BIPARTITE GRAPHS AND THE SHAPLEY VALUE

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ABSTRACT. We provide a cooperative game-theoretic structure to analyze bipartite graphs where we have a set of employers and a set of workers. Links can form between a employer and a worker and there is no link either between employers or between workers. As in Myerson [3], a cooperation structure in our model can be represented by a set of bilateral links. However, unlike Myerson [3], in this paper bilateral links can only be formed between some employers and some workers. Moreover, the bilateral links are given at the outset. In this scenario we characterize the Shapley Value.

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1. INTRODUCTION

Myerson [3] introduces notions from graph theory to model cooperation structures in games. The central notion is that the players in a game can cooperate by forming bilateral links between them. For characteristic function games, Myerson [3] studies fair allocation rules for games with cooperation structures and characterizes the Myerson value. In the current paper we take a related but different approach to study a class of games with cooperation structures. We consider a model where there are two disjoint sets of players. The model that we consider is a two sided matching model. One economic interpretation of our model is that one set of agents is the set of employers and the other is the set of workers. The matching model that we consider is a many-to-many matching model—each employer can employ a number of workers and likewise each worker can simultaneously work for more than one employer. We model interactions between agents by means of bilateral links between an employer and a worker. We restrict attention to the case where there cannot be any link between two employers or two workers. If a particular worker i is able to work with an employer j, then we say that i and j are connected by a link. The existence of the link *per se* does not mean that there is active interaction between the worker and the employer. What it means is that the channel of communication between the employer and the worker is available. Whether they decide to activate the link/

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channel of communication is completely a decision for the respective employer and the worker to take.

In our model there are two disjoint sets of agents, E and W, $E \cap W = \emptyset$. There is a set of pre-existing links between E and W. The set of links describes a *bipartite* graph g between E and W. If there is no link present between nodes i and j we denote the empty link as $(ij) = \emptyset$. The implication for bipartite graph is that for any $j, j' \in E$, $(jj') = \emptyset$. Likewise for any $i, i' \in W$, $(i, i') = \emptyset$. As in Myerson [3], a cooperation structure in our model can be represented by a set of bilateral links. However, unlike Myerson, in this paper bilateral links can only be formed between some employers and some workers. Moreover, the bilateral links are given at the outset. In this paper we do not consider the set of all possible bipartite cooperation structures.

Given a graph g, if a set of workers is linked to employer j, and they agree to cooperate, they produce a surplus v_j. The subscript j denotes that initially the surplus accrues to employer j. For each employer j we have such a surplus v_j. If all links are active, the vector of surplus generated is $\{v_1, \cdots, v_J\}$. We are concerned with the issue of finding a fair procedure for dividing this vector of surplus. The cooperation structure, represented by the bipartite graph, puts restriction on the available allocations of the vector of surplus. The restriction is that no transfer is possible between two players when there is no direct link between them. In our model there is no direct link between two employers or two workers. As a consequence allocation of the vector of surplus $\{v_1, \cdots, v_J\}$ is not the same as allocating the total surplus $\sum_{j \in E} v_j$. If the agents who are connected through the graph g decide to cooperate a vector of worths $\{v_1, \cdots, v_J\}$ is generated (here J is the number of employers). As mentioned above, a major difference with the Myerson model is that unless two agents l and k are directly connected, there cannot be any transfer between l and k. As a consequence, ours is *not* a fully transferable utility model. Of course, when there is only one employer, our model reduces to a standard transferable utility game with communication structure. Indeed for any coalition of one employer and the workers who are connected to the employer through the graph g, it is a standard transferable utility game. We exploit this feature of our model in our characterization.

We describe the Shapley Value (see Shapley [4]) in our context and present an axiomatic characterization. The axioms are standard Shapley axioms except one. We utilize a version of additivity axiom that is stronger than the Additivity axiom in Shapley's characterization (see Shapley [4]). The reason for the strengthening is that, now, for each coalition we have a vector of worths instead of a single scalar number. The rest of the axioms are standard.

The paper is organized as follows: in Section 2 we outline the model and the axioms. In Section 3 we introduce 'bipartite unanimity games' and analyze its properties that are used in Section 4 to characterize the Shapley Value.

2. The Model

We consider a two sided market model. There are two disjoint sets of agents. For clarity we describe one set of agents as employers and denote the set by E. The other set of agents is described as the set of workers, denoted by W. Both E and W are finite. Let $E = \{1, \cdots, j, \cdots J\}$ and $W = \{1, \cdots, i, \cdots, I\}$. Let $N = E \cup W$. For any $i \in W$ and $j \in E$, an unordered link between i and j is denoted by (ij). In the case where there is no link between two agents l and k, we say $(lk) = \emptyset$. A bipartite graph g is a collection of unordered links between elements of E and W, such that $(lk) \neq \emptyset$ only if $l \in W$ and $k \in E$. Let BG denote the set of all bipartite graphs over N. Generic elements of BG are denoted by g, g' etc. For our analysis the bipartite graph g is held fixed throughout. Fix a graph $g \in BG$ and any $j \in E$. Let $W(j, g)$ be the set of workers that form a link with employer j under the graph g; that is, $W(j, g) = \{i \in W | (ji) \neq \emptyset\}$. Similarly let $E(i, g)$ denote the set of employers with whom worker i is linked, that is, $E(i, g) = \{j \in E | (ji) \neq \emptyset\}$. We say that an employer j is *not linked* under g if $W(j, g) = \emptyset$ and a worker i is *not linked* under g if $E(i, g) = \emptyset$. Given a graph $g \in BG$, a subgraph g' of g, denoted by $g' \subseteq g$, is a graph whose nodes are a subset of the nodes of g and the edges are subsets of the edges of g. In particular if $g' \subseteq g$ and $(ij) \subseteq g$ then $g' \cup (ij) \subseteq g$, that is, the subgraph relation is closed under union operation.

2.1. **Coalitions.** A coalition S is a subset of the set of players $N = E \cup W$. For any $j \in E$ and any $W_j \subseteq W(j, g)$, let $g(j, W_j)$ denote the subgraph with nodes $\{j \cup W_j\}$. For a coalition $S \subseteq N$, let $w_j(S, g) = S \cap W(j, g)$. In other words, $w_j(S, g)$ is the set of workers in coalition S who are connected to employer j under the graph g. For any $S \subseteq N$, the graph g induces a *cover* for S which groups each employer in S with the workers in S who are connected to the aforementioned employer. We denote this cover by $\{j \cup \{w_j(S, g)\}\}_{j \in S \cap E}$. For sure, $\bigcup_{j \in S \cap E} \{j \cup \{w_j(S, g)\}\} = S$.

