An integrable subbdle $E \subset TM$ is a foliation of M

E integrable if $E = \ker df$ locally, f ~~$: M \to N$~~ submersion

A foliation gives submersions $s_\alpha : U_\alpha \to \mathbb{R}^q$ s.t.

$$\ker ds_\alpha \,|\, U_\alpha \cap U_\beta = \ker ds_\beta \,|\, U_\alpha \cap U_\beta \quad \forall \alpha, \beta$$

Moreover, if $x \in U_\alpha \cap U_\beta$, it has a nbd $W \subset U_\alpha \cap U_\beta$ s.t.

$$s_\alpha^{-1} s_\beta (x) \cap W = s_\beta^{-1} s_\alpha (x) \cap W$$

Conversely, such a structure determines a unique foli on M

Not every subbdle $E \subset TM$ is integrable

Example. $M = \mathbb{R}^3 - 0$

$\varphi = x\,dy + y\,dz + z\,dx$ nowhere zero on M

So $E = \ker \varphi$ is a 2-dim subbdle of TM

Suppose E is integrable

Then $\exists$ submersions $s_\alpha : U_\alpha \to \mathbb{R}$ s.t.

$$\ker ds_\alpha = \ker \varphi \,|\, U_\alpha \quad \forall \alpha$$

So $\varphi = h\, ds_\alpha$, $h : U_\alpha \to \mathbb{R}$ C^∞ nowhere zero

on U_α $\quad d\varphi = dh \wedge ds_\alpha = \frac{dh}{h} \wedge h\, ds_\alpha = \eta \wedge \varphi$

Therefore $\varphi \wedge d\varphi = \varphi \wedge \eta \wedge \varphi = 0$

So $\varphi \wedge d\varphi = 0$ identically on M

On the other hand

$$\varphi \wedge d\varphi = (x + y + z)\, dx \wedge dy \wedge dz$$

Not zero.

Theorem (Deshma, Clebsch, Frobenius)

$E \subset TM$ locally given as a simultaneous Ker of everywhere independent 1-forms $\omega_1, \dots, \omega_q$

that is $E_x = \{ v \in T_x(M) : \omega_1(v) = \dots = \omega_q(v) = 0 \}$

$x \in$ domains of $\omega_1, \dots, \omega_q$

Then E is integrable iff each $d\omega_i \in$ Ideal generated by $\omega_1, \dots, \omega_q$

Equivalently

Theorem E integrable iff E is involutive

$$X, Y \in \Gamma(E) \Rightarrow [X, Y] \in \Gamma(E)$$

These give a complete and satisfying answer to local question of integrability of $E \subset TM$

Global Problem

$E \subset TM$ subbundle

Does there exist an integrable subbdle

$E' \subset TM$

such that $E \cong E'$ as vector bundles?

No complete answer is given so far.

In this lecture we discuss topological obstructions to the existence of E' in terms of characteristic classes of E.

Theorem (Deahna, Clebsch, Frobenius)

$E \subset TM$ locally given as a simultaneous ker of everywhere independent 1-forms $\omega_1, \dots, \omega_q$

that is $E_x = \{ v \in T_x(M) : \omega_1(v) = \cdots = \omega_q(v) = 0 \}$

$x \in$ domain of $\omega_1, \dots, \omega_q$

then E is integrable iff each $d\omega_i \in$ Ideal generated by $\omega_1, \dots, \omega_q$

Equivalently

Theorem E integrable iff E is involutive

$$X, Y \in \Gamma(E) \Rightarrow [X, Y] \in \Gamma(E)$$

These give a complete and satisfying answer to local question of integrability of $E \subset TM$

Global Problem

$E \subset TM$ subbundle

Does there exist an integrable subbdle

$E' \subset TM$

such that $E \simeq E'$ as vector bundles?

No complete answer is given so far.

In this lecture we discuss topological obstructions to the existence of E' in terms of characteristic classes of E.

