1 Generalities from homological algebra

Let us fix a commutative unital ring k.

Definition 1.1 *A Chain complex of k-modules is a (finite or infinite at one or both ends) sequence C_n of k-modules together with module maps $d_n : C_n \to C_{n-1}$ for each n satisfying $d_{n-1} \circ d_n = 0$. We often denote each d_n by just d , unless there's chance of any confusion, and call the map d the differential. Elements of C_n are called chain of degree or dimension n. Since $B_n \equiv B_n(C) := \mathrm{Im}(d_{n+1}) \subseteq Z_n \equiv Z_n(C) := \mathrm{Ker}(d_n)$, we consider the quotient modules $H_n \equiv H_n(C) := Z_n/B_n$, called the n-th homology group. The sequence $H_* \equiv (H_n)_n$ is called the homology of the chain complex (C, d). An element of B_n (respectively Z_n) is called a boundary (resp. cycle) of dimension of degree n.*

Similarly, by a cochain complex we mean a sequence C^n of k-modules equipped with maps $d_n : C^n \to C^{n+1}$ satisfying $d_{n+1} \circ d_n = 0$. We call an element of C^n a cochain of degree or dimension n. Similarly, we define cocycle and coboundary and also the n-th cohomology group $H^n(C) \equiv H^n :=$ $\mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1})$.

A chain (resp. cochain) complex is called acyclic if its homology (resp. cohomology) groups are all trivial, i.e. (0).

Let us give some more basic definitions and state some standard results (sometimes with brief skectch of proof) for chain complexes. However, note that all these definitions and results have the obvious counterparts for cochain complexes.

Definition 1.2 *Let (C, d) and (C', d') be chain complex. A family of maps $f_n : C_n \to C'_n$ is called a morphism of complexes (written symbolically as $f : C \to C'$) if $f \circ d = d' \circ f$, i.e. $f_{n-1} \circ d_n = d'_n \circ f_n$ for all n. Two such morphisms f and g are said to be (chain) homotopic if there is a famliy $h = (h_n)$ of module maps $h_n : C_n \to C'_{n+1}$ such that*

$$f - g = dh + hd,$$

i.e. $f_n - g_n = d'_{n+1}h_n + h_{n-1}d_n$ for all n. We write $f \sim g$ in this case.

A complex C is called contractible if $\mathrm{id_C} \sim 0$, where 0 denotes the identically zero morphism from C to C. The corresponding homotopy is called a contracting homotopy.

Lemma 1.3 *Let* (C,d), (C',d') *be chain complexes. A morphism* $f : C \to C'$ *induces a group homomirphism* $f_* : H(C) \to H(C')$. *Moreover, if* $f \sim g$, *then* $f_* = g_*$.

Sketch of proof :
It is obvious from the definition of morphism of complexes that $f(Z_n) \subseteq Z'_n := \mathrm{Ker}(d'_n)$, $f(B_n) \subseteq B'_n := \mathrm{Im}(d'_{n+1})$, and the map $x + B_n \to f(x) + B'_n$ (where $x \in Z_n$) is well-defined group homomorhism. Now, if f, g are chain homotopic via a family of maps $h = (h_n)$, we have $f_n(x) - g_n(x) = d'_{n+1}(h_n(x)) \in B'_n$, for $x \in Z_n$, since $d_n(x) = 0$. Thus $f_n(x)+B'_n = g_n(x)+B'_n$, i.e. $f_*(x) = g_*(x)$. By a short exact sequence of complexes we mean a sequence of the form $0 \to C \xrightarrow{f} C' \xrightarrow{g} C'' \to 0$, where f, g are morphisms of complexes.

Theorem 1.4 *Given a short exact sequence of complexes as above, there exists a long exact sequence of homology groups given by*

$$... \to H_n(C) \xrightarrow{f_*} H_n(C') \xrightarrow{g_*} H_n(C'') \xrightarrow{\partial} H_{n-1}(C)...$$

where $\partial \equiv (\partial_n)$ *is a family of maps (called the connecting map) such that* $\partial_n : H_n(C'') \to H_{n-1}(C)$.

Sketch of proof :
Take $y \in Z_n(C'')$. By the surjectivity of g_n, given $y \in Z_n(C'')$, we can find $x \in Z_n(C')$ such that $y = g_n(x)$. Since $g_{n-1}d'(x) = d''g_n(x) = 0$, we have $d'(x) \in \mathrm{Ker}(g_{n-1}) = \mathrm{Im}(f_{n-1})$, so there exists $z \in C_{n-1}$ such that $d'(x) = f_{n-1}(z)$. Indeed, $z \in Z_{n-1}(C)$, i.e. $dz = 0$, since $f_{n-2}(d(z)) = d'(f_{n-1}(z)) = (d')^2(x) = 0$, and f_{n-2} is one-to-one. We set $\partial(y + B_n(C'')) = x + B_{n-1}(C)$. It is routine to verify that this is indeed well-defined and the resulting long sequence of homology groups is exact.