2.2. **Bipartite Graph Problems.** If an employer j is linked to a workers i, under g, then they can generate some surplus $v_j((ij), g) \in \mathfrak{R}$. Observe that (ij) is a subgraph of g. Fix a $j \in E$ and $W_j \subseteq W(j, g)$ and let $g(j, W_j)$ denote the subgraph of g such that the set of nodes for $g(j, W_j)$ is $W_j \cup \{j\}$. If the set of agents $W_j \cup \{j\}$ decide to cooperate, the surplus generated is $v_j(g(j, W_j), g) \in \mathfrak{R}$. In particluar we denote $v_j(g(j, W(j, g)), g) = v_j(g, g) \equiv v_j(g)$.

Given an employer $j \in E$ and a graph g we have defined the family of functions $\mathrm{V}_j \equiv \{v_j(g(j, W_j), g)\}_{g(j, W_j) \subseteq g(j, W(j, g))}$ that represent all possible surpluses generated by employer j under all possible subgraphs $g(j, W_j) \subseteq g(j, W(j, g))$. Given a graph g, for each $j' \in E$ we now have defined the surplus function $\mathrm{V}_{j'}$. Let $\mathcal{V}$ be the class of all possible surplus functions $\mathrm{V}_{j'}$. We denote the tuple $\mathrm{V} = (\mathrm{V}_1, \ldots, \mathrm{V}_J)$ as the surplus vector. We denote by $\mathcal{V}^E = \times_{j \in E} \mathcal{V}$ the set of all vector of surplus functions. The collection (V, g) is called a *bipartite graph problem*.

2.3. **Bipartite Games.** Let (V, g) be a bipartite graph problem. We now formulate a *bipartite-graph cooperative game* from the bipartite graph problem (V, g).

Definition 2.1. For a bipartite graph problem (V, g), the associated *bipartite cooperative game* is the triple (N, g, V_g) where $N = E \cup W$ is the set of agents, g is the associated bipartite graph specifying the cooperation structure between the employers and workers and $V_g : 2^{|N|} \to \Re^{|E|}$ is the vector-valued characteristic function defined as,

$$V_g(S) = \begin{cases} (0, \ldots, 0) & \text{if } S = \emptyset \\ (v_1(g(1, w_1(S, g)), g), \cdots, v_J(g(J, w_J(S, g)), g)) & \text{if } S \neq \emptyset. \end{cases}$$

Note that, $v_j(g(j, w_j(S, g)), g) = 0$ whenever $g(j, w_j(S)) \not\subseteq g$. For each $\mathrm{V} = (\mathrm{V}_1, \ldots, \mathrm{V}_J) \in \mathcal{V}^E$ and $g \in BG$, we can associate a bipartite cooperateive game (N, g, V_g). Let $\mathcal{B}_g$ denote the class of all such games.

2.4. **The Shapley Value.** Our purpose in this paper is to a single valued allocation rule for our specific cooperative game with communication structure. One of the more important single-valued solution concepts in the case of cooperative games is the Shapley Value. The idea of the Shapley Value hinges crucially on the notion of marginal contributions of players.

Consider $S \subseteq N$. The associated subgraph is $g(S) \subset g$. Let us now consider an agent $k \in N \setminus S$. If k joins the coalition S, the associated subgraph is $g(S \cup \{k\}) \subset g$. In our bipartite graph model, agent k may have multiple direct links with other agents. If k is a worker, k may be linked with multiple employers under g. If more than one employer linked to k are in the coalition S, then measuring the change in surplus due to k joining S is an issue. We make the behavioral assumption that when k joins coalition S, all the links that k has with agents in S become active simultaneously. For example, suppose that $k \in W$ and that k is connected to two employers j and j'. Also suppose that $j, j' \in S$. When k joins S both surpluses v_j and $v_{j'}$ are affected. Formally, when agent k joins coalition S, the associated worth vector is

$$V_g(S \cup \{k\}) = \big(v_j(g(j, w_j(S \cup \{k\})), g)\big)_{j \in E}$$

Therefore, the difference in the worth-vector as a result of k joining coalition S is

$$\begin{aligned} & V_g(S \cup \{k\}) - V_g(S) \\ &= \big(v_j(g(j, w_j(S \cup \{k\})), g)\big)_{j \in E} - \big(v_j(g(j, w_j(S)), g)\big)_{j \in E} \\ &= \big(v_j(g(j, w_j(S \cup \{k\})), g) - v_j(g(j, w_j(S)), g)\big)_{j \in E} \end{aligned} \tag{2.1}$$

The expression in equation 2.1 is the vector of marginal contributions for player k when she joins coalition S.

Definition 2.2. Given a bipartite game (N, g, V_g), the marginal contribution of any player $k \in N$ to a coalition $S \subseteq N \setminus \{k\}$ is $mc_k(S) = [V_g(S \cup \{k\}) - V_g(S)]$. The j^{th} component of the vector, $mc_k^j(S)$ is the addition to the surplus of the j^{th} employer when k joins coalition S.

Note that given a graph g, if either $S \not\ni j$ and $k \neq j$ or $S \ni j$ and $k \notin W(j, g)$, then $mc_k^j(S) = 0$. We can now define an allocation rule as follows:

Definition 2.3. An allocation rule is a mapping $\phi : \mathcal{B}_g \to \Re^{J+I}$. That is, for each $(N, g, V_g) \in \mathcal{B}_g$, $\phi(N, g, V_g) = \big(\phi_k(N, g, V_g)\big)_{k \in N} \in \Re^{J+I}$.

The Shapley Value is an allocation rule that assigns to each player, his *average marginal contribution* to the worth of all possible coalitions. Given that any bipartite game (N, g, V_g) generates, for every coaltion $S \subseteq N$, a surplus vector $V_g(S) \in \Re^J$, the definition of Shapley Value in this context is given below.

For any $T \subset E$, define $\mathbf{1}_T$ as the $J \times 1$ column vector whose j^{th} row element is 1 if $j \in T$ and is 0 if $j \in E \setminus T$.

