Lie algebroids 101

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1. Principal bundles

Definition: A *principal bundle* (P, M, G, π) consists of a *total space* P, a *base space* M, a *structure group* G which acts freely on P to the right, $P \times G \to P$, and a *projection* $\pi\colon\ P \to M$, such that;

(1) for $u, v \in P$,

$\pi(u) = \pi(v)$ if and only if $\exists\, g \in G : v = ug$. "*Orbits equal fibres*"

(2) M is covered by open sets U_i which are the domains of local sections $\sigma_i\colon\ U_i \to P$.

The gauge groupoid of a principal bundle

Let G act on $P \times P$ diagonally, and let $\Omega = \frac{P \times P}{G}$ be the quotient manifold, Denote orbits by $\langle v, u\rangle$.

Define groupoid structure:

$$\beta(\langle v, u\rangle) = \pi(v), \qquad \alpha(\langle v, u\rangle) = \pi(u),$$
$$\langle w, v'\rangle\, \langle v, u\rangle = \langle w, ug\rangle \text{ where } v' = vg.$$

2. ‘Transitive’ Lie groupoids

Current terminology defines a Lie groupoid Ω to be *transitive* if $(\beta, \alpha)\colon \Omega \to M \times M$ is a surjective submersion. (These were originally called *locally trivial Lie groupoids* and I usually keep to this.)

Define $\Omega_m = \alpha^{-1}(m)$. Since α is a surjective submersion, Ω_m is a regular submanifold.

Also, $\Omega_m = (\beta, \alpha)^{-1}(M \times \{m\})$. So Ω_m is a complete pre-image of a regular submanifold, and therefore the restriction $(\beta, \alpha)\colon \Omega_m \to (M \times \{m\})$ is a surjective submersion. Dropping the second factor, the restriction $\beta\colon \Omega_m \to M$ is a surjective submersion.

$\Omega_m^n := (\beta, \alpha)^{-1}(n, m)$ is a regular submanifold of Ω and hence of Ω_m.

$\Omega_m^m \times \Omega_m^m$ is a regular submanifold of $\Omega * \Omega$ and so the restriction of the multiplication is smooth. Hence Ω_m^m is a Lie group.

3. ‘Essentially equivalent’

Prop: Let $P(M, G, \pi)$ be a principal bundle. Choose $u_0 \in P$ and write $m_0 = \pi(u_0)$. Then the map

$$P \to \left.\frac{P \times P}{G}\right|_{m_0}, \quad u \mapsto \langle u, u_0 \rangle$$

is a diffeomorphism and the map

$$G \to \left.\frac{P \times P}{G}\right|_{m_0}^{m_0}, \quad g \mapsto \langle u_0 g, u_0 \rangle$$

is an isomorphism of Lie groups. Together they form an isomorphism of principal bundles over M.

Let Ω be a locally trivial Lie groupoid on M, and choose $m \in M$. Then the map

$$\frac{\Omega_m \times \Omega_m}{\Omega_m^m} \to \Omega, \quad \langle \eta, \xi \rangle \mapsto \eta \xi^{-1},$$

to Ω from the gauge groupoid of the vertex bundle at m is an isomorphism of Lie groupoids over M.

4. But...

The isomorphisms on the previous frame depend on choosing reference points.

For many purposes this does not matter.

However, care is required with automorphisms. Let $h \in G$ and define an automorphism (preserving M) of $P(M, G, \pi)$ by

$$u \mapsto uh, \; g \to h^{-1}gh.$$

The corresponding automorphism of $\frac{P \times P}{G}$ is the identity.

5. The Atiyah construction

Given a principal bundle $P(M, G, \pi)$, lift the action of G on P to TP and form the quotient manifold $\frac{TP}{G}$.

Thus elements of $\frac{TP}{G}$ are $\langle X \rangle$ where $X \in T_u(P)$ and $\langle X \rangle = \langle Y \rangle$ iff there exists $g \in G$ such that $Y = T(R_g)(X)$.