We now define a bicomplex and related notions.

Definition 1.5 *A bicomplex is given by* $C = (C_{pq})$ *of a double sequence (finite or infinite in all possible directions) of* k*-modules together with two families of module maps* $d^h \equiv (d^h_{pq})$, $d^v \equiv (d^v_{pq})$, *where* $d^h_{pq} : C_{pq} \to C_{p-1,q}$, $d^v_{pq} : C_{pq} \to C_{p,q-1}$ *and*

$$d^v d^v = d^h d^h = d^v d^h + d^h d^v = 0.$$

Given such a bicomplex, we can form a complex (called the total complex of the bicomplex) $\mathrm{Tot}(C) = (C'_n)$*, where* $C'_n := \oplus_{p+q=n} C_{pq}$*, and the differential map* $d : C' \to C'$ *given by* $d = d^h + d^v$*. The homology* $H_*(\mathrm{Tot}(C))$ *is also called the homology of the bicomplex* C *and sometimes denoted also by* $H_*(C)$*.*

We state a useful result without proof.

Theorem 1.6 *Let* $C = (C_{pq})_{p,q\geq 0}$ *be a bicomplex (indexed by nonnegative integeres) such that for all* $p \geq 0$ *and* $q > 0$*, we have* $H_q(C_{p,*}) = 0$*. Let* $K_n := H_0(C_{n,*})$*, with* d^v *inducing a differential to make* K_n *into a chain complex. Then* $H_n(\mathrm{Tot}(C)) \cong H_n(K_*)$ *for all* $n \geq 0$*.*

For a complex (C, d), and an integer p, we form a complex called the *shifted complex*, denoted by $C[p]$, which is given by $C[p]_n = C_{n-p}$ and the differential is given by $(-1)^p d$. Given two complexes (C, d) and $C', d')$, we form a complex called the *tensor product* of C and C', denoted by $C \otimes C'$, whose n-th module is given by $\oplus_{p+q=n} C_p \otimes_k C'_q$, with the differantial given by $\tilde{d}(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y)$, where $\deg(x)$ is the degree or dimension of the chain x.
Exercise : Verify that $\tilde{d}^2 = 0$.

We We state an important result (called the *Kunneth theorem*) about the tensor product of complexes without proof.

Theorem 1.7 *If* $Z_n(C), H_n(C)$ *are projective then there is a canonical isomorphism*

$$\oplus_{p+q=n} H_p(C) \otimes H_q(C') \cong H_n(C \otimes C').$$

Theorem 1.8 *Let* (C, d) *be a complex with* $C_n = A_n \oplus B_n$*,* $d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$*, w.r.t the direct sum decomposition of* C_n*. Assume also that* $\delta^2 = 0$ *and* (B, δ) *is acyclic, i.e. has trivial homology groups, with a contracting homotopy* $h : B_* \to B * + 1$*. Let* $\epsilon := \alpha - \beta h \gamma$*. Then* $\epsilon^2 = 0$*, and the complex* (A, ϵ) *has the same homology as that of* (C, d)*, i.e.* $H_*(C, d) \cong H_*(A, \epsilon)$*.*

Sketch of proof:
From the relations $d^2 = 0$ and $\delta^2 = 0$ we get

$$\gamma\alpha = -\delta\gamma, \quad \gamma\beta = 0, \quad \alpha\beta = -\beta\delta,$$

which allows one to deduce $\epsilon^2 = 0$ by routine calculation. Define $\phi = (\mathrm{id}, -h\gamma) : A_* \to C_*$. It is straightforward to verify that it is a morphism of complexes, hence $\mathrm{Coker}(\phi) := C_*/\mathrm{Im}(\phi)$ is a chain sub-complex of C_*. Let $\pi_n^1 : C_n \to A_n$ be the projection map onto the first direct summand; then it is easy to see that $\pi_n^1 \circ \phi_n = \mathrm{id}$, hence $\psi_n := \phi_n \circ \pi_n^1$ is an idempotent and so each C_n decomposes into a direct sum $\mathrm{Ker}(\psi_n) \oplus \mathrm{Im}(\psi_n)$. Moreover, $\mathrm{Im}\psi_n) = \mathrm{Im}(\phi_n)$ and $\mathrm{Ker}(\psi_n) = \{(0, b) : \ b \in B_n\} \cong B_n$, i.e. $\mathrm{Coker}(\phi)$ is isomorphic with (B, δ). Consider the short exact sequence $0 \to A \stackrel{\phi}{\to} C \to \mathrm{Coker}(\phi) \stackrel{q}{\to} 0$ (where q is the quotient map from C onto $C/\mathrm{Im}(\phi)$) and the induced long exact homology sequence

$$...0 \to H_n(A) \stackrel{\phi_*}{\to} H_n(C) \stackrel{q_*}{\to} H_n(\mathrm{Coker}(\phi)) \cong H_n(B) = 0 \to ...,$$

from which it follows that ϕ_* is an isomorphism of abelian groups $H_*(A)$ and $H_*(C)$.