Definition 2.4. The Shapley Value is a mapping $\psi : \mathcal{B}_g \to \Re^{J+I}$ such that for any given $(N, g, V_g) \in \mathcal{B}_g$, and for any $k \in N$

$$\psi_k(N, g, V_g) = \sum_{S \subseteq N \setminus \{k\}} \left[\frac{|S|!(|N| - |S| - 1)!}{|N|!} \right] mc_k(S) \mathbf{1}_E$$

$$= \sum_{j \in E} \sum_{S \subseteq N \setminus \{k\}} \left[\frac{|S|!(|N| - |S| - 1)!}{|N|!} \right] mc_k^j(S) \tag{2.2}$$

In our framework, there are two different types of agents, employers and workers. While an employer can influence only one element of the surplus vector, a worker i, on the other hand, can be actively employed by more than one employer. As a consequence, the Shapley Values of the two types of players will be different. The following proposition makes clear the difference.

Proposition 2.5. Consider a bipartite game $(N, g, V_g) \in \mathcal{B}_g$. Then,

(i) the Shapley Value of any employer $j \in E$ is given by

$$\psi_j(N, g, V_g) = \sum_{W_j \subseteq W(j,g)} \left[\frac{|W_j|!|W(j, g) \setminus W_j|!}{|W(j, g) \cup \{j\}|!} \right] mc_j^j(W_j) \text{ and}$$

(ii) the Shapley Value of any worker $i \in W$ is given by

$$\psi_i(N, g, V_g) = \sum_{j \in E(i,g)} \psi_i^j(N, g, V_g)$$

where $\psi_i^j(N, g, V_g) = \sum_{W_j \subseteq W(j,g) \setminus \{i\}} \left[\frac{|W_j \cup \{j\}|!|W(j,g) \setminus W_j \cup \{i\}|!}{|W(j,g) \cup \{j\}|!} \right] mc_i^j(W_j \cup \{j\})$.

PROOF: Before proving the result we state a combinatorial identity. For any given positive integer r

$$\sum_{k=0}^{n-1-p} \binom{r+k}{r} \binom{n-1-(r+k)}{p-r} = \binom{n}{p+1}. \tag{2.3}$$

The proof of identity (2.3) is very standard and is hence omitted (see, for example Lovász [2], pages 18 and 172).

First consider any employer $j \in E$. Clearly, adding employer j to a coalition S can lead to an change in surplus only if S includes some workers $W_j \subseteq W(j,g)$. Morever, given any $W_j \subseteq W(j,g)$, for all S such that $W_j \subset S$ and $S \setminus W_j \subseteq N \setminus W(j,g) \cup \{j\}$,

$$mc_j(S) = \left(mc_j^j(W_j), 0 \cdots, 0 \right).$$

Thus, the Shapley Value of employer is a weighted sum of $\{mc_j^j(W_j)\}_{W_j \subseteq W(j,g)}$. Using these observations we get $\psi_j(N,g,V_g) = \sum_{W_j \subseteq W(j,g)} \eta(W_j) mc_j^j(W_j)$ where,

$$\eta(W_j) = \sum_{k=0}^{|N \setminus W(j,g) \cup \{j\}|} \binom{|N \setminus W(j,g) \cup \{j\}|}{k} \left[\frac{(k+|W_j|)!(|N \setminus W_j \cup \{j\}| - k)!}{|N|!} \right]$$

$$= \left[\frac{|N \setminus W(j,g) \cup \{j\}|!|W_j|!|W(j,g) \setminus W_j|!}{|N|!} \right] \sum_{k=0}^{|N \setminus W(j,g) \cup \{j\}|} \binom{|W_j|+k}{|W_j|} \binom{|N \setminus W_j \cup \{j\}| - k}{|W(j,g) \setminus W_j|}$$

$$= \left[\frac{|N \setminus W(j,g) \cup \{j\}|!|W_j|!|W(j,g) \setminus W_j|!}{|N|!} \right] \binom{|N|}{|W(j,g) \cup \{j\}|}$$

$$= \left[\frac{|W_j|!|W(j,g) \setminus W_j|!}{|W(j,g) \cup \{j\}|!} \right] \tag{2.4}$$

The penultimate equality follows from identity (2.3). From the last step we get $\eta(W_j) = \left[\frac{|W_j|!|W(j,g) \setminus W_j|!}{|W(j,g) \cup \{j\}|!} \right]$ for all $W_j \subseteq W(j,g)$ and (i) follows.

We now consider part (ii) of the proposition. Observe that worker i contributes only in coalitions that includes employers from the set $E(i,g)$ who are linked to worker i under g. Using this observation we get ,

$$\psi_i(N,g,V_g) = \sum_{j \in E(i,g)} \psi_i^j(N,g,V_g)$$

where $\psi_i^j(N,g,V_g) = \sum_{S \subseteq N \setminus \{i\}} \left(\frac{|S|!(|N|-|S|-1)!}{|N|!} \right) mc_i^j(S)$. We consider any $j \in E(i,g)$ and calculate $\psi_i^j(N,g,V_g)$. For any $S \subset N \setminus \{i\}$ such that $j \notin S$, $mc_i^j(S) = 0$. For any $S \subseteq N \setminus \{i\}$ such that $\{j\} \cup W_j \subseteq S$, $W_j \subseteq W(j,g) \setminus \{i\}$ and $S \setminus W_j \cup \{j\} \in N \setminus W(j,g) \cup \{j\}$, we have $mc_i^j(S) =$

$mc_i^j(W_j \cup \{j\})$. Therefore, $\psi_i^j(N, g, V_g) = \sum_{W_j \subseteq W(j,g) \setminus \{i\}} \zeta(W_j) mc_i^j(W_j \cup \{j\})$ where,

$$\zeta(W_j) = \sum_{k=0}^{|N \setminus W(j,g) \cup \{j\}|} \binom{|N \setminus W(j,g) \cup \{j\}|}{k} \left[\frac{(k + |W_j \cup \{j\}|)!(|N \setminus W_j \cup \{j,i\}| - k)!}{|N|!}\right]$$

$$= \left[\frac{|N \setminus W(j,g) \cup \{j\}|!|W_j \cup \{j\}|!|W(j,g) \setminus W_j \cup \{j\}|!}{|N|!}\right] \sum_{k=0}^{|N \setminus W(j,g) \cup \{j\}|} \binom{|W_j \cup \{j\}| + k}{|W_j \cup \{j\}|} \times \binom{|N \setminus W_j \cup \{j,i\}| - k}{|W(j,g) \setminus W_j \cup \{j\}|}$$

$$= \left[\frac{|N \setminus W(j,g) \cup \{j\}|!|W_j \cup \{j\}|!|W(j,g) \setminus W_j \cup \{j\}|!}{|N|!}\right] \binom{|N|}{|W(j,g) \cup \{j\}|}$$

$$= \left[\frac{|W_j \cup \{j\}|!|W(j,g) \setminus W_j \cup \{j\}|!}{|W(j,g) \cup \{j\}|!}\right] \tag{2.5}$$