Make $\frac{TP}{G}$ a vector bundle over M. Define the projection to be $\langle X_u \rangle \mapsto \pi(u)$. For addition, suppose $\langle X \rangle$ and $\langle Y \rangle$ are in the same fibre. Then $X \in T_u P$ and $Y \in T_{ug} P$ for some g and the addition is:

$$\langle X \rangle + \langle Y \rangle = \langle T(R_g)(X) + Y \rangle.$$

Sections of $\frac{TP}{G}$ correspond to G-invariant vector fields on P. Given a G-invariant vector field $\mathscr{X}$, define $X \in \Gamma\left(\frac{TP}{G}\right)$ by

$$X(m) = \langle \mathscr{X}(u) \rangle$$

where $\pi(u) = m$.

6. The Atiyah construction, p2

Denote the G-invariant vector field corresponding to $X \in \Gamma\left(\frac{TP}{G}\right)$ by $\overline{X}$. Transfer the bracket of vector fields to $\Gamma\left(\frac{TP}{G}\right)$ by

$$\overline{[X, Y]} = [\overline{X}, \overline{Y}].$$

This is a Lie algebroid bracket on $\frac{TP}{G}$ with anchor π_* defined by

$$\pi_*(\langle X \rangle) = T(\pi)(X).$$

The kernel of π_* is $\frac{T^\pi P}{G}$. We'll return to this shortly.

7. Associated vector bundles and representations

Let $\rho\colon\ G \to GL(V)$ be a representation of G on a vector space V. The *associated vector bundle* $E = \frac{P\times V}{G}$ is defined as the quotient of $P \times V$ over the action of G by $(u, a)g = (ug, \rho(g^{-1})(a))$.

Proposition: *Associated vector bundles correspond to representations of the gauge groupoid.*

Definition: A *representation* of a Lie groupoid $\Omega \rightrightarrows M$ on a vector bundle $E \to M$ is a morphism of Lie groupoids $\Omega \to \Phi(E)$.

Let $\Omega = \frac{P\times P}{G}$ be the gauge groupoid of $P(M, G, \pi)$ and let $E = \frac{P\times V}{G}$ be the associated vector bundle above. Define a representation $\widetilde{\rho}$ of Ω on E by

$$\widetilde{\rho}(\langle v, u' \rangle)(\langle u, a \rangle) = \langle v, \rho(g^{-1})(a) \rangle$$

where $u' = ug$.

8. The Atiyah sequence

Since G acts on P (to the right), there is an infinitesimal action $\mathfrak{g} \to \mathscr{V}(P)$, $X \mapsto X^\dagger$. Define

$$P \times \mathfrak{g} \to T^\pi P, \ (u, X) \mapsto X^\dagger(u).$$

This induces an isomorphism of vector bundles $\frac{P\times\mathfrak{g}}{G} \to \frac{T^\pi P}{G}$ and so there is an exact sequence

$$\frac{P \times \mathfrak{g}}{G} \rightarrowtail \frac{TP}{G} \twoheadrightarrow TM.$$

This is the *Atiyah sequence* of $P(M, G, \pi)$ and is the Lie algebroid of $\frac{P\times P}{G}$.

9. Connections

There are several equivalent definitions of a connection in $P(M, G, \pi)$.

First, a *connection* in $P(M, G, \pi)$ is a horizontal invariant distribution $\mathscr{H}$ on P.

Horizontal means that $TP = \mathscr{H} \oplus T^{\pi} P$.

Invariant means that $\mathscr{H}_{ug} = T(R_g)(\mathscr{H}_u)$ for all $u \in P,\ g \in G$.

Because $\mathscr{H}$ is invariant, it quotients to $\frac{\mathscr{H}}{G} \subseteq \frac{TP}{G}$. Then $\frac{TP}{G} = \frac{\mathscr{H}}{G} \oplus \frac{T^{\pi} P}{G}$.

A complement to $\frac{T^{\pi} P}{G}$ in $\frac{TP}{G}$ corresponds to a right-inverse $\gamma\colon\ TM \to \frac{TP}{G}$ such that the image of γ is $\frac{\mathscr{H}}{G}$.

The converse is also true: a right inverse to $\frac{TP}{G} \to TM$ defines an invariant horizontal distribution on P.

10. Connections, p2

Definition: A Lie algebroid is *transitive* is the anchor is surjective,

A *connection* in a transitive Lie algebroid A is a right-inverse $\gamma\colon TM \to A$ to the anchor,

The *curvature* of a connection γ is $R_\gamma\colon TM \oplus TM \to L$ defined by

$$R_\gamma(X, Y) = \gamma[X, Y] - [\gamma(X), \gamma(Y)]$$

for $X, Y \in \mathscr{V}(M)$.

