Affinoids and double groupoids

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1. Affinoids

Suppose given a manifold M and two surjective submersions $h\colon M \to Q_H$ and $v\colon M \to Q_V$ to manifolds Q_H and Q_V. Call a pair $(x, y) \in M^2$ *vertical* if $h(x) = h(y)$; that is, if it belongs to

$$R(h) = \{(x, y) \in M^2 \mid h(x) = h(y)\}.$$

Similarly say that $(x, y) \in M^2$ is *horizontal* if $v(x) = v(y)$; that is, if it belongs to $R(v)$.

Further suppose that we are given a regular submanifold $\Lambda \subseteq M^4$, whose elements are called *parallelograms*; display $(x, y, z, w) \in \Lambda$ as

$$\begin{array}{ccc} w & \text{———} & z \\ | & & | \\ y & \text{———} & x \end{array}$$

Assume that if $(x, y, z, w) \in \Lambda$ then (x, z) and (y, w) are vertical and (y, x), (w, z) are horizontal. Axioms follow:

2. Axioms

Axiom I. *Given any* $(x, y, z) \in M^3$ *such that* (y, x) *is horizontal and* (x, z) *is vertical, there exists a unique* $w \in M$ *such that* $(x, y, z, w) \in \Lambda$*.*

Now define two elements (x, z), $(y, w) \in R(h)$ to be *v-parallel* if $(x, y, z, w) \in \Lambda$. Similarly, define (y, x), $(w, z) \in R(v)$ to be *h-parallel* if $(x, y, z, w) \in \Lambda$.

Axiom II. *The relations of* v*-parallelism and* h*-parallelism just defined are equivalence relations on* $R(h)$ *and* $R(v)$ *respectively.*

Definition: (Weinstein, 1990) An *affinoid structure* on a manifold M consists of two surjective submersions $h\colon M \to Q_H$ and $v\colon M \to Q_V$, together with a regular submanifold $\Lambda \subseteq M^4$, which satisfy Axioms I and II above and for which the bijection $\Lambda \to R(h) * R(v)$ is a diffeomorphism.

Call Q_H and Q_V the *bases* of the affinoid structure. A manifold equipped with an affinoid structure is an *affinoid space.*

3. Simple examples

- $M = Q_V \times Q_H$ for any two manifolds Q_V, Q_H, and the projections.

- $M = G$ a Lie group. Let $Q_H = Q_V = \{\cdot\}$ and define (x, y, z, w) to be in Λ if $w = yx^{-1}z$.

- M the *total* space of a principal bundle $P(B, G, \pi)$ with $h = \pi\colon\ P \to B$ and v the constant map $P \to \{\cdot\}$. Parallograms in Λ are all

$$\begin{array}{ccc} y & \text{———} & xg^{-1} \\ | & & | \\ yg & \text{———} & x \end{array}$$

where $x, y \in P$ and $g \in G$.

4. Groupoids G_h and G_v

Denote the set of equivalence classes of horizontal pairs modulo h-parallelism by G_h and the set of equivalence classes of vertical pairs modulo v-parallelism by G_v, and denote the projections by $\widetilde{h}\colon R(v) \to G_h$ and $\widetilde{v}\colon R(h) \to G_v$.

The kernel pair of $\widetilde{h}$ is precisely $\Lambda \subseteq R(h) \times R(h)$. Since Λ is a submanifold of $R(h) \times R(h)$ and the projection $\Lambda \to R(h),\ (x, y, z, w) \mapsto (x, z),$ is a surjective submersion, it follows that G_h has a manifold structure making $\widetilde{h}$ a submersion.

Then G_h is a Lie groupoid over Q_H, with $\widetilde{h}$ a morphism over h.

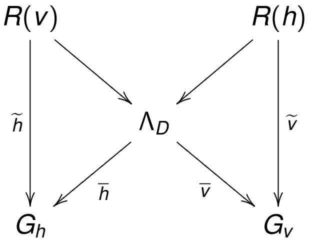
Likewise for the vertical structure.

5. Diagonal structure

Λ itself has a groupoid structure, on base M. Denote this Λ_D.

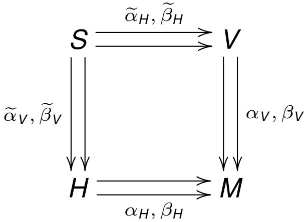
Then $\overline{h}$ and $\overline{v}$ are groupoid morphisms from Λ_D to G_h and G_v. Both are fibrations.

What are the kernels ?



6. Double Lie groupoids

A *double Lie groupoid* consists of a manifold S equipped with two Lie groupoid structures on bases H and V, each of which is a Lie groupoid on base M, such that the structure maps (source, target, multiplication, identity, inversion) of each groupoid structure on S are morphisms with respect to the other, and such that the *double source map* $\alpha_2\colon\ S \to H \times_M V,\ s \mapsto (\widetilde{\alpha}_V(s), \widetilde{\alpha}_H(s))$ is a surjective submersion.



A double Lie groupoid is *vacant* if the double source map is a diffeomorphism $\alpha_2\colon\ S \to H \times_M V$.

7. Matched pairs

Let H and V be Lie groups. They form a *matched pair* if H acts on V to the left and V acts on H to the right, such that

$$(h_2 h_1)^v = h_2^{({}^{h_1}v)} h_1{}^v, \qquad {}^h(v_2 v_1) = {}^h v_2 \, {}^{h^{v_2}} v_1.$$

Theorem: A vacant double groupoid, the side groupoids of which are groups, induces a matched pair strucure on them. Conversely, given a matched pair structure on two groups, there is an associated vacant double Lie groupoid.

A vacant double groupoid admits a diagonal strucure $S_D \rightrightarrows \cdot$ which is the bicrossproduct of the side groups. The side groups are injected into the diagonal group as subgroups.

All this goes through for general vacant double groupoids abd matched pairs of groupoids.

8. References

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