Again the penultimate equality follows from (2.3). Since the selection of $j \in E(i, g)$ was arbitrary, from the last step it follows that, for all $j \in E(i, g)$,

$$\psi_i^j(N, g, V_g) = \sum_{W_j \subseteq W(j,g) \setminus \{i\}} \left[\frac{|W_j \cup \{j\}|!|W(j,g) \setminus W_j \cup \{j\}|!}{|W(j,g) \cup \{j\}|!}\right] mc_i^j(W_j \cup \{j\}).$$

This proves part (ii) of the proposition. □

We now introduce some axioms on allocation rules. The first axiom is Pareto Efficiency. This property simply says that the allocation made by the rule must exhaust all the surplus generated.

Definition 2.6. (*Pareto Efficiency*): An allocation rule $\phi : \mathcal{B}_g \to \Re^{J+I}$ satisfies *Pareto efficiency* if for all $(N, g, V_g) \in \mathcal{B}_g$, $\sum_{k \in N} \phi_k(N, g, V_g) = V_g(N)\mathbf{1}_E$, i.e.,

$$\sum_{k \in N} \phi_k(N, g, V_g) = \sum_{j \in E} v_j(g)$$

In order to describe the other axioms some definitions are in order.

Definition 2.7. For a bipartite game (N, g, V_g) we define the following properties.

(1) Player $k \in N$ is a *dummy player* if, for all $S \subseteq N \setminus \{k\}$, $mc_k(S) = (mc_k^1(S), \cdots, mc_k^J(S)) = (0, \ldots, 0)$.
(2) Players $k, k' \in N$ are *symmetric* if, for all $S \subseteq N \setminus \{k, k'\}$, $mc_k(S) = mc_{k'}(S)$.

We now introduce two very important axioms. The first axiom is a *dummy* axiom.

Definition 2.8. (*Dummy*): An allocation rule $\phi : \mathcal{B}_g \to \Re^{J+I}$ satisfies *dummy property* if, for a dummy player $k \in N$ in any $(N, g, V_g) \in \mathcal{B}_g$, $\phi_k(N, g, V_g) = 0$.

The second axiom is *Symmetry*. This is a fairness requirement that states, equals are treated equally.

Definition 2.9. (*Symmetry*): An allocation rule $\phi : \mathcal{B}_g \rightarrow \Re^{J+I}$ satisfies *symmetry* if for any pair of symmetric players $k, k' \in N$ in any $(N, g, V_g) \in \mathcal{B}_g$, $\phi_k(N, g, V_g) = \phi_{k'}(N, g, V_g)$.

The third axiom that we impose is *Additivity*. Additivity is a standard axiom in the characterization of Shapley Value for TU games. Since for any bipartite game (N, g, V_g) and any coalition S, $V_g(S) \in \Re^J$, our notion of additivity is stronger.

Fix a bipartite graph g and consider two bipartite games, (N, g, V_g) and (N, g, V'_g). Define $(V_g + V'_g)$ as follows: for all $S \subseteq N$, $(V_g + V'_g)(S) = V_g(S) + V'_g(S)$. Observe that for all $S \subseteq N$, $(V_g + V'_g)(S) \in \Re^J$. Hence, $(N, g, V_g + V'_g) \in \mathcal{B}_g$.

Definition 2.10. (*Additivity*): An allocation rule $\phi : \mathcal{B}_g \rightarrow \Re^{J+I}$ satisfies *additivity* if for any pair of bipartite games $(N, g, V_g), (N, g, V'_g) \in \mathcal{B}_g$, and for all $k \in N$,

$$\phi_k(N, g, V_g + V'_g) = \phi_k(N, g, V_g) + \phi_k(N, g, V'_g)$$

It is well known that in the standard transferable utility games the unique value that satisfies Pareto, dummy, symmetry and additivity properties is the Shapley Value (see Shapley [4]). However, as mentioned above, ours is a strong version of the original additivity axiom. The reason we impose a strong additivity axiom is that we are looking at cases where the coalitional worth is a vector and not a scalar.

3. Bipartite Unanimity Games

In the context of TU games an important notion is that of *unanimity* games.

Definition 3.1. A TU game (N, u_Q) is a unanimity game if

$$u_Q(S) = \begin{cases} 1 & \text{if } Q \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

We propose a similar notion for bipartite graph games. We call these games *bipartite unanimity games*.

Definition 3.2. Fix a bipartite graph g, $j \in E$ and $W_j \subset W(j, g)$. A *bipartite unanimity game* is a triple $(N, g, U^{W_j \cup \{j\}})$ where ,

(i) for all $S \subseteq N$,

$$U^{W_j \cup \{j\}}(S) = (\mathrm{u}_1^{W_j \cup \{j\}}(S), \ldots, \mathrm{u}_J^{W_j \cup \{j\}}(S)) \in \{0, 1\}^J$$

(ii) The specific binary functions $\mathbf{u}_j^{W_j\cup\{j\}} : 2^{|N|} \to \{0,1\}$ for all $j' \in E$ satisfy the following restrictions:

$$\mathbf{u}_{j'}^{W_j\cup\{j\}}(S) = \begin{cases} 1 & \text{if } j' = j \text{ and } W_j \cup \{j\} \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Define by $\mathcal{U}_g \equiv \{\{(N, g, U^{W_j\cup\{j\}})\}_{W_j\subseteq W(j,g), W_j\neq\emptyset}\}_{j\in E, W(j,g)\neq\emptyset}$ the family of bipartite unanimity games. It is obvious that $\mathcal{U}_g \subset \mathcal{B}_g$.