Let us now introduce simplicial modules, which are pre-cursors of cycilc modules which will be at the heart of our discussion of cycilc homology theory later on.

Definition 1.9 *A pre-simplicial module is a family $C = (C_n)$ of k-modules together with module maps $d_i \equiv d_{n,i} : C_n \to C_{n-1}$ $(i = 0, 1, ..., n)$ (called face maps) satisfying*

$$d_i d_j = d_{j-1} d_i, 0 \leq i < j \leq n.$$

A map or morphism from a pre-simplicial module C to another such module C' is a family $f = (f_n)$ of module maps $f_n : C_n \to C'_n$ satisfying $f_{n-1} d_i = d'_i f_n$, where d'_i s are the face maps for C'. A pre-simplicial homotopy between two pre-simplicial maps ,g from C to C' is a family $(h_i \equiv h_{n,i})$ of maps $h_i : C_n \to C'_{n+1}$ $(i = 0, ..., n)$ sastisfying

$$d_i h_j = h_{j-1} d_i \quad \text{for} \quad i < j$$

$$d_i h_i = d_i h_{i-1} \quad \text{for} \quad 0 < i \leq n;$$

$$d_i h_j = h_j d_{i-1} \quad \text{for} \quad i > j + 1;$$

$$d_0 h_0 = f, \quad d_{n+1} h_n = f.$$

Lemma 1.10 *Given a pre-simplicial module C as above, the map $d := \sum_{i=0}^n (-1)^i d_i$ satisfies $d^2 = 0$, i.e. (C_*, d) is a chain complex. Moreover, any* *pre-simplical map f from a pre-simplcial module C to another such module C' induces a morphism of complexes from (C_*, d) to (C'_*, d'). Given a pre-simplcial homotopy (h_i) between two pre-simplicial maps f and g, we have a chain homotopy given by $h :=_{i=0}^{n} (-1)^i h_i$ between f and g.*

Proof :
We have

$$d^2 = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i d_j.$$

Breaking it into sum over $i < j$ and $i \geq j$, and observing that the term $(-1)^{i+j} d_i d_j$ in the first part cancels with the term $(-1)^{j-1+i} d_{j-1} d_i$ in the second, we conclude that $d^2 = 0$. A similar calculation allows us to prove that $dh + hd = f - g$.

Definition 1.11 *A simplicial module is given by a family of modules M_n together with maps d_i, s_i, $i = 0, ..., n$, where $(M, (d_i))$ is a pre-simplicial module and the maps $s_i : M_n \to M_{n+1}$ (called the degeneracy maps) satisfy $s_i s_j = s_{j+1} s_i$ for $i \leq j$, and moreover,*

$$\begin{aligned} d_i s_j = \\ & s_{j-1} d_i \ \text{ for } i < j, \\ & \mathrm{id}_M \ \text{ for } i = j, j+1, \\ & s_j d_{i-1} \ \text{ for } i > j+1. \end{aligned}$$

A simplicial module M is said to have an *extra degeneracy* if there there is a map $s_{n+1} : M_n \to M_{n+1}$ satisfying similar relations as $s_i, i = 0, ..., n$ as in the above definition. If such an extra degeneracy exists, we have $d_i s_{n+1} = s_n d_i$ for $0 \leq i \leq n$ and $d_{n+1} s_{n+1} = \mathrm{id}$. Thus, $ds_{n+1} - s_n d = (-1)^{n+1}\mathrm{id}$, i.e. $(-1)^n s_n$ is a contracting homotopy, hence (M, d) is acyclic.

We shall now introduce a convenient simplification of a simplicial module which is called the *normalization* of the given simplicial module. Consider the submodule D_n of M_n spanned by the images of $s_0, ..., s_{n-1}$, i.e. $D_n := s_0 M_{n-1} + ... s_{n-1} M_{n-1}$. It is easy to see that d maps D_n into D_{n-1}, hence (D_*, d) can be viewed as a subcomplex of (M_*, d), and we can consider the quotient complex $(M_*/D_*, d)$, which is called the *normalized complex* obtained from (M, d).

