusses a famous example due to Halkin (1974) which constructs an infinite horizon problem for which the transversality condition is not a necessary characterization of optimality.

Before stating the details of the example, let us go back to (31) and analyse the reason why it leads to (32). The inequality (31) is a relationship between the shadow price of investment q_t^* and all subsequent shadow prices over infinite time. Note that, given the objective function and the technology, (32) holds because optimality imposes nontrivial restrictions on the behaviour of the capital accumulation path for all $t > t_0$. (See Proposition 1 above.) Halkin, on the other hand, constructs an objective function which leaves the path of accumulation unrestricted.

To get a feel for Halkin's example, consider an agent engaged in wealth accumulation. Her lifetime utility depends on the difference between the terminal (i.e. limiting) value of her wealth and the initial wealth she owns. Suppose that the maximum possible value of the terminal wealth is $\bar{K}$ and that her initial wealth is K_0. Then, the optimum value of her welfare is $\bar{K} - K_0$. The important characteristic of this objective function is that the agent's welfare is independent of the path followed for approaching $\bar{K}$. The marginal social product of a rise in K_0 is thus $q(0) = \partial(\bar{K} - K_0)/\partial K_0 = -1$, which is independent of the marginal social productivities of K along the way to the optimum $\bar{K}$. Consequently, the value of the shadow price at $t = 0$ does not put any restriction on future values of the shadow price. The same argument holds for the shadow price at any later point in time. In other words, $q(t) = \partial(\bar{K} - K(s))/\partial K(s) = -1 \;\; \forall s > t$. Hence, for this problem, the co-state variable does not converge to zero.

Let us now state and work out the example algebraically. The problem is stated as follows:

$$\text{Maximize } \textstyle\int_0^{\infty} (1 - y)\; u \; dt$$

subject to

$$\dot{y} = (1 - y)\ u,$$

$$y(0) = 0,$$

$$u \in [0, 1].$$

Obviously, u and y are respectively the control and state variables for this problem.

Solution:

Substituting the state equation in the objective function,

$$\begin{aligned} \int_0^\infty \dot{y}\ dt &= y|_0^\infty \\ &= \lim{}_{t \to \infty}\ y(t). \end{aligned}$$

The problem thus reduces to maximizing $\lim{}_{t \to \infty}\ y(t)$. To find the upper bound of y, we solve the equation

$$\dot{y} + (y - 1)\ u = 0.$$

Substituting $z = y - 1$, the equation reduces to

$$\dot{z} + zu = 0.$$

The solution to this equation is

$$z(t) = be^{-\int_0^t u(\nu)d\nu},\ b = \text{constant},$$

or,

$$y(t) = 1 + b\ e^{-\int_0^t u(\nu)\ d\nu}.$$

At $t = 0$, $y(0) = 0 = 1 + b$, or, $b = -1$. Hence, the general solution is $y(t) = 1 - e^{-\int_0^t u(\nu) d\nu}$. Writing $\int_0^t u(\nu) d\nu = h(t) \geq 0$, the solution is $y(t) = 1 - e^{h(t)}$, whence $y(t) \in [0, 1)$. Thus, the upper bound of $y(t)$ is unity and any path leading to it is a solution to the problem. There is no unique optimum path. Indeed, any constant $u \in (0, 1)$ is a solution to the problem. Suppose such a constant u^* is selected.

The Hamiltonian for the problem is

$$\begin{aligned} \mathcal{H} &= (1 - y)\ u + \lambda\ (1 - y)\ u \\ &= ((1 - y)(1 + \lambda))\ u. \end{aligned}$$

The *FOC*'s are:

$$\dot{y} = u\ (1 - y),$$

$$\dot{\lambda} = (1 + \lambda)\ u,$$

$$(1 - y)(1 + \lambda) = 0.$$

Choose $\lambda^* = -1\ \forall\ t$. Then $(u, \lambda) = (u^*, -1)$ satisfies all the optimality conditions, but $\lim_{t \to \infty} \lambda(t) \neq 0$. Note that the value of the co-state variable tallies with the one we obtained above from purely economic arguments.

This completes the counter-example.