Consider a bipartite game (N, g, V_g). Fix a j and consider the subset of agents $\{j\} \cup W(j,g)$. The subgraph associated with the above set of agents is $g(\{j\} \cup W(j,g))$. We now consider the subgame $(\{j\} \cup W(j,g), g(\{j\} \cup W(j,g)), V_g\mathbf{1}_j)$. Observe that the subgame described above is a transferable utility game. We call this subgame the j^{th} *partial transferable utility game* (jPTU). Observe that, the j^{th} partial TU game $(N, g, U^{W_j\cup\{j\}}\mathbf{1}_j)$ defined above, is a TU unanimity game.

Proposition 3.3. Given a graph g, the family of bipartite unanimity games $\mathcal{U}_g$ are linearly independent.

PROOF: Let us define $\bar{g}$ to be the complete bipartite graph, i.e., for any $i \in W$ and for any $j \in E$, $(ij) \neq \emptyset$. Therefore, for any $j \in E$, $W(j, \bar{g}) = W$ and for any $i \in W$, $E(i, \bar{g}) = E$. Let $\mathcal{U}_{\bar{g}}$ be the set of all possible bipartite unanimity games associated with this complete bipartite graph $\bar{g}$. In other words, $\mathcal{U}_{\bar{g}} = \left\{\left\{(N, \bar{g}, U^{W'\cup\{j\}})\right\}_{W'\subseteq W,\, W'\neq\emptyset}\right\}_{j\in E}$. Since $g \subseteq \bar{g}$, for any $g \in BG$, we have $\mathcal{U}_g \subseteq \mathcal{U}_{\bar{g}}$. To prove the proposition we first claim that the family of unanimity games in $\mathcal{U}_{\bar{g}}$ are linearly independent. Suppose that the claim is false. Then there exists a $j \in E$, a set $W' \subseteq W$, a bipartite unanimity game $(N, g, U^{W'\cup\{j\}}) \in \mathcal{U}_{\bar{g}}$, and a collection of sets, and a collection of sets $\mathcal{T} \equiv \{T_k\}_{k=1}^K$ such that,

(i) $U^{T_k} \in \mathcal{U}_{\bar{g}}$ for all $k = 1, \ldots, K$ and,
(ii) $U^{w\cup\{j\}} = \sum_{k=1}^{K} a_{T_k} U^{T_k}$ where for all k, $a_{T_k} \neq 0$

Observe that, for any k, the set T_k is of the form $\hat{W} \cup \{j'\}$ where $\hat{W} \subseteq W$ and $j' \in E$. We claim that if $j' \neq j$, then for any $\hat{W} \subseteq W$, $\hat{W} \cup \{j'\} \notin \mathcal{T}$. To see this, suppose to the contrary that there exists a $j' \neq j$ and a $\hat{W} \subseteq W$ such that $\hat{W} \cup \{j'\} \in \mathcal{T}$. Let $\underline{W} \subseteq \hat{W}$ be a subset of $\hat{W}$ with the smallest cardinality. If there is more than one subset of $\hat{W}$ with the smallest cardinality we pick any one of them. Let $S = \underline{W} \cup \{j'\}$. Observe that the game $(N, g, U^{W'\cup\{j\}}\mathbf{1}_{j'})$ is j'-th partial TU game. Moreover, it is a TU unanimity game. Therefore we get, $0 = U^{W'\cup\{j\}}(S)\mathbf{1}_{j'} = \sum_{k=1}^{K} a_{T_k} U^{T_k}(S)\mathbf{1}_{j'} = a_{\underline{W}\cup\{j'\}} U^{\underline{W}\cup\{j'\}}(S)\mathbf{1}_{j'} = a_{\underline{W}\cup\{j'\}} \neq 0$ and we have a contradiction to our earlier claim. Therefore we have that $\mathcal{T} \subset \left\{\{\hat{W} \cup \{j\}\}\right\}_{\hat{W}\subseteq W, \hat{W}\neq\emptyset}$ and $U^{W'\cup\{j\}} = \sum_{t\in\mathcal{T}} a_t U^t$. Observe that $U^{W'\cup\{j\}} = \sum_{t\in\mathcal{T}} a_t U^t \Rightarrow U^{W'\cup\{j\}}(S)\mathbf{1}_j = \sum_{t\in\mathcal{T}} a_t U^t(S)\mathbf{1}_j$ for all $S \subseteq N$. Moreover $\{(N, g, U^t\mathbf{1}_j)\}_{t\in\mathcal{T}\cup\{W'\cup\{j\}\}}$ is a collection from the set of partial TU games that are also unanimity games. From Shapley [4] we know that the collection of all unanimity games is linearly independent. Therefore, it is impossible to have $U^{W'\cup\{j\}} = \sum_{t\in\mathcal{T}} a_t U^t$ since that would imply $U^{W'\cup\{j\}}\mathbf{1}_j = \sum_{t\in\mathcal{T}} a_t U^t \mathbf{1}_j \Rightarrow u_{W'\cup\{j\}} = \sum_{t\in\mathcal{T}} a_t u_t$ and that would imply linear dependence between unanimity games. This proves that the family of bipartite unanimity games included in $\mathcal{U}_{\bar{g}}$ are linearly independent. The result follows since for any graph $g \in BG$, $\mathcal{U}_g \subseteq \mathcal{U}_{\bar{g}}$. □

Proposition 3.4. Given g, any $(N, g, V_g) \in \mathcal{B}_g$ can be written as a linear combination of the family of bipartite unanimity games $\mathcal{U}_g$. In particular, $V_g = \sum_{j\in E} \sum_{W_j\subseteq W(j,g)} \alpha_{W_j\cup\{j\}} U^{W_j\cup\{j\}}$ where $\alpha_{W_j\cup\{j\}} = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|} V_g(Q\cup\{j\})\mathbf{1}_j = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|} mc^j_j(Q)$.