Theorem 1.12 *The homology of the normalized complex is isomorphic with that of* (M, d)*.*

We refer the reader to [3] for a proof using spectral sequences. Given two simplicial modules $(M, (d_i^M), (s_i^M))$ and $(N, (d_i^N), (s_i^N))$, we can define a simplicial module $M \times N$ (called the product of M and N) by setting $(M \times N)_n = M_n \otimes_k N_n$, $d_i = d_i^M \otimes d_i^N$, $s_i = s_i^M \otimes s_i^N$. Note that this is different from the tesnor product complex $M_* \otimes N_*$ which is defined earlier for general chain complexes without any simplicial structure. However, there is a natural map, called the *shuffle*, denoted by sh from $M_* \otimes N_*$ to $(M \times N)_*$, given as follows. For $a \in M_p$, $b \in M_q$, define

$$\mathrm{sh}(a \otimes b) := \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu) s_{\nu_q} ... s_{\nu_1}(a) \otimes s_{\mu_p} ... s_{\mu_1}(b),$$

where the summation is taken over all '$p - q$-shuffles', i.e. partition of $\{0, 1, ..., p + q - 1\}$ into two disjoint sets $\mu = \{\mu_1 < \mu_2 < ... < \mu_p\}$ and $\nu = \{\nu_1 < ... < \nu_q\}$, and where $\mathrm{sgn}(\mu, \nu)$ is the sign of the permutation determined by $\{\mu_1, ..., \mu_p, \nu_1, ...\nu_q\}$.

Exercise : Check that the above gives a morphism of complexes from $M \otimes N$ to $M \times N$.

We also need little bit of category theoretical abstract nonsense, which we introduce now. Assuming readers' familiarity with the basic terminologies of cateogory theory, we briefly recall the definition of an abelian cateogory and the notion of derived functors. One important remark : the homology /cohomology theory discussed above is valid in an abstract abelian category. We leave it as an exercise for the interested and patient reader to work out all the details.

Definition 1.13 *A category* $\mathcal{C}$ *is said to be an additive category if the following are satisfied :*
(i) $\mathrm{Hom}(E, F)$ *is an abelian group for all* $E, F \in \mathrm{Obj}(\mathcal{C})$*;*
(ii) composition of morphisms is a bilinear map with respect to the group structure of the Hom *sets;*
(iii) there is a zero object, i.e. $0 \in \mathrm{Obj}(\mathcal{C})$ *such that* $\mathrm{Hom}(E, 0)$ *and* $\mathrm{Hom}(0, E)$ *are singleton sets for every object* E*;*
(iv) finite products and coproducts exist in $\mathcal{C}$*.*
An additive cateogory $\mathcal{C}$ *is called abelian if the following additional axioms*

are satisfied :
(v) 'kernel' exists in $\mathcal{C}$ in the sense that for every $E, F \in \mathrm{Obj}(\mathcal{C})$ and $f : E \to F$ (i.e. $f \in \mathrm{Hom}(E,F)$), there exists an object K and morphism $g : K \to E$ such that $f \circ g$ is the zero of the abelian group $\mathrm{Hom}(K,F)$, and for every object K' and morphism $g' : K' \to E$ satisfying $f \circ g' = 0$, there is a unique $u : K' \to K$ such that $g \circ u = g'$ (here K is denoted by $\mathrm{Ker}(f)$);
(vi) similarly, 'cokernel' exists in $\mathcal{C}$, i.e. for every $E, F \in \mathrm{Obj}(\mathcal{C})$ and $f : E \to F$ (i.e. $f \in \mathrm{Hom}(E,F)$), there exists an object L and morphism $h : F \to L$ such that $h \circ f = 0$, and for every object L' and morphism $h' : F \to L'$ satisfying $h' \circ f = 0$, there is a unique morphism $v : L \to L'$ such that $h' = v \circ h$ (L is denoted by $\mathrm{Coker}(f)$).

Clearly, the category $\mathrm{Mod}-R$ (resp. $R-\mathrm{Mod}$) of right (resp. left) R-modules (where R is a possibly noncommutative ring) is abelian category.

One can imitate the definitions of projective/ injective modules and resolutions for modules to generalize these concepts for an abelian cateogory. Fix an abelian category $\mathcal{C}$. We say that an object P is *projective* if the following holds : given any two objects E, M and morphisms $f : E \to M$, $g : P \to M$ such that $\mathrm{coker}(f) = 0$, there is a unique morphism $h : P \to M$ satisfying $f \circ h = g$. Similarly, we can define an *injective* object by reversing the arrows and replacing cokernel by kernel in the definition of a projective object. We say that $\mathcal{C}$ has enough projectives (resp. enough injectives) if for any object C, there is a projective object P (resp. injective object I) and a morphism $f : P \to C$ (resp. $f : M \to I$) such that $P \xrightarrow{f} C \to 0$ (resp. $0 \to C \xrightarrow{f} I$) is exact. It can be easily seen that the categories $R - \mathrm{Mod}$ and $\mathrm{Mod} - R$ have enough projectives as well as enough injectives. If $\mathcal{C}$ has enough projectives (injectives) then every object admits a *projective resolution* (resp. *injective resolution*) , i.e. an exact sequence $P \equiv ... \to P_n \to P_{n-1} ... \to P_1 \to C \to 0$ (resp. $0 \to C \to P_1 \to ...$) with each P_n projective (injective). The following result which state without proof asserts that any projective (injective) resolotion of a given object is unique upto homotopy.