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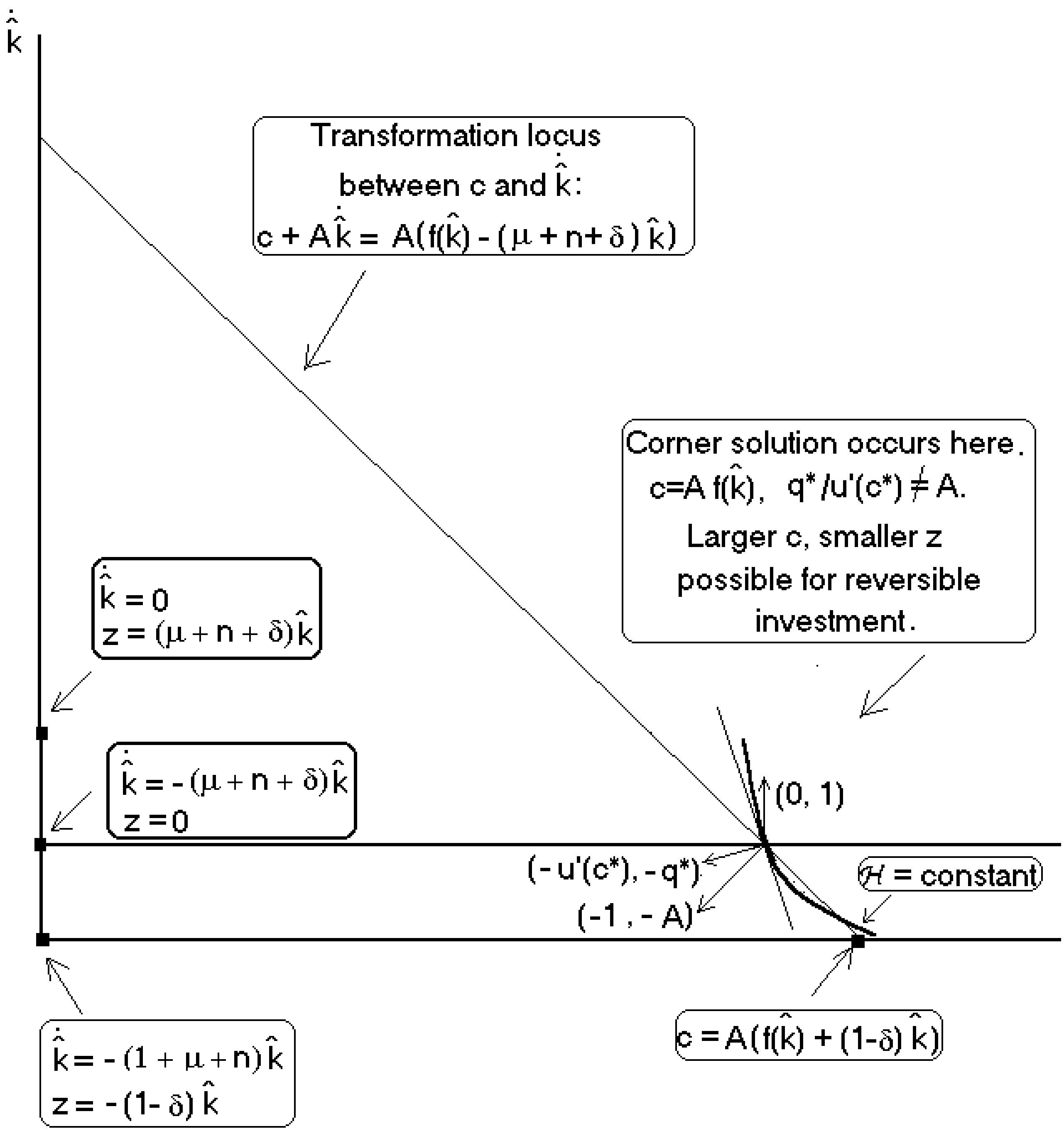

$\dot{\hat{k}}$
Transformation locus between c and $\dot{\hat{k}}$: $c + A\dot{\hat{k}} = A(f(\hat{k}) - (\mu + n + \delta)\hat{k})$
Corner solution occurs here. $c = A f(\hat{k})$, $q^*/u'(c^*) \neq A$. Larger c, smaller z possible for reversible investment.
$\dot{\hat{k}} = 0$
$z = (\mu + n + \delta)\hat{k}$
$\dot{\hat{k}} = -(\mu + n + \delta)\hat{k}$
$z = 0$
(0, 1)
$(-u'(c^*), -q^*)$
(-1 , - A)
$\mathcal{H}$ = constant
$c = A(f(\hat{k}) + (1-\delta)\hat{k})$
$\dot{\hat{k}} = -(1 + \mu + n)\hat{k}$
$z = -(1-\delta)\hat{k}$

Figure 1

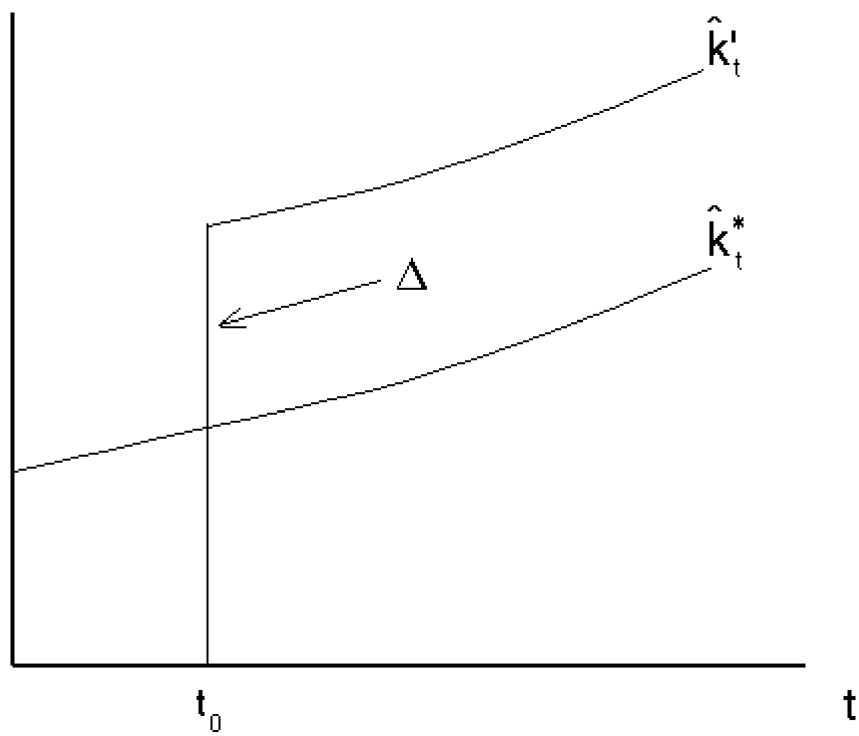

Figure 2