PROOF: It is enough to show that given a g and a $(N, g, V_g) \in \mathcal{B}_g$, for each $j \in E$, $[V_g(S)]\mathbf{1}_j = \sum_{W_j\subseteq W(j,g)} \alpha_{W_j\cup\{j\}} U^{W_j\cup\{j\}}(S)\mathbf{1}_j$ for all $S \subseteq N$ and $\alpha_{W_j\cup\{j\}} = \sum_{T\subseteq W_j} (-1)^{|W_j\setminus T|}[V_g(T\cup\{j\})]\mathbf{1}_j$.[1] Note that $(N, g, V_g\mathbf{1}_j)$ is the j-th partial TU game for the bipartite game (N, g, V_g). From Shapley ([4]) we know that, the collection of unanimity games acts as a basis for TU games. In particular, $V_g\mathbf{1}_j = \sum_{T\subseteq N} \alpha_T u_T$ where $\alpha_T = \sum_{Q\subseteq T}(-1)^{|T\setminus Q|} V_g(Q)\mathbf{1}_j$. Take a bipartite game (N, g, V_g), fix a $j \in E$ and let us consider the j-th partial TU game $(N, g, V_g\mathbf{1}_j)$. In order to prove our proposition it suffices to show the following:

(a1) For all $k \in W(j, g) \cup \{j\}$, $\alpha_{\{k\}} = 0$.
(a2) For all T such that $T \subseteq W \setminus W(j, g)$, $\alpha_T = 0$.
(a3) For all T such that $T \subseteq E \setminus \{j\}$, $\alpha_T = 0$.
(a4) For all $W_j \subseteq W(j, g)$, $\alpha_{W_j\cup\{j\}} = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|} mc^j_j(Q)$

Step (a1) is obvious. For Step (a2) consider any T such that $i \in T$ and $i \in W \setminus W(j, g)$. Then we get $\alpha_T = \sum_{Q\subseteq T} (-1)^{|T\setminus Q|} V_g(Q)\mathbf{1}_j = \sum_{Q\subseteq T\setminus\{i\}} (-1)^{|T\setminus[Q\cup\{i\}]|}[V_g(Q\cup\{i\}) - V_g(Q)]\mathbf{1}_j = \sum_{Q\subseteq T\setminus\{i\}} (-1)^{|T\setminus[Q\cup\{i\}]|} mc^j_i(Q) = 0$. The last step follows from the fact that worker i is not linked to j and hence $mc^j_i(Q) = 0$ for all $Q \subseteq T \setminus \{i\}$. For (a3) consider any set T such that $j' \in T$ and $j' \in E \setminus \{j\}$. Then $\alpha_T = \sum_{Q\subseteq T} (-1)^{|T\setminus Q|} V_g(Q)\mathbf{1}_j = \sum_{Q\subseteq T\setminus\{j'\}} (-1)^{|T\setminus[Q\cup\{j'\}]|}[V_g(Q\cup\{j'\}) - V_g(Q)]\mathbf{1}_j = \sum_{Q\subseteq T\setminus\{j'\}} (-1)^{|T\setminus[Q\cup\{j'\}]|} mc^j_{j'}(Q) = 0$. The last step follows from the fact that employer j' is not linked to employer j. For (a4) note that for each $W_j \cup \{j\}$ with $W_j \subseteq W(j, g)$ and $W_j \neq \emptyset$, $\alpha_{W_j\cup\{j\}} = \sum_{Q\subseteq W_j\cup\{j\}} (-1)^{|W_j\cup\{j\}\setminus Q|} V_g(Q)\mathbf{1}_j = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|}[V_g(Q\cup\{j\}) - V_g(Q)]\mathbf{1}_j = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|} V_g(Q\cup\{j\})\mathbf{1}_j = \sum_{Q\subseteq W_j} (-1)^{|W_j\setminus Q|} mc^j_j(Q)$.

[1] Recall that for $(N, g, U^{W_j\cup\{j\}}) \in \mathcal{U}_g$, the j-th partial TU game $\left(N, g, U^{W_j\cup\{j\}}\mathbf{1}_{j'}\right)$ is a null game for all $j' \in E \setminus \{j\}$.

From steps (a1)-(a4) it follows that $V_g(S)\mathbf{1}_j = \sum_{W_j \subseteq W(j,g)} \alpha_{W_j \cup \{j\}} u_{W_j \cup \{j\}}(S)$ for all $S \subseteq N$ and $\alpha_{W_j \cup \{j\}} = \sum_{Q \subseteq W_j} (-1)^{|W_j \setminus Q|} mc_j^j(Q)$. Since $(N, g, U^{W_j \cup \{j\}}\mathbf{1}_j) = (N, u_{W_j \cup \{j\}})$ for all non-empty $W_j \subseteq W(j,g)$ and since for all $j' \in E \setminus \{j\}$, $(N, g, U^{W_j \cup \{j\}}\mathbf{1}_j)$ is a null game the result follows. □

4. THE CHARACTERIZATION RESULT

The central result of this paper is a characterization of the Shapley Value for any bipartite graph game (N, g, V_g) using the axioms stated above.

Theorem 4.1. For a graph g and its associated family of bipartite games $(N, g, V_g) \in \mathcal{B}_g$, the unique allocation rule that satisfies dummy property, symmetry, additivity and Pareto efficiency is the Shapley Value defined in Proposition 2.5.

PROOF: We prove the result in two steps. In step 1 we show that any allocation rule that satisfies dummy, symmetry, additivity and Pareto efficiency is unique. In step 2 we show that the Shapley Value satisfies dummy, symmetry, additivity and Pareto efficiency.

STEP 1: A value $\phi : \mathcal{B}_g \to \Re^{J+I}$ that satisfies dummy property, symmetry, additivity and Pareto efficiency is unique.

PROOF OF STEP 1: Fix a $j \in J$, and consider the bipartite unanimity game $(N, g, U^{W_j \cup \{j\}}) \in \mathcal{U}_g$. Take $\alpha \neq 0$ and consider the game $(N, g, \alpha U^{W_j \cup \{j\}})$. Consider any allocation rule $\phi : \mathcal{B}_g \to \Re^{J+I}$ that satisfies dummy property, symmetry, additivity and Pareto efficiency. For all $k \in N \setminus [W_j \cup \{j\}]$, $mc_k(S) = (0, 0, \cdots, 0)$ for all $S \subseteq N \setminus \{k\}$. Since ϕ satisfies dummy property,

$$\phi_k(N, g, \alpha U^{W_j \cup \{j\}}) = 0 \text{ for all } k \in N \setminus [W_j \cup \{j\}].$$