Lemma 1.14 *Suppose that an abelian cateogory $\mathcal{C}$ has enough projectives and let $P \to C \xrightarrow{\pi} 0$ and $P'C \xrightarrow{\pi'} 0$ be two projective resolutions of an object C. Then there is a morphism of complexes $f : P \to P'$ such that $\pi' \circ f = \pi$, and any two such morphisms are homotopic. More generally, given any morphism $\phi : C \to D$ (where C, D are objects in $\mathcal{C}$) and projective* *resolutions $P \to C \to 0$, $Q \to D \to 0$, we can get a (unique upto homotopy) morphism of complexes from P to Q which commutes with ϕ.*

A similar result holds for a categroy with enough injectuves, if we replace projective by injective in the above.

In particlar, we see that any two projective resolutions of a given bject C have the same homology : given two projective resolutions P, P', we can find morphisms $f : P \to P'$ and $g : P' \to P$ by the preceeding lemma, and then apply the lemma again on $f \circ g$ and $g \circ f$ to conclude that they are homotopic to id_C. If F is an additive covariant functor from $\mathcal{C}$ to another abelian category $\mathcal{D}$ (say), then the homotopy of P and P' implies the same for $F(P)$ and $F(P')$, hence the cohomology of $F(P)$ is same as that of $F(P')$. In other words, $H^*(F(P))$ depends only on C and not on any specific choice of P. This allows one to define the so-called *left drived functor* L^nF, by setting $L^nF(C) = H^n(F(P))$. We leave it to the reader as a simple exercise to write down the definition of $L^nF(f)$ for a morphism f between two objects in $\mathcal{C}$. In particular, taking $\mathcal{C}$ to be the category of (right) R-modules, and $F \equiv F_B = \cdot \otimes_R B$ (where B is a given left R module) we get the Tor-functor $\mathrm{Tor}^n_R(\cdot, B) := L^nF_B(\cdot)$.

In a similar way, we can define right derived functor R_nG for a contravariant for a category with enough projectives, replacing cohomology by homology in the above. Moreover, for a category with enough injectives, we get left/ right derived functors for contravariant/covariant functors. In particular, the contravariant Hom functor gives rise to the well-known functor called Ext. These materials can be found in any standard textbooks on homological algebra, for example, [4]

2 Hochschild homology and cohomology

We begin by defining Hochschild homology and cohomology. Let $\mathcal{A}$ be a unital k-algebra and M be an $\mathcal{A}$-$\mathcal{A}$ bimodule. Let $C_n \equiv C_n(\mathcal{A}, M) := M \otimes_k \mathcal{A}^n$ (where $\mathcal{A}^n$ denotes n-fold tensor product over k). We equip it with a simplicial module structure given by $d_0(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) := ma_1 \otimes a_2 \otimes ... \otimes a_n$, $d_i(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) := m \otimes a_1 ... \otimes a_i a_{i+1} \otimes ... a_n$ for $1 \leq i < n$, $d_n(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) := a_n m \otimes a_1 \otimes ... \otimes a_{n-1}$, $s \equiv s_0(m \otimes a_1 \otimes ... \otimes a_n) := m \otimes 1 \otimes a_1 \otimes ... \otimes a_n$, $s_j(m \otimes a_1 \otimes ... \otimes a_n) = m \otimes a_1 \otimes ... \otimes a_j \otimes 1 \otimes a_{j+1} \otimes ... \otimes a_n$ for $j = 1, ..., n$.
Exercise : Verify that the above defines a simplicial structure.
Thus, $b := \sum_{i=0}^{n} (-1)^i d_i$ defines a differential map so that (C_*, b) is a chain complex.

Definition 2.1 *The above complex* (C_*, b) *is called the Hochschild chian complex and its homology is called the Hochschild homology groups of* $\mathcal{A}$ *with coeffients in* M*, and is denoted by* $H_*(\mathcal{A}, M)$*. In case* $\mathcal{A} = M$*, it is called the Hochschild homology of* $\mathcal{A}$*, denoted by* $HH_*(\mathcal{A})$*.*

Moreover, by Theorem 1.12, it is sometimes convenient to consider the normalization of the above simplicial module (which has the same homology), which is called the *normalized Hochschild complex*, denoted by $\bar{C}(\mathcal{A}, M)$.
Exercise: Verify that the normalized Hochshild complex is given by

$$(\bar{C})_n(\mathcal{A}, M) = M \otimes_k (\mathcal{A}/k) \otimes_k ... \otimes_k (\mathcal{A}/k),$$

where $\mathcal{A}/k$ denotes the quotient of the unital algebra $\mathcal{A}$ by the k-submodule generated by the unit 1.