$\mathcal{H}$ separable real Hilbert space
BGL_n set of n-dim subspaces of $\mathcal{H}$ topologized by a metric which is distance between planes

<u>Classification Thm</u>

X paracompact $\quad \mathrm{Vect}_n(X) \longleftrightarrow [X, BGL_n]$

<u>Theorem</u> Modulo torsion

$$H^*(BGL_n) \cong \mathbb{Z}[p_1, \dots, p_{[\frac{n}{2}]}] \quad \text{poly ring}$$

$p_i \in H^{4i}(BGL_n)$ ith Pontryagin class canonically defined

As an algebra over $\mathbb{R}$, $H^*(BGL_n; \mathbb{R}) = \mathbb{R}[p_1, \dots, p_{[\frac{n}{2}]}]$

$$\pi : E \longrightarrow M \qquad g_E : M \longrightarrow BGL_n$$

$$p_i(E) = g_E^*\left(H^*(BGL_n; \mathbb{R}) \subset H^*(M; \mathbb{R})\right)$$

$\mathrm{Pont}^*(E) = g_E^*\left(H^*(BGL_n; \mathbb{R}) \subset H^*(M, \mathbb{R})\right)$
real Pontryagin ring of E

<u>THEOREM</u> $E \subset TM$ integrable, $Q = TM/E$ $\dim Q = q$
Then $\mathrm{Pont}^k(Q) = 0$ if $k > 2q$
This is a global integrability condition

Reformulation

Total Pontryagin class $P(E) = 1 + p_1(E) + \cdots + p_{[\frac{n}{2}]}(E)$

$\in H^{**}(M;\mathbb{R})$ ring of formal infinite series

$a_0 + a_1 + \cdots + a_n + \cdots$, $a_i \in H^i(M;\mathbb{R})$

$P(E)$ is invertible, as its leading term 1 invertible $\in H^{**}(M;\mathbb{R})$

If $E' \subset E$ subbundle, the basic "duality" formula

$$P(E/E') = P(E).P(E')^{-1} \text{ holds}$$

Therefore the Pontryagin classes of E/E' depend only on isomorphism classes of E and E', and not on the embedding $E' \subset E$

THEOREM If $E \subset T(M)$ is isomorphic to an integrable subbundle $E' \subset TM$ and $Q = TM/E$ then $\mathrm{Pont}^k(Q) = 0$ for $k > 2q$ where $q = \dim Q$.

Connection of vector bundle $\pi : E \to M$ C^∞

$\Gamma(E)$ vector space over $\mathbb{R}$ and a module over the alg $C^\infty(M, \mathbb{R})$

$$\text{Write } \mathcal{X} = \Gamma(TM)$$

Def. A connection of E is an $\mathbb{R}$-bilinear map

$\nabla : \mathcal{X} \times \Gamma(E) \to \Gamma(E)$ writing $\nabla(X, \sigma) = \nabla_X(\sigma)$

(i) $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X(\sigma)$ (2) $\nabla_{fX}(\sigma) = f\nabla_X(\sigma)$

Existence of ∇

If $U_\alpha \subset M$ open, $(\sigma_\alpha^1, \ldots, \sigma_\alpha^q)$ smooth frame on U_α

$\sigma_\alpha^i \in \Gamma(E|U_\alpha)$, for each $x \in U_\alpha$ $\pi^{-1}(x) \subset E$ has basis

$$\sigma_\alpha^1(x), \ldots, \sigma_\alpha^q(x)$$

then there is a $q \times q$ matrix $\omega^\alpha = (\omega_{ij}^\alpha)$ of 1-forms on U_α

$$\nabla_X(\sigma_\alpha^i) = \sum_{j=1}^{q} \omega_{ij}^\alpha(X)\, \sigma_\alpha^j$$

Consider ∇ as $C^\infty(M,\mathbb{R})$ - linear map of modules

$$\nabla : \mathfrak{X} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(E)) \qquad X \longrightarrow \nabla_X$$

Not a homo of Lie algebras, $[\nabla_X, \nabla_Y] \neq \nabla_{[X,Y]}$

<u>Def</u>. The curvature $K : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(E))$

$$K(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

For a C^∞ local frame σ^i_α on U_α, there is a $q \times q$ matrix $\Omega^\alpha = (\Omega^\alpha_{ij})$, $\Omega^\alpha_{ij} \in \Lambda^2(M)$ such that

$$K(X,Y)(\sigma^i_\alpha) = \sum_{j=1}^{q} \Omega^\alpha_{ij}(X,Y)\, \sigma^j_\alpha .$$

Ω^α and Ω^β on $U_\alpha \cap U_\beta$ are related by

$$\Omega^\alpha = g_{\alpha\beta}\, \Omega^\beta\, g_{\alpha\beta}^{-1}$$

$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_q$ are cocycles for the bdle $\pi : E \longrightarrow M$.