Consider now any two agents $k, k' \in W_j \cup \{j\}$. Given the definition of $U^{W_j \cup \{j\}}$, we have for any $S \subseteq N \setminus \{k, k'\}$

$$mc_k(S) = mc'_k(S) = (0, 0, \cdots, 0).$$

Moreover, consider any $T \subseteq N \setminus [W_j \cup \{j\}]$. Define, $T_k = T \setminus \{k\}$ and $T_{k'} = T \setminus \{k'\}$. Observe that,

$$mc_k^{j'}(T_k \cup \{k\}) = mc_{k'}^{j'}(T_{k'} \cup \{k'\}) = \begin{cases} \alpha & \text{if } j = j' \\ 0 & \text{otherwise} \end{cases}$$

Hence players in $W_j \cup \{j\}$ are all symmetric players and are not dummy players. Since ϕ satisfies symmetry, $\phi_k(N, g, \alpha U^{W_j \cup \{j\}}) = x$ for all $k \in W_j \cup \{j\}$. Moreover, since ϕ satisfies

Pareto efficiency,

$$[\alpha U^{W_j \cup \{j\}}(N)]\mathbf{1}_E = \alpha U^{W_j \cup \{j\}}(N)\mathbf{1}_j = \alpha$$
$$= \sum_{k \in N} \phi_k(N, g, \alpha U^{W_j \cup \{j\}}) = \sum_{k \in W_j \cup \{j\}} \phi_k(N, g, \alpha U^{W_j \cup \{j\}})$$

From the above conditions we therefore get that,

$$\phi_k(N, g, \alpha U^{W_j \cup \{j\}}) = \begin{cases} \frac{\alpha}{|W_j \cup \{j\}|} & \text{if } k \in W_j \cup \{j\} \\ 0 & \text{otherwise.} \end{cases}$$

From Proposition 3.4 we know that any game $(N, g, V_g) \in \mathcal{B}_g$ can be written as a linear combination of bipartite unanimity games from $\mathcal{U}_g$. Applying additivity of ϕ it follows that for all $k \in N$, we have

$$\phi_k(N, g, V_g) = \phi_k \left(N, g, \sum_{j \in E} \sum_{W_j \subseteq W(j,g)} \alpha_{j,W_j} U^{W_j \cup \{j\}} \right) = \sum_{R \in \bar{R}, R \ni k} \frac{\alpha_R}{|R|}$$

where $\bar{R} = \{\{j \cup W_j\}_{W_j \subseteq W(j,g), \ni W_j \neq \emptyset}\}_{j \in E, \ni W(j,g) \neq \emptyset}$. From Proposition 3.3 we also know that the bipartite unanimity games in $\mathcal{U}_g$ are linearly independent. Hence,

$$\phi(N, g, V_g) = \phi \left(N, g, \sum_{j \in E} \sum_{W_j \subseteq W(j,g)} \alpha_{j,W_j} U^{W_j \cup \{j\}} \right)$$

obtained by applying dummy property, symmetry, additivity and Pareto efficiency is unique.

STEP 2: The Shapley Value ψ satisfies dummy property, symmetry, additivity and Pareto efficiency.

PROOF OF STEP 2: From the expressions of Shapley Value in (i) and (ii) in Proposition 1, it follows naturally that the expressions in (i) and (ii) satisfy dummy property. In order to show that the Shapley Value satisfies symmetry we need to consider three distinct cases:

- : (a) the players $k, k' \in J$,
- : (b) the players $k, k' \in W$, and
- : (c) $k \in J$ and $k' \in W$.

Case (a): If $k, k' \in E$, the only way k and k' can be symmetric is if for any $S \subset N \setminus \{k, k'\}$,

$$v_k\left(g(k, w_k(S \cup \{k\}, g))\right) = v_{k'}\left(g(k', w_{k'}(S \cup \{k\}, g))\right) \tag{4.1}$$

But if (4.1) holds then Proposition 2.5(i) we get $\psi_k(N, g, V_g) = \psi_{k'}(N, g, V_g)$.

Case (b): Suppose that $k, k' \in W$. Observe that in this case the set of employers with whom k and k' are actively connected to must be the same, i.e., $E(k, g) = E(k', g)$. Moreover, if k and k' are symmetric, it also implies that for all $j \in E(k, g) = E(k', g)$, for all $W_j \subset W(j, g) \setminus \{k, k'\}$, $mc_k^j(W_j \cup \{j\}) = mc_{k'}^j(W_j \cup \{j\})$. Therefore, $\psi_k(N, g, V_g) = \psi_{k'}(N, g, V_g)$.

Case (c): Suppose $k \in E$ and $k' \in W$. The only class of games where k and k' will be symmetric is where either (i) k' is not connected to any other employer or (ii) the marginal contribution of k' to any employer $j \neq k$ is 0 i.e., for all $j \neq k$ and for all $W_j \subset W(j,g)$, $mc^j_{k'}(W_j \cup \{j\}) = 0$. Moreover, for any $W \subset W(k,g) \setminus \{k'\}$, it has to be the case that, $mc^k_k(W) = v_k(g(k,W),g) = 0 = mc^k_{k'}$. In addition, if $k \in E$ and $k' \in W$ are symmetric, then for any $T \subset N$ such that $\{k,k'\} \notin T$, $mc^k_k(T \cup \{k'\}) = v_k(g(k, w_k(T \cup \{k'\}), g)) = mc^k_{k'}(T \cup \{k\})$. Therefore, $\psi_k(N,g,V_g) = \psi_{k'}(N,g,V_g)$.

To show additivity, consider (N,g,V_g), (N,g,V'_g) and $(N,g,V_g+V'_g)$. Note that, for any $k \in N$ and $S \subset N \setminus \{k\}$, $(V_g+V'_g)(S \cup \{k\}) = \Big(v_j(g(j,w_j(S \cup \{k\}),g) + v'_j(g(j,w_j(S \cup \{k\})),g)\Big)_{j \in E}$. Suppose $k \in E$. Notice that, for all $j \neq k$, $v_j(g(j,w_j(S \cup \{k\}),g) = v'_j(g(j,w_j(S \cup \{k\}),g) = 0$ since $g(j,w_j(S \cup \{k\}) \subsetneq g$. Therefore from equation (2.2) we get

$$\psi_k(N,g,V_g+V'_g) = \sum_{j \in E} \sum_{S \subseteq N \setminus \{k\}} \left[\frac{|S|!(|N|-|S|-1)!}{|N|!}\right] \times$$