For a connection in $P(M, G, \pi)$, curvature is often defined as a 2-form Ω on P with values in $\mathfrak{g}$. That is, $\Omega\colon TP \oplus TP \to P \times \mathfrak{g}$.

This Ω is equivariant (or ‘pseudo-tensorial’), so it quotients down to $\frac{TP}{G} \oplus \frac{TP}{G} \to \frac{P\times\mathfrak{g}}{G}$.

Ω vanishes when one or more arguments is in $T^\pi P$ (Ω is ‘horizontal’), so it descends to $TM \oplus TM \to \frac{P\times\mathfrak{g}}{G}$.

11. Connections, p3

All the 'infinitesimal' theory of connections can be conducted in an abstract transitive Lie algebroid: the Bianchi identities, covariant derivatives, ...

The Lie algebroid $A\Phi(E)$ of the frame groupoid of a vector bundle E has sections which are differential operators $D\colon \Gamma E \to \Gamma E$ of order $\leqslant 1$ and are such that there exists a vector field X on M such that

$$D(f\mu) = fD(\mu) + X(f)\mu,$$

for $\mu \in \Gamma E$, $f \in C^\infty(M)$.

The standard concept of connection in a vector bundle assigns to each $X \in \mathscr{V}(M)$ a differential operator ∇_X such that $\nabla_X(f\mu) = f\nabla_X(\mu) + X(f)\mu$, So it is a map $\mathscr{V}(M) \to \Gamma A\Phi(E)$. Further, $\nabla_{fX}(\mu) = f\nabla_X(\mu)$ for all $f \in C^\infty(M)$, so ∇ is a vector bundle map $TM \to A\Phi(E)$ and is a connection in the Lie algebroid $A\Phi(E)$.

12. Cohomology

de Rham cohomology for a manifold M is defined in terms of the cochain complex of real k-forms and the differential

$$d\omega(X_1,\dots,X_{k+1}) = \sum_{r=1}^{k+1}(-1)^{r+1}X_r(\omega(X_1,\dots,\hat{X}_r,\dots,X_{k+1})) + \sum_{r<s}(-1)^{r+s}\omega([X_r,X_s],X_1,\dots,\hat{X}_r,\dots,\hat{X}_s,\dots,X_{k+1})$$

Cohomology of a Lie algebroid A on M with coefficients in a representation $\rho\colon A \to A\Phi(E)$ can be defined by a similar formula:

$$d\omega(X_1,\dots,X_{k+1}) = \sum_{r=1}^{k+1}(-1)^{r+1}\rho(X_r)(\omega(X_1,\dots,\hat{X}_r,\dots,X_{k+1})) + \sum_{r<s}(-1)^{r+s}\omega([X_r,X_s],X_1,\dots,\hat{X}_r,\dots,\hat{X}_s,\dots,X_{k+1})$$

13. Cohomology, p3

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{r=1}^{k+1} (-1)^{r+1} \rho(X_r)(\omega(X_1, \ldots, \hat{X}_r, \ldots, X_{k+1}))$$
$$+ \sum_{r<s} (-1)^{r+s} \omega([X_r, X_s], X_1, \ldots, \hat{X}_r, \ldots, \hat{X}_s, \ldots, X_{k+1})$$

In this formula, the ω are skew-symmetric forms $A \oplus \cdots \oplus A \to E$; that is, they are $C^\infty(M)$-multilinear. This defines the *Lie algebroid cohomology of A with coefficients in ρ (or E).*

Particular cases:

If $A = TM$ then a representation of A on E is a flat connection in E and gives a local system of coefficients in E.

If $A = \frac{TP}{G}$ and $E = \frac{P \times V}{G}$ then the cohomology is an equivariant de Rham cohomology of P with coefficients in V.

The cases where A arises from a Poisson structure or is an action Lie algebroid also yield known cohomologies.

14. References

All this material can be found in

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids,* London Mathematical Society Lecture Note Series, no. 213, Cambridge University Press, 2005.

Chapters 1, 3 and 7.