For the case of complex vector bdles, chern classes can be defined in complex de Rham cohomology $H^*_{DR}(M;\mathbb{C})$ by exactly the same techniques (using smooth complex p-forms $\Lambda^p_{\mathbb{C}}(M)$. If E has complex dim n, one obtains

$$c(E) = 1 + c_1(E) + \cdots + c_n(E), \quad c_i(E) \in H^{2i}_{DR}(M,\mathbb{C}),$$

and again $c(E \oplus E') = c(E)\cdot c'(E)$.

analytic

For complex holomorphic bdles on complex holomorphic manifolds, holomorphic connections do not generally exist, although they can be defined locally. As a consequence, there is if one uses these local holomorphic connections, generally no way of producing forms in the de Rham complex representing chern classes. But it is possible to define in this way representative cocycles in the double complex $\check{C}^*(\mathcal{U}, \Lambda^*_c(M)) = K^*(\mathcal{U})$

Let $\check{H}^*(M,\mathbb{R})$ be the standard Čech cohomology of M with coeffs in $\mathbb{R}$

Theorem (de Rham) There is a canonical isomorphism of graded algebras

$$\theta_M : H^*_{DR}(M) \longrightarrow \check{H}^*(M;\mathbb{R})$$

Moreover, if $f: M \to N$ smooth, then

$$\begin{tikzcd} H^*_{DR}(N) \arrow[r, "\theta_N"] \arrow[d, "f^*"'] & \check{H}^*(N;\mathbb{R}) \arrow[d, "f^*"] \\ H^*_{DR}(M) \arrow[r, "\theta_M"'] & \check{H}^*(M;\mathbb{R}) \end{tikzcd}$$

Commutative.

Pf (due to André Weil) $\mathcal{U}$ open covering of M

For integers (≥ 0) (p,q), write

$$K^{p,q}(\mathcal{U}) = \check{C}^p(\mathcal{U}; A^q)$$

which is the pth Čech cochain module with values in the q-forms.

Each $c \in K^{p,q}(\mathcal{U})$ is a function which assigns to each ordered $(p+1)$-tuple $(U_{\alpha_0}, \dots, U_{\alpha_p})$, $U_{\alpha_i} \in \mathcal{U}$, a q-form

$$c_{\alpha_0 \alpha_1 \cdots \alpha_p} \in \Lambda^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$$

$(c = 0 \text{ if } U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p} = \phi)$

The Čech coboundary

$$\delta : K^{p,q}(\mathcal{U}) \longrightarrow K^{p+1,q}(\mathcal{U}) \text{ is given by}$$

$$|\delta(c)|_{\alpha_0 \cdots \alpha_{p+1}} = \sum (-1)^i \varphi_{\alpha_i}(c_{\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1}})$$

where φ_{α_i} is the restriction map

$$\varphi_{\alpha_i} : \Lambda^q(U_{\alpha_0} \cap \cdot \cap \hat{U}_{\alpha_i} \cap \cdots \cap U_{\alpha_{p+1}}) \to \Lambda^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_{p+1}})$$

As usual $\delta^2 = 0$

The de Rham coboundary

$$d : K^{p,q}(\mathcal{U}) \longrightarrow K^{p,q+1}(\mathcal{U}) \text{ is given by}$$

$$|d(c)|_{\alpha_0, \dots, \alpha_p} = d(c_{\alpha_0, \dots, \alpha_p}).$$

Then $d\delta = \delta d$ on $K^{*,*}(\mathcal{U})$.