<u>Lemma</u> $\Omega^\alpha = d\omega^\alpha - \omega^\alpha \cdot \omega^\alpha$ (matrix product)

Polynomial fn. $\varphi : gl_q \longrightarrow \mathbb{R}$ is invariant if

$$\varphi(m) = \varphi(g m g^{-1}) \quad \forall g \in GL_q,\ m \in gl_q$$

Invariant polynomials form an algebra over $\mathbb{R}$
(sums and products of inv polynomials are invariant)

Basic invariant polynomials are $\Sigma_0, \Sigma_1, \Sigma_2, \cdots$

$$\Sigma_p(m) = \mathrm{trace}(m^p)$$

they are generators of the alg of invariant polynomials

$\pi : E \to M$ with connection ∇, K curvature of ∇

φ an inv polynomial of deg r, Ω^α curv matrix of 2-forms on U_α

then $\varphi(\Omega^\alpha) \in \Lambda^{2r}(U_\alpha)$. Moreover on $U_\alpha \cap U_\beta$

$$\varphi(\Omega^\beta) = \varphi(g_{\alpha\beta}\Omega^\beta g_{\alpha\beta}^{-1}) = \varphi(\Omega^\alpha).$$

Thus the 2-forms fit together to give an ele $\varphi(K) \in \Lambda^{2r}(M)$

$\varphi(K)$ is closed $\quad d(\varphi(K)) = 0 \quad [\varphi(K)] \in H^{2r}_{DR}(M)$

<u>Def.</u> If φ an inv polynomial of deg r

$$\varphi(E) = [\varphi(K)] \in H^{2r}_{DR}(M)$$

is called the Pontryagin class of E corres to φ.

$\mathrm{Pont}^*(E) \subset H^*_{DR}(M)$ graded subalg of all such Pontryagin classes.

<u>Prop 1</u> The cohomology class $\varphi(E)$ is independent of the choice of the connection ∇ of E

<u>Pf.</u> ∇^0, ∇^1 two connections of E with curvatures K^0, K^1

Consider vector bdle $\pi \times \mathrm{id} : E \times \mathbb{R} \to M \times \mathbb{R}$

Define a connection $\tilde{\nabla}$ on this bundle:

Take a section $\sigma' \in \Gamma(E \times \mathbb{R})$ that is constant in $\mathbb{R}$-direction, that is $\tilde{\nabla}_{\partial/\partial t}(\sigma') = 0$

and take $X \in T_{(x,t)}(M \times \{t\})$. Define

$$\tilde{\nabla}_X(\sigma') = (1-t)\nabla^0_X(\sigma') + t\,\nabla^1_X(\sigma')$$

This can be extended to all sections of $E \times \mathbb{R}$, because every section σ is a function-linear combination of sections σ' constant in $\mathbb{R}$-direction.

It is easy to check that $\tilde{\nabla}$ is a connection.

Thus if $\tilde{K}$ = curv of $\tilde{\nabla}$, $[\varphi(\tilde{K})] \in H^{2r}_{DR}(M \times \mathbb{R})$.

Now if $i_0, i_1 : M \to M \times \mathbb{R}$, then $i_0^* = i_1^*$

$$i_0^*[\varphi(\tilde{K})] = [\varphi(K^0)], \quad i_1^*[\varphi(\tilde{K})] = [\varphi(K^1)] \quad \square$$

Prop 2 $\mathrm{Pont}^j(E) = 0$ if j is not a ~~multiple of 4~~

Pf. clear if j is not divisible by 2

Put a smooth inner product on E

(first define locally then piece together with a smooth PU)

Define a connection of E such that

$$(*) \qquad X\langle \sigma_1, \sigma_2\rangle = \langle \nabla_X \sigma_1, \sigma_2\rangle + \langle \sigma_1, \nabla_X \sigma_2\rangle \quad \forall\, X \in \mathfrak{X}, \ \sigma_1, \sigma_2 \in \Gamma(E)$$