Let us now concentrate on the case $M = \mathcal{A}$ and consider $b' : \mathcal{A}^{n+1} \to \mathcal{A}^n$ gien by $b' := \sum_{i=0}^{n-1} d_i = b - (-1)^n d_n$. Let $s : \mathcal{A}^n \to \mathcal{A}^{n=1}$ be given by $s(a_1 \otimes ... \otimes a_n) := 1 \otimes a_1 \otimes ... \otimes a_n$. It is easy to check that $d_i s = s d_{i-1}$ for $i = 1, ..., n-1$ and $d_0 s = \text{id}$. From this, it follows that $b's + sb' = \text{id}$, so we have the following

Theorem 2.2 *The complex* $(\mathcal{A}^{n+1}), b'$ *is acyclic.*

Now we consider a particular resolution of the algebra $\mathcal{A}$. Consider the opposite algebra $\mathcal{A}^{op}$ of $\mathcal{A}$ and the algebra $\mathcal{A}^e := \mathcal{A} \otimes_k \mathcal{A}^o p$. $\mathcal{A}$ can be viewed as a left $\mathcal{A}^e$ module given by $(a \otimes a').b := aba'$. Similarly, M can be thought of as a right $\mathcal{A}^e$ module. It is easy to see that $M \otimes_{\mathcal{A}^e} \mathcal{A}^{n+2} \cong M \otimes_k \mathcal{A}^n$. Let C_*^{bar} be a complex given by $C_n^{bar} = \mathcal{A}^{n+2}$, and the differential is b'. Augment this at the right end by $\mathcal{A}^2 \to \mathcal{A} \to 0$ where the map from $\mathcal{A}^2$ to $\mathcal{A}$ is the multiplication μ. Clearly, the image of b'_1 $(= d_0 - d_1)$ is in the kernel of μ. Conversely, if a finite sum $\sum_k a_k \otimes b_k$ is in the kernel of μ, then $\sum_k a_k b_k = 0$, so $\sum_k a_k \otimes b_k = b'_1(\sum(1 \otimes a_k \otimes b_k))$. Thus, we have proved the following :

Lemma 2.3 *The sequence* $C_*^{bar} \xrightarrow{\mu} \mathcal{A} \to 0$ *is exact, i.e.* C_*^{bar} *is a resolution of* $\mathcal{A}$ *in the category of (left)* $\mathcal{A}^e$*-modules.*

From this, we get an alternative definition of $H_*(\mathcal{A}, M)$.

Theorem 2.4 *If* $\mathcal{A}$ *is* k*-projective then* $H_n(\mathcal{A}, M) = \mathrm{Tor}_n^{\mathcal{A}^e}(M, \mathcal{A})$.

Proof :
Clearly, C_*^{bar} is a projective resolution in this case, hence the theorem follows by observing that $M \otimes_{\mathcal{A}^e} \mathcal{A}^{n+2} \cong M \otimes_k \mathcal{A}^n$.

We recall that two unital k-algebras R and S are called *Morita equivalent* if there exist finitely generated $R-S$ and $S-R$ bimodules P and Q (respectively) such that $P \otimes_S Q \cong R$ and $Q \otimes_R P \cong S$ as (resp) $R-R$ and $S-S$ bimodules. It can be shown (see [3]) that the isomorphisms above, say u and v resp, can be chosen so that $qu(p \otimes q') = v(q \otimes p)q'$, $pv(q \otimes p') = u(p \otimes q)p'$ for all $p, p' \in P$, $q, q' \in Q$.

Theorem 2.5 *If* R *and* S *are Morita equivalent, then for any* $R-R$ *bimodule* M, $H_*(R, M) \cong H_*(S, Q \otimes_R M \otimes_S P)$. *In particular,* $HH_*(R) \cong HH_*(S)$.

Sketch of proof :
Choose and fix $p_i, q_i, i = 1, .., s$ and $p'_j, q'_j, j = 1, ..., t$ such that $u(\sum p_i \otimes q_i) = 1_R$, $v(\sum_j q'_j \otimes p'_j) = 1_S$. For $n \geq 0$ define $\psi_n : M \otimes R^n \to Q \otimes M \otimes P \otimes S^n$ and $\phi_n : Q \otimes M \otimes P \otimes S^n \to M \otimes R^n$ as follows :

$$\psi_n(m \otimes a_1 \otimes ... \otimes a_n) := \sum_{1 \leq i_l \leq s} (q_{i_0} \otimes m \otimes p_{i_1})(q_{i_1} \otimes a_1 p_{i_2}) ... \otimes v(q_{i_n} \otimes a_n p_{i_0}),$$

$$\phi_n(q \otimes m \otimes p \otimes b_1 ... \otimes b_n) := \sum_{1 \leq j_k \leq t} u(p_{j_0} \otimes q) m u(p \otimes q_{j_1}) \otimes u(p_{j_1} \otimes b_1 q_{j_2}) \otimes ... \otimes u(p_{j_n} \otimes b_n q_{j_0}).$$

Verify that $\phi\psi \sim \mathrm{id}$ (homotopic) via the map $h = \sum(-1)^l h_l$, where

$$h_l(m \otimes a_1 \otimes ... \otimes a_n) := \sum (mu(p_{i_0} \otimes q'_{j_0})) \otimes (u(p'_{j_0} \otimes q_{i_0}) a_1 u(p_{i_1} \otimes q'_{j_1})) \otimes ... \otimes (u(p'_{j_l} \otimes q_{i_l})) \otimes a_l \otimes ... \otimes a_n.$$

Similarly, a homotopy can be produced to show that $\psi\phi \sim \mathrm{id}$.

A particular case of the above deserves special attention, namely the Morita equivalent algebras $M_r(\mathcal{A}) \equiv \mathcal{A} \otimes M_r(k)$ and $\mathcal{A}$. Let M be an $\mathcal{A}$-$\mathcal{A}$ bimodule. We can consider $M_r(M) := M \otimes_k M_r(k)$ as an $M_r(\mathcal{A})$-bimodule in a natural way. Let us denote by E_{ij} the k-valued $r \times r$ matrix whose all but $(i-j)$-th entry are 0, and $(i-j)$-th entry is 1. Also, for $\alpha \in M_r(\mathcal{A})$ or $M_r(M)$, we denote by α_{ij} its $(i-j0$-th entry, which is in $\mathcal{A}$ or M respectively. Now, for any positive integer n, define the *generalized trace map* $\mathrm{Tr} : M_r(M) \otimes M_r(\mathcal{A})^{\otimes^n} \to M \otimes \mathcal{A}^{\otimes^n}$ by

$$Tr(\alpha^{(0)} \otimes \alpha^{(1)} \otimes ... \otimes \alpha^{(n)}) := \sum \alpha^{(0)}_{i_0 i_1} \otimes \alpha^{(1)}_{i_1 i_2} \otimes ... \otimes \alpha^{(n)}_{i_n i_0},$$

where the sum is over all choices of $(i_0, ..., i_n)$, such that each $i_k \in \{1, ..., r\}$.

Lemma 2.6 *The map* Tr *is a morphism of chain complex from* $C_*(M_r(\mathcal{A}), M_r(M))$ *to* $C_*(\mathcal{A}, M)$*.*

Sketch of proof :
Clearly, elements of the form $ua \equiv u \otimes a$, with $u \in M_r(k)$, $a \in M$ (or $a \in \mathcal{A}$) generate $M_r(M)$ (respectively $M_r(\mathcal{A})$, and it is sufficient to verify that Tr is a pre-simplicial morphism, i.e. $\mathrm{Tr} d_i = d_i \circ \mathrm{Tr}$ for all i, on elements of the form $(u_0 a_0 \otimes u_1 a_1 \otimes ... \otimes u_n a_n)$, where we have used the notation d_i for the face maps of both the complexes $C_*(M_r(\mathcal{A}), M_r(M))$ and $C_*(\mathcal{A}, M)$. To this end, first observe that

$$\mathrm{Tr}(u_0 a_0 \otimes ... \otimes u_n a_n) = \mathrm{tr}(u_0 ... u_n) a_0 \otimes a_1 \otimes ... \otimes a_n,$$

where $u_i \in M_r(k)$, $a_0 \in M$, $a_i \in \mathcal{A}$ for $i \geq 1$, and tr denotes the usual k-valued matrix trace on $M_r(k)$. The rest can be done by straightforward algebraic calculation using the property of the matrix-trace tr, i.e. $\mathrm{tr}(uv) = \mathrm{tr}(vu)$.

Thus, it makes sense to consider the induced map Tr_*. We also consider a natural map in the opposite direction, namely $\mathrm{inc} : \mathcal{A} \to M_r(\mathcal{A})$ by sending a to $E_{11} a \equiv E_{11} \otimes a$, i.e. the matrix with a as the diagonal entry, and rest being 0. Similarly, we can define $\mathrm{inc} : M \to M_r(M)$, and denote by the same symbol inc also the map from $M \otimes \mathcal{A}^{\otimes^n}$ to $M_r(M) \otimes M_r(\mathcal{A})^{\otimes^n}$ which sends $a_0 \otimes a_1 \otimes ... \otimes a_n$ to $\mathrm{inc}(a_0) \otimes ... \otimes \mathrm{inc}(a_n)$. It is easy to see that inc is a morphism of Hochschild complexes, so inc_* is a map at the homology level.

Theorem 2.7 *The maps* Tr_* *and* inc_* *are inverses of each other, hence* $\mathrm{Tr}_* : HH_*(M_r(\mathcal{A}), M_r(M)) \cong HH_*(\mathcal{A}, M)$.

Sketch of proof :
It is clear that $\mathrm{Tr} \circ \mathrm{inc} = \mathrm{id}$, so we prove that $\mathrm{inc} \circ \mathrm{Tr}$ is homotopic to identity, which will complete the proof. Define $h_i : M_r(M) \otimes M_r(\mathcal{A})^{\otimes^n} \to M_r(M) \otimes M_r(\mathcal{A})^{\otimes^{n+1}}$, $i = 0, 1, ..., n$, as follows :

$$h_i(\alpha^{(0)} \otimes \alpha^{(1)} \otimes ... \otimes \alpha^{(n)})$$
$$= \sum (E_{j_0 1}\alpha^{(0)}_{j_0 j_1}) \otimes (E_{11}\alpha^{(1)}_{j_1 j_2}) \otimes ... \otimes (E_{11}\alpha^{(i)}_{j_i j_{i+1}}) \otimes (E_{1 j_{i+1}} \otimes 1) \otimes \alpha^{(i+1)} \otimes ... \otimes \alpha^{(n)},$$

where the sum is over all $(j_0, ..., j_{i+1})$, with $j_p \in \{1, ..., r\}$. It can be verified that $h := \sum_{i=0}^n (-1)^i h_i$ satisfies

$$hb + bh = \mathrm{id} - \mathrm{inc} \circ \mathrm{Tr}.$$

Remark 2.8 *In Theorem 2.7, the map* inc *could be replaced by inclusion map through any other diagonal position, say* inc_k*, where* $k = 1, ..., r$*, which is the map* $\mathcal{A} \ni a \mapsto E_{kk} \otimes a$*. Thus,* $(\mathrm{inc}_k)_* = (\mathrm{Tr}_*)^{-1} = (\mathrm{inc}_1))_*$ *for any* k.

We shall see an interesting application of the above remark. Let $g \in \mathcal{A}$ be an invertible element, and consider the corresponding *conjugation* map, denoted by γ^g, from $M \otimes \mathcal{A}^{\otimes^n}$ to $M \otimes \mathcal{A}^{\otimes^{n+1}}$ for any n, given by

$$\gamma^g(a_0 \otimes a_1 \otimes ... \otimes a_n) := g a_0 g^{-1} \otimes g a_1 g^{-1} \otimes ... \otimes g a_n g^{-1},$$

where $a_0 \in M$, $a_i \in \mathcal{A}$ for $i \geq 1$. Clearly, this is a morphism of the Hochschild complex (**Exercise** : verify!), so induces a map $\gamma^g_* : HH_*(\mathcal{A}) \to HH_*(\mathcal{A})$. Similarly, let $w := \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \in M_2(\mathcal{A})$ be an invertible element, so we have $\gamma^w_* : HH_*(M_2(\mathcal{A})) \to HH_*(M_2(\mathcal{A}))$. However, it is a simple observation that

$$\gamma^w \circ \mathrm{inc}_1 = \mathrm{inc}_1, \quad \gamma^w \circ \mathrm{inc}_2 = \mathrm{inc}_2 \circ \gamma^g,$$

where $\mathrm{inc}_1, \mathrm{inc}_2 : \mathcal{A} \to M_2(\mathcal{A})$ are as in the Remark 2.8. Since $\mathrm{inc}_{1*} = \mathrm{inc}_{2*} = \mathrm{Tr}_*^{-1}$ by Remark 2.8, we have

$$\begin{aligned} &\gamma^g_* \\ &= \mathrm{inc}_{1*}^{-1} \circ \gamma^w_* \circ \mathrm{inc}_{1*} \\ &= \mathrm{inc}_{1*}^{-1} \mathrm{inc}_{1*} = \mathrm{id}. \end{aligned}$$

We record this important observation as a theorem for future use.

Theorem 2.9 *The map γ_*^g induced by the conjugation is identity at the level of Hochschild homology.*

We now discuss briefly the cohomological counterpart of the above.

Definition 2.10 *The Hochschild cohomology of the algebra with coefficients in the $\mathcal{A}$-$\mathcal{A}$ bimodule M, denoted by $H^*(\mathcal{A}, M)$, is defined as follows :*

$$H^n(\mathcal{A}, M) := H_n(\mathrm{Hom}_{\mathcal{A}^e}(C_*^{bar}(\mathcal{A}), M)).$$

Note that the n-th coboundary map of the complex $\mathrm{Hom}(C_*^{bar}(\mathcal{A}), M)$ is given by $\phi \mapsto (-1)^{n+1}\phi \circ b'$.

A useful alternative description of the Hochshild cochains is obtained as follows. Given a k-module map $f : \mathcal{A}^n \to M$ we consider an element $\tilde{f} \in \mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^{n+2}, M)$ given by

$$\tilde{f}(a_0, ...a_{n+1}) = a_0 f(a_1, ..., a_n)a_{n+1}.$$

We leave it as an exercise to verify that the above correspondence between $\mathrm{Hom}_k(\mathcal{A}^n, M)$ and $\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^{n+2}, M)$ is a bijection. We shall use this identification quite often without explicitly mentioning it. **Exercise :** If $\mathcal{A}$ is projective as a module over k then $H^n(\mathcal{A}, M) \cong \mathrm{Ext}^n_{\mathcal{A}^e}(\mathcal{A}, M)$.
Exercise : State and prove the cohomological analogues of Theorem