Reference

A. Weil, Sur les théorèmes de de Rham, Comment. Math. Helv. 26 (1952), 119 – 145

set $D' = \delta$, $D'' = (-1)^q d$, $D = D' + D''$

then $D^2 = 0$

Set $K^n(\mathcal{U}) = \bigoplus_{p+q=n} K^{p,q}(\mathcal{U})$

then $D : K^n(\mathcal{U}) \longrightarrow K^{n+1}(\mathcal{U})$

Define a multiplication

$$K^{p,q}(\mathcal{U}) \times K^{r,s}(\mathcal{U}) \longrightarrow K^{p+r,\, q+s}(\mathcal{U}) \text{ by}$$

$$(\omega\eta)_{\alpha_0 \cdots \alpha_{p+r}} = (-1)^{qr}\, \omega_{\alpha_0 \cdots \alpha_p} \cdot \eta_{\alpha_p, \cdots, \alpha_{p+r}}$$

(dot indicates exterior multiplication of forms)

This also defines a multiplication

$$K^n(\mathcal{U}) \times K^m(\mathcal{U}) \to K^{n+m}(\mathcal{U}).$$

Note if $\omega \in K^{p,q}$, $\eta \in K^{r,s}$, then

$$D'(\omega\eta) = \delta((-1)^{rq}\, \omega \cdot \eta).$$

using this one can show

$$D'(\omega\eta) = D'(\omega)\eta + (-1)^{p+q}\omega D'(\eta)$$

and $D''(\omega\eta) = D''(\omega)\eta + (-1)^{p+q}\, \omega D''(\eta)$

therefore $D : K^*(\mathcal{U}) \to K^*(\mathcal{U})$ is an antiderivation —

Note Ker of $D' : K^{0,q}(\mathcal{U}) \to K^{1,q}(\mathcal{U})$ is $\Lambda^q(M)$

and Ker of $D'' : K^{p,0}(\mathcal{U}) \to K^{p,1}(\mathcal{U})$ is $\check{C}^p(\mathcal{U}; \mathbb{R})$.

$$\delta = D' : K^{0,q}(\mathcal{U}) \longrightarrow K^{1,q}(\mathcal{U})$$

c

$$c : U_{\alpha_0} \longrightarrow \Lambda^q(U_{\alpha_0}) \quad c_{\alpha_0}$$

$$c : U_{\alpha_1} \longrightarrow \Lambda^q(U_{\alpha_1}) \quad c_{\alpha_1}$$

$$(\delta c)_{\alpha_0,\alpha_1} = \varphi_{\alpha_0}(c_{\alpha_0}) - \varphi_{\alpha_1}(c_{\alpha_1})$$

q-form

$$\left.\begin{aligned} \varphi_{\alpha_0} &: \Lambda^q(U_{\alpha_0}) \longrightarrow \Lambda^q(U_{\alpha_0} \cap U_{\alpha_1}) \\ \varphi_{\alpha_1} &: \Lambda^q(U_{\alpha_1}) \longrightarrow \Lambda^q(U_{\alpha_0} \cap U_{\alpha_1}) \end{aligned}\right\} \text{restrictions}$$

$$\ker \delta = \Lambda^q(M)$$

$$D'' = (-1)^0 d = d : K^{p,0}(\mathcal{U}) \longrightarrow K^{p,1}(\mathcal{U})$$

$$c : (U_{\alpha_0}, \dots, U_{\alpha_p}) \longrightarrow C^\infty(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathbb{R}) \quad c_{\alpha_0 \cdots \alpha_p}$$

$$dc : (U_{\alpha_0}, \dots, U_{\alpha_p}) \longrightarrow \Lambda^1(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}) \quad dc_{\alpha_0 \cdots \alpha_p}$$

$$\ker D'' = \check{C}^p(\mathcal{U}; \mathbb{R}).$$

Inclusion of the kernels define maps

$$\alpha : \Lambda^*(M) \longrightarrow K^*(\mathcal{U})$$
$$\beta : \check{C}^*(\mathcal{U}, \mathbb{R}) \longrightarrow K^*(\mathcal{U})$$

which are homomorphisms of cochain complexes and homomorphisms of graded algebras

<u>Def.</u> An open covering $\mathcal{U}$ is called simple if every finite non-empty intersection of elements of $\mathcal{U}$ is a contractible set.

<u>Lemma.</u> Every open covering of M has a refinement that is a simple covering.