$$\left\{\Big(v_j(g(j,w_j(S \cup \{k\}),g) + v'_j(g(j,w_j(S \cup \{k\})),g)\Big) - \Big(v_j(g(j,w_j(S),g) + v'_j(g(j,w_j(S)),g)\Big)\right\}$$

$$= \sum_{S \subseteq N \setminus \{k\}} \left[\frac{|S|!(|N|-|S|-1)!}{|N|!}\right] \left(v_j(g(j,w_j(S \cup \{k\}),g) - v_j(g(j,w_j(S),g)\right)$$

$$- \sum_{S \subseteq N \setminus \{k\}} \left[\frac{|S|!(|N|-|S|-1)!}{|N|!}\right] \Big(v'_j(g(j,w_j(S \cup \{k\}),g) - v'_j(g(j,w_j(S),g)\Big)$$

$$= \psi_k(N,g,V_g) + \psi_k(N,g,V'_g) \tag{4.2}$$

Suppose now $k \in W$. As above one can show that for a $k \in W$, $\psi_k(N,g,V_g) + \psi_k(N,g,V'_g)$.

To show that Pareto efficiency is satisfied, we consider any $(N,g,V_g) \in \mathcal{B}_g$, an employer $j \in E$ and the set $W(j,g)$. Consider the jPTU game $(N,E,V_g\mathbf{1}_j)$. Note that in this TU game $V_g(S \cup \{k\})\mathbf{1}_j) - V_g(S)\mathbf{1}_j = 0$ for all $k \in N \setminus [W(j,g) \cup \{j\}]$. Therefore, all $k \in N \setminus [W(j,g) \cup \{j\}]$ are dummy players and hence their average marginal contribution in the TU game $(N,g,V_g\mathbf{1}_j)$ is zero. Note that marginal contribution of any $k \in W(j,g) \cup \{j\}$ to any coalition $S \subseteq N \setminus \{k\}$ is $[V_g(S \cup \{k\}) - V_g(S)]\mathbf{1}_j$. Using the proof of Proposition 2.5 it follows that the average marginal contribution of employer j in the game $(N,g,V_g\mathbf{1}_j)$ is $\psi_j(N,g,V_g)$ and that of any worker $i \in W(j,g)$ is $\psi^j_i(N,g,V_g)$. Given $V_g(N)\mathbf{1}_j = v_j(g)$ and using the fact that Shapley Value for a TU game satisfies efficiency we get $\psi_j(N,g,V_g) + \sum_{i \in W(j,g)} \psi^j_i(N,g,V_g) = v_j(g)$. Since the selection of $j \in E$ was arbitrary, $V_g(N)\mathbf{1}_E = \sum_{j \in E} v_j(g) = \sum_{j \in E} \left\{\psi_j(N,g,V_g) + \sum_{i \in W(j,g)} \psi^j_i(N,g,V_g)\right\} = \sum_{k \in N} \psi_k(N,g,V_g)$ and we see that Pareto efficiency is satisfied. This completes the proof of Step 2.

Step 1 shows that any allocation rule that satisfies dummy property, symmetry, additivity and Pareto efficiency has to be unique. Step 2 shows that the Shapley Value satisfies dummy property, symmetry, additivity and Pareto efficiency. Hence the Shapley Value is the unique allocation rule that satisfies dummy property, symmetry, additivity and Pareto efficiency. □

4.1. **Component Balancedness.** In our setting it is a natural requirement that the allocation rule should not lead to cross-subsidization across principals and that each worker can receive a non-zero allocation only from those principals with which they are linked under g. This desirable property of the allocation rule is formally defined below.

Definition 4.2. An allocation rule $\phi : \mathcal{B}_g \to \Re^{J+I}$ is *component balanced* if for each $(N, g, V_g) \in \mathcal{B}_g$ and each $i \in W$ there exists $\{\{\phi_i^j(N, g, V_g)\}_{j \in E(i,g)}\} \in \Re^{|E(i,g)|}$ having the following properties.

(c1) For each $i \in W$, $\phi_i(N, g, V_g) = \sum_{j \in E(i,g)} \phi_i^j(N, g, V_g)$.

(c2) For each $j \in E$, $\phi_j(N, g, V_g) + \sum_{i \in W(j,g)} \phi_i^j(N, g, V_g) = v_j(g)$.

Proposition 4.3. For a graph g and its associated family of bipartite games $(N, g, V_g) \in \mathcal{B}_g$, the Shapley Value is component balanced.

PROOF: From Proposition 2.5 (ii) it follows that for a graph g and any associated bipartite game $(N, g, V_g) \in \mathcal{B}_g$, the Shapley Value of any worker $i \in W$ is given by $\psi_i(N, g, V_g) = \sum_{j \in E(i,g)} \psi_i^j(N, g, V_g)$ where $\psi_i^j(N, g, V_g) = \sum_{W_j \subseteq W(j,g) \setminus \{i\}} \left[\frac{|W_j \cup \{j\}|! |W(j,g) \setminus W_j \cup \{i\}|!}{|W(j,g) \cup \{j\}|!}\right] mc_i^j(W_j \cup \{j\})$. Thus, for each $i \in W$, the set $\{\{\psi_i^j(N, g, V_g)\}_{j \in E(i,g)}\} \in \Re^{|E(i,g)|}$ satisfies property (c1) of component balancedness. To verify property (c2) recall that the average marginal contribution of any employer j in the jPTU game $(N, g, V_g \mathbf{1}_j)$ is $\psi_j(N, g, V_g)$ and that of any worker $i \in W(j, g)$ is $\psi_i^j(N, g, V_g)$. Given $V_g(N)\mathbf{1}_j = v_j(g)$ and using the fact that Shapley Value for a TU game satisfies efficiency we get $\psi_j(N, g, V_g) + \sum_{i \in W(j,g)} \psi_i^j(N, g, V_g) = v_j(g)$. Since the selection of employer j was arbitrary, property (c2) of component balancedness also holds. □

Young [5] characterized the Shapley Value in the TU game set up using strong monotonicity. Hart and Mas-Colell [1] showed that in the TU game context there exists only one real valued function called potential function with the property that the payoff vector resulting from this potential function coincides with the Shapley Value. Both Young, and Hart and Mas-Colell used unanimity games to derive their results. Since we have introduced bipartite unanimity games we conjecture that the results of Young [5] and Hart and Mas-Colell [1] can be naturally extended to our context.

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