How to do this?

on U_α choose a smooth orthonormal frame σ_α^i by Gram-Schmidt

then define ∇^α on U_α by requiring $\nabla_X^\alpha(\sigma_\alpha^i) = 0$, $i = 1, \dots, q$

It is easy to check ∇^α has the property $(*)$

Now obtain ∇ by piecing together these local connections by a C^∞ PU

It is an easy exercise that the curvature K of this ∇ has the property

$$\langle K(X,Y)\sigma_1, \sigma_2\rangle = \langle \sigma_1, -K(X,Y)\sigma_2\rangle.$$

As consequence the matrix Ω^α is anti-symmetric

Therefore if $n > 0$ is odd, then $(-\Omega^\alpha)^n$ is anti-symmetric,

hence $\Sigma_n(\Omega^\alpha) = \mathrm{trace}((-\Omega^\alpha)^n) = 0$. Thus $\Sigma_n(K) = 0$.

But any inv poly of odd deg is a linear combination of polynomials of the form $\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_k}$ in which some of the indices i_j are odd.

Therefore $\deg(\phi)$ is odd. This implies $\phi(K) = 0$ $\square$

This enables us to define the total Pontryagin class of E

$$P(E) = 1 + P_1(E) + \cdots + P_{[\frac{n}{2}]}(E)$$

$$= \det\left[I + \frac{\sqrt{-1}}{2\pi} K\right], \quad p_j(E) \in H_{DR}^{4i}(E).$$

No imaginary components ~~are~~ arise because of the Prop.

From this formula it is easy to prove

$$P(E \oplus E') = P(E)\cdot P(E').$$

Characteristic classes and integrability

Suppose $E \subset TM$ integrable, $Q = TM/E$ with fibre dim q

Want to define a connection of Q (basic connection)

Since E integrable, for all $X, Y \in \Gamma(E)$, $[X, Y] \in \Gamma(E)$

Let $\pi : TM \to Q$ canonical projection

$$0 \longrightarrow E \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0$$

If $Z \in \Gamma(Q)$ then $Z = \pi(\tilde{Z})$ for some $\tilde{Z} \in \Gamma(TM) = \mathfrak{X}$

$\tilde{Z}$ is well-defined modulo elements of $\Gamma(E)$.

For $X \in \Gamma(E)$ and $Z \in \Gamma(Q)$, $\pi[X, \tilde{Z}] \in \Gamma(Q)$

(Note $[X, \tilde{Z}] \in \Gamma(TM)$ and it is the Lie derivative $L_X \tilde{Z}$ of $\tilde{Z}$ w.r.t. X. But $\pi[X, \tilde{Z}] \notin \Gamma(E)$)

Write this as $\nabla'_X(Z)$

$$\nabla' : \Gamma(E) \times \Gamma(Q) \longrightarrow \Gamma(Q)$$
$$(X, Z) \longrightarrow \pi[X, \tilde{Z}]$$

Easy to verify

(1) $\nabla'_X(fZ) = X(f)Z + f\,\nabla'_X(Z)$

(2) $\nabla'_{fX}(Z) = f\,\nabla'_X(Z)$

(i) comes from $L_X(f\tilde{Z}) = X(f)\tilde{Z} + L_X(\tilde{Z})$

$$\Rightarrow \pi L_X(f\tilde{Z}) = X(f)Z + L_X(\tilde{Z})$$

But this is not a full-fledged connection of Q because the variable X is restricted to range over $\Gamma(E) \subset \mathfrak{X}$, it is not any vector field $\in \mathfrak{X}$.

To complete ∇' to a connection of Q, fix a Riem metric on TM and split TM into a direct sum of E and an orthogonal complement bdle to E in TM. This complement is iso to Q. So by the choice of Riemannian metric, we get an iso

$$TM \cong E \oplus Q.$$

Let $\overline{\nabla}$ be any connection of Q

For $X \in \mathfrak{X} = \Gamma(E) \oplus \Gamma(Q)$, write $X = X_E \oplus X_Q$

then define $\quad \nabla_X Z = \nabla'_{X_E}(Z) + \overline{\nabla}_{X_Q}(Z) \quad \forall Z \in \Gamma(Q)$

It is trivial to check that ∇ is a connection of Q.

Note if $X \in \Gamma(E) \subset \mathfrak{X} = \Gamma(E) \oplus \Gamma(Q)$, then $X_Q = 0$ and $\nabla_X(Z) = \nabla'_{X_E}(Z)$, we get the previous definition.

<u>Def</u>. A basic connection ∇ of Q is one such that

$$\nabla_X(Z) = \pi(X, \tilde{Z}) \quad \forall X \in \Gamma(E)$$

where $\tilde{Z} \in \mathfrak{X}$ is such that $\pi(\tilde{Z}) = Z$.

<u>Lemma 1</u> Under the assumption that E is integrable there is a basic connection of Q.

<u>Lemma 2</u> If ∇ is a basic connection of Q, then its curvature K satisfies

$$K(X, Y) = 0 \quad \forall X, Y \in \Gamma(E).$$

<u>Pf</u>. Let $Z \in \Gamma(Q)$ and $\tilde{Z} \in \mathfrak{X}$ with $\pi(\tilde{Z}) = Z$. Then

$$K(X,Y)(Z) = \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) - \nabla_{[X,Y]}(Z)$$

$$= \nabla_X(\pi(Y, \tilde{Z})) - \nabla_Y(\pi(X, \tilde{Z})) - \pi\big[[X,Y], \tilde{Z}\big]$$

can choose $\widetilde{\pi(X, \tilde{Z})} = [X, \tilde{Z}]$ and $\widetilde{\pi(Y, \tilde{Z})} = [Y, \tilde{Z}]$

$$= \pi[X, [Y, \tilde{Z}]] - \pi[Y, [X, \tilde{Z}]] - \pi\big[[X,Y], \tilde{Z}\big]$$

$$= 0 \quad \text{by Jacobi identity.} \qquad \square$$

Lemma 3. $U_\alpha \subset M$ simultaneously common trivializing nbd for Q and E

σ_α smooth frame on U_α for Q

$I_\alpha(E)$ ideal in $\Lambda^*(U_\alpha)$ generated by 1-forms which vanish on $\Gamma(E|U_\alpha)$

Let Ω^α curvature matrix associated to the frame σ_α by a basic connection of Q.

then each entry $\Omega^\alpha_{ij} \in I_\alpha(E)$.

Pf. $E|U_\alpha$ can be described as

$$E = \{ X \in TM : \omega_1(X) = \cdots = \omega_q(X) = 0 \}$$

where $\omega_1, \ldots, \omega_q$ are 1-forms linearly independent at each pt of U_α

In particular, $I_\alpha(E)$ is generated by $\omega_1, \ldots, \omega_q$

Complete this to ~~the~~ a basis $\omega_1, \ldots, \omega_q, \omega_{q+1}, \ldots, \omega_n$ of 1-forms on M

the 1-forms $\omega_{q+1}, \ldots, \omega_n$ restrict to a basis of E^*_x, $x \in U_\alpha$.

Then for a non-trivial form

$$\omega = \sum_{q+1 \le i < j \le n} f_{ij}\, \omega_i \wedge \omega_j$$

there exist $X, Y \in \Gamma(E|U_\alpha)$ such that

$$\omega(X, Y) \neq 0.$$

therefore, it follows by Lemma 2 that

each $\Omega^\alpha_{ij} \in I_\alpha(E)$. $\square$

PROOF OF THE MAIN THEOREM

Any $\omega \in I_\alpha(E)$ is of the form

$$\omega = \sum_{i=1}^{q} \theta_i \wedge \omega_i, \quad \theta_i \in \Lambda^*(U_\alpha)$$

and $\omega^{q+1} = 0$, thus $I_\alpha(E)^{q+1} = 0$.

If φ is an inv poly of deg $> q$, then by Lemma 3 and this remark

$$\varphi(\Omega^\alpha) = 0$$

Therefore $\varphi(K) = 0$ for $\deg \varphi > q$. $\square$

This theorem follows immediately by our def of Pontryagin algebra.