<u>Lemma</u> If $\mathcal{U}$ is a simple covering, then

$$\alpha^* : H^*_{DR}(M) \longrightarrow H^*(K^*(\mathcal{U}), D)$$
$$\beta^* : \check{H}^*(\mathcal{U}; \mathbb{R}) \rightarrow H^*(K^*(\mathcal{U}), D)$$

are isomorphisms

These are true for every simple $\mathcal{U}$. Therefore passing to direct limits over open coverings we get

$$\theta_M = (\beta^*)^{-1} \circ \alpha^* : H^*_{DR}(M) \longrightarrow \check{H}^*(M; \mathbb{R})$$

Note that domain of α^* has already attained its limit value., and

$$\mathcal{U} > \mathcal{V} \Rightarrow \check{H}^*(\mathcal{U}; \mathbb{R}) \longrightarrow \check{H}^*(\mathcal{V}; \mathbb{R})$$
$$\varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}; \mathbb{R}) = \check{H}^*(M; \mathbb{R})$$

the Čech cohomology of M.

The naturality of θ_M can be proved easily.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots \\
\Lambda^q(M) \arrow[u, "d"] \arrow[r, "\alpha"] & K^{0,q}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r, "D'"] & K^{1,q}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r, "D'"] & \cdots \arrow[r, "D'"] & K^{p,q}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r] & \cdots \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "D''"'] & \vdots \arrow[u, "D''"] & & \vdots \arrow[u, "D''"'] \\
\Lambda^1(M) \arrow[u, "d"] \arrow[r, "\alpha"] & K^{0,1}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r, "D'"] & K^{1,1}(\mathcal{U}) \arrow[u, "D''"] \arrow[r, "D'"] & \cdots \arrow[r, "D'"] & K^{p,1}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r] & \cdots \\
\Lambda^0(M) \arrow[u, "d"] \arrow[r, "\alpha"] & K^{0,0}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r, "D'"] & K^{1,0}(\mathcal{U}) \arrow[u, "D''"] \arrow[r, "D'"] & \cdots \arrow[r, "D'"] & K^{p,0}(\mathcal{U}) \arrow[u, "D''"'] \arrow[r] & \cdots \\
& \check{C}^0(\mathcal{U}, \mathbb{R}) \arrow[u, "\beta"'] \arrow[r, "\delta"'] & \check{C}^1(\mathcal{U}, \mathbb{R}) \arrow[u, "\beta"'] \arrow[r, "\delta"'] & \cdots \arrow[r, "\delta"'] & \check{C}^p(\mathcal{U}, \mathbb{R}) \arrow[u, "\beta"'] \arrow[r, "\delta"'] & \cdots
\end{tikzcd}
$$

<u>Lemma</u> The rows are exact, and if $\mathcal{U}$ is a simple covering the columns are also exact.

Pf. Let $\{\lambda_\alpha\}$ smooth partition of unity subordinate to the covering $\mathcal{U} = \{U_\alpha\}$.

If $c \in \check{C}^p(\mathcal{U}, \Lambda^q)$, define $L(c) \in \check{C}^{p-1}(\mathcal{U}, \Lambda^q)$ by

$$(L(c))_{\alpha_0 \cdots \alpha_p} = \sum_\alpha \lambda_\alpha \cdot c_{\alpha \alpha_0 \cdots \alpha_{p-1}}$$

where each $\lambda_\alpha c_{\alpha\alpha_0 \cdots \alpha_{p-1}} \in \Lambda^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_{p-1}})$

If $D'(c) = 0$, a straightforward computation gives $c = D' L(c)$. This proves the rows are exact.

If $\mathcal{U}$ is simple, then the ⊕ columns are exact, by Poincaré Lemma

$$H^k_{DR}(\mathbb{R}^q) = H^k_{DR}(\text{Point}) = \mathbb{R} \text{ if } k=0$$
$$= 0 \text{ if } k>0$$

$$0 \to \mathbb{R} \xrightarrow{i} \Lambda^0(\mathbb{R}^q) \xrightarrow{d} \Lambda^1(\mathbb{R}^q) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^q(\mathbb{R}^q) \xrightarrow{d} \cdots$$

$\downarrow$ constant functions on $\mathbb{R}^q$, $i(r)$ = constant function r

<u>Lemma</u> α^*, β^* are 1-1 OK

α^*, β^* onto

<u>Pf</u>. We prove β^* is onto. Proof for α^* similar.

Let $z \in K^m = K^{0,m} \oplus K^{1,m} \oplus \cdots \oplus K^{m,0}$

$$z = \sum_{p=0}^{m} z_p, \quad z_p \in K^{p,m-p} = \text{Compon of } z \text{ in } K^{p,m-p}$$

Let $Dz = 0$. Now $D'z + D''z = 0 \Rightarrow D'z = 0$, $D''z = 0$

In parti $D''z_0 = 0$

By exactness, $\exists\, u \in K^{0,m-1}$ s.t. $D''u = z_0$. then

$$\begin{tikzcd} z_0 \; K^{0,m} \arrow[u, "D''"] \\ u \; K^{0,m-1} \arrow[u, "D''"] \arrow[r, "D'"] & K^{1,m-1} \end{tikzcd}$$

$$z - Du = z_0 + z_1 + \cdots - D'u - \underbrace{D''u}_{= z_0}$$
$$= (z_1 - D'u) + z_2 + \cdots + z_m$$

thus $z - Du$ has 0-component in $K^{0,m}$

Du is a D-cocycle, cohomologous to z

Repeating this ~~way~~ process, we finally get a D-cocycle $w \in K^{n,0}$, cohomologous to z

Now $D(w) = 0 \Rightarrow D'(w) = 0$, $D''(w) = 0$

So $\exists\ v \in \check{C}^n(\mathcal{U}, \mathbb{R})$ such that

$$w = \beta(v)$$

and $\delta(v) = 0$

v is a δ-cocycle

$$\begin{tikzcd}
{} & {} \\
w \in K^{n,0} \arrow[u, "D''"] \arrow[r, "D'"] & 0 \\
v \in \check{C}^n(\mathcal{U}, \mathbb{R}) \arrow[u, "\beta"] \arrow[r, "\delta"] & 0 \arrow[u]
\end{tikzcd}$$

this shows β^* is onto.

Let $\mathcal{U} = \{U_\alpha\}$ a simple open covering of M so that each U_α is a trivializing nbd of the complex holomorphic bdle $E \to M$.
Define a holo connection ∇^α on $E|U_\alpha$ by taking it as trivial on a holo frame field. Let Ω^α be curvature form on U_α.
If $\varphi : gl(n, \mathbb{C}) \to \mathbb{C}$ is an inv polynomial of deg r, then $\varphi(\Omega^\alpha) \in \Lambda^{2r}_{\mathbb{C}}(U_\alpha)$. Define $\varphi^0 \in \check{C}^0(\mathcal{U}, \Lambda^{2r}_{\mathbb{C}})$ by

$$\varphi^0_\alpha = \varphi(\Omega^\alpha).$$

On $U_\alpha \cap U_\beta$, there are two connections ∇^α and ∇^β. Put on $(U_\alpha \cap U_\beta) \times \mathbb{C}$ the connection $(1-z)\nabla^\alpha + z\nabla^\beta$ on the bdle $E|(U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$
(like the proof of Proposition 1)
Let $\Omega^{\alpha\beta}$ be the corresponding curvature form.
Then $\varphi(\Omega^{\alpha\beta}) \in \Lambda^{2r}((U_\alpha \cap U_\beta) \times \mathbb{C})$.
Let Δ^p standard p-simplex. then

$$(U^\alpha \cap U^\beta) \times \Delta^1 \subset (U^\alpha \cap U^\beta) \times \mathbb{R} \subset (U^\alpha \cap U^\beta \times \mathbb{C})$$

the projection $\pi^{\Delta^1} : (U^\alpha \cap U^\beta) \times \Delta^1 \to U^\alpha \cap U^\beta$ defines (via integration along the fibre)

$$\pi^{\Delta^1}_* : \check{C}^1(\mathcal{U}, \Lambda^{2r}_{\mathbb{C}}) \to \check{C}^1(\mathcal{U}, \Lambda^{2r-1}_{\mathbb{C}}).$$

Integration along the ~~fine~~ fibre (Review)

<u>Proposition</u> For $\pi : V \times \Delta^p \to V$, there is a canonical homomorphism $\pi_* : \Lambda^r(V \times \Delta^p) \to \Lambda^{r-p}(V)$, $r \geq 0$, which is zero for $r < p$ and satisfies

$$(**) \qquad \pi_* \circ d + (-1)^{p+1} d \circ \pi_* = \pi^{\partial}_* \circ i^* \quad \text{for } r \geq p$$

where $\pi^{\partial} : V \times \delta\Delta^p \to V$, $i : V \times \delta\Delta^p \hookrightarrow V \times \Delta^p$

<u>Pf.</u> one can show the existence of a unique π_* satisfying

$$\int_{V \times \Delta^p} \psi \cdot \pi^*(\varphi) = \int_V \pi_*(\psi) \cdot \varphi$$

$\forall\, \psi \in \Lambda^r(V \times \Delta^p)$, $\varphi \in \Lambda_0^{m+p-r}(V)$, $m = \dim V$

$$\begin{tikzcd}
\Lambda^r(V \times \Delta^p) \arrow[rr, "d"] \arrow[dd, "\pi_*"'] \arrow[dr, "i^*"] & & \Lambda^{r+1}(V \times \Delta^p) \arrow[dd, "\pi_*"] \\
& \Lambda^r(V \times \partial\Delta^p) \arrow[dr, "\pi^{\partial}_*"] & \\
\Lambda^{r-p}(V) \arrow[rr, "d"] & & \Lambda^{r+1-p}(V)
\end{tikzcd}$$

Let $\omega \in \Lambda^r(V \times \Delta^p)$, $\varphi \in \Lambda_0^{m+p-r-1}(V)$

$$\int_V (\pi_* d\omega + (-1)^{p+1} d\pi_*\omega) \cdot \varphi = \int_M \pi_*(d\omega) \cdot \varphi$$

$$+ (-1)^{p+1} \int_V [d(\pi_*(\omega) \cdot \varphi) - (-1)^{r-p} \pi_*(\omega) \cdot d\varphi]$$

$$= \int_V \pi_*(d\omega) \cdot \varphi + (-1)^r \int_V \pi_*(\omega) \cdot d\varphi$$

$$= \int_{V \times \Delta^p} d\omega \cdot \pi^*(\varphi) + (-1)^r \int_{V \times \Delta^p} \omega \cdot \pi^*(d\varphi)$$

$$= \int_{V \times \Delta^p} d(\omega \cdot \pi^*(\varphi)) = \int_{V \times \partial\Delta^p} i^*(\omega) \cdot \pi^{\partial *}(\varphi)$$

$$= \int_V \pi^{\partial}_* i^*(\omega) \cdot \varphi$$ this proves the identity.

Thus we have $\pi_*^{\Delta^1} : \check{C}^1(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n}) \to \check{C}^1(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n-1})$

Now define $\varphi^1 \in \check{C}^1(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n-1})$ by

$$\varphi^1_{\alpha\beta} = \pi_*^{\Delta^1}(\varphi(\Omega^{\alpha\beta})).$$

On $U_\alpha \cap U_\beta \cap U_\gamma$ we work with three connections and convex combination of these over $(U_\alpha \cap U_\beta \cap U_\gamma) \times \mathbb{C}^2$ and we use

$$\pi^{\Delta^2} : (U_\alpha \cap U_\beta \cap U_\gamma) \times \Delta^2 \longrightarrow U_\alpha \cap U_\beta \cap U_\gamma$$

to define $\varphi^2 \in \check{C}^2(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n-2})$ by

$$\varphi^2_{\alpha\beta\gamma} = \pi_*^{\Delta^2}(\varphi(\Omega^{\alpha\beta\gamma})).$$

Continuing this way, we produce

$$\varphi^0 \in \check{C}^0(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n}) = K^{0,2n}(\mathcal{U})$$
$$\varphi^1 \in \check{C}^1(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n-1}) = K^{1,2n-1}(\mathcal{U})$$
$$\dots$$
$$\varphi^m \in \check{C}^2(\mathcal{U}, \Lambda_{\mathbb{C}}^{2n-m}) = K^{m,2n-m}(\mathcal{U})$$

and hence

$$\bar{\varphi} = (\varepsilon_0 \varphi^0, \varepsilon_1 \varphi^1, \dots, \varepsilon_m \varphi^m) \in \text{direct sum } K^{2n}(\mathcal{U})$$

where $\varepsilon_i = (-1)^{\left[\frac{i+1}{2}\right]}$

this $\bar{\varphi}$ is a cocycle

$$D\bar{\varphi} = 0 \quad , D = D' + D'' , \ D' = \delta \ , \ D'' = (-1)^q d$$

because of the identity (* *)

$$\pi_*^{\Delta^p} \circ d + (-1)^{p+1} d \circ \pi_*^{\Delta^p} = \pi_*^{\partial\Delta^p} \circ i^*$$

$$= \sum_{j=0}^{p} (-1)^j \pi_*^{\Delta^{p-1}(j)}$$

where $\Delta^{p-1}(j)$ is the jth face of Δ^p. the last equality is by the combinatorial version of Stoke's theorem [Sternberg, Diff. Geom. p.109]

Since $d\varphi(\Omega^{\alpha_0 \cdots \alpha_p}) = 0$, this relation gives

$$d(\varphi^0) = 0$$

and for $p > 0$

$$d(\varphi^p) = (-1)^{p+1} \sum_{j=0}^{p} (-1)^j \pi_*^{\Delta^{p-1}(j)} \varphi_*^p = (-1)^{p+1} \delta(\varphi^p)$$

therefore $D'(\varphi^p) = (-1)^{p+1} d(\varphi^p)$

and $D''(\varphi^p) = (-1)^{2n-p} d(\varphi^p) = (-1)^p d(\varphi^p)$

therefore $D(\varphi^p) = 0$

and we obtain the chern class

$$\varphi(E) = [\bar{\varphi}] \in H^{2n}(K^*(\mathcal{U}, D)) = H^{2n}_{DR}(M, \mathbb{C}).$$

Theorem M complex analytic manifold
TM holomorphic tangent bundle
$E \subset TM$ a complex subbundle which is isomorphic to a holomorphic integrable subbundle $E' \subset TM$, $Q = TM/E$, $\dim_{\mathbb{C}} Q = q$.
Then $\mathrm{Chern}^k(Q) = 0$ for $k > 2q$ -

Application $\mathbb{C}P^n$ complex projective n-space
$T = T(\mathbb{C}P^n)$ holomorphic tangent bdle
If n is odd, then T contains a holomorphic sub-bdle of complex ~~dim~~ codim one.
Moreover, if $n > 1$, no holomorphic sub bundle of T of complex codim one is integrable.

Pf. Let $[n+1]$ = Trivial bdle $\mathbb{C}P^n \times \mathbb{C}^{n+1} \to \mathbb{C}P^n$.
Each $x \in \mathbb{C}P^n$ is a 1-dim subspace of $\mathbb{C}^{n+1}$.
Therefore can define holomorphic bundles

$$S = \{(x,v) \in [n+1] : v \in x\}, \quad \dim_{\mathbb{C}} S = 1$$

$$Q = \{(x,v) \in [n+1] : v \perp x\} \quad \dim_{\mathbb{C}} Q = n$$

$$[n+1] = S \oplus Q$$

There's a canonical isomorphism of hol bdles

$$\begin{tikzcd} T \arrow[dr] & \cong & \mathrm{Hom}(S,Q) \arrow[dl] \\ & \mathbb{C}P^n & \end{tikzcd}$$

given $\varphi \in \mathrm{Hom}(S_x, Q_x)$, $x \in \mathbb{C}P^n$,
define $\sigma : U \to \mathbb{C}P^n$, $U = \{z \in \mathbb{C} : |z| = 1\}$ by
$\sigma(z) = $ 1-dim subspace of $\mathbb{C}^{n+1}$ containing
$$(1+z)v + z\varphi(v), \quad v \in S_x$$
then σ is a holomorphic curve with
$\sigma(0) = $ 1-dim subspace of $\mathbb{C}^{n+1}$ containing $v = x$
Denote the tangent vector to this curve $\sigma(z)$ at x
by $\alpha(\varphi)$. In this way one obtains an isomorphism
$$\alpha : \mathrm{Hom}(S, Q) \to T(\mathbb{C}P^n).$$
If n is odd, choose a basis $\theta_1, \dots, \theta_{n+1}$ of $(\mathbb{C}^{n+1})^*$
and form the non-deg anti-symmetric
bilinear form $\omega : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$
$$\omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 + \dots + \theta_n \wedge \theta_{n+1}$$
[note.
$$\theta_1 \wedge \theta_2 (v_1, v_2) = \frac{1}{2} \begin{vmatrix} \theta_1(v_1) & \theta_1(v_2) \\ \theta_2(v_1) & \theta_2(v_2) \end{vmatrix}$$
]

Let $H = S^*$ and H^2 symmetric square of H

H^2_x is the space of all sym bilinear forms on S_x, an element $\eta \in H^2_x$ may be called an indefinite metric: it assigns to each $v \in S_x$ its distance $\eta(v, v)$, i.e.

η is a homogeneous function $S_x \to \mathbb{C}$ of deg 2.

thus each $\eta \in H^2_x$ is simply a homogeneous fn of deg 2 from $S_x \to \mathbb{C}$.

H^2 is a line bundle

Define $w_x : \mathrm{Hom}(S_x, Q_x) \to H^2_x$, $x \in \mathbb{C}P^n$,

by $w_x(\varphi)(v) = w(v, \varphi(v))$, $v \in S_x$.

Since w_x is non-degenerate and $\varphi(v)$ ranges over orthogonal complement of $v \neq 0$,

w_x is surjective for all $x \in \mathbb{C}P^n$, and hence defines a surjective bundle map

$$w_* : T = \mathrm{Hom}(S, Q) \to H^2$$

then $\mathrm{Ker}\, w_*$ is a holomorphic subbdle of T of complex codim one.

Now suppose $n > 1$, and let $E \subset T$ a holomorphic subbundle of codim one.

If E is integrable,

$$c_1(T/E)^2 \in \text{Chern}^4(T/E) = 0$$

Since $H^*(\mathbb{C}P^n; \mathbb{C})$ is well known to be generated by 1 and by $u = c_1(S^*) \in H^2(\mathbb{C}P^n, \mathbb{C})$ with the single relation $u^n = 0$, the above together with $n > 1$ imply that $c_1(T/E) = 0$.

But $c(T) = c(E) \cdot c(T/E)$

Therefore $c_n(T) = 0$.

On the other hand,

$$T \oplus [1] = \text{Hom}(S, Q \oplus S) = \text{Hom}(S, [n+1])$$

$$= S^* \oplus \cdots \oplus S^* \quad (n+1 \text{ times})$$

So

$$c(T) = c(T \oplus [1]) = c(S^*)^{n+1} = (1+u)^{n+1}$$

showing that $c_n(T) \neq 0$.

This contradiction proves E cannot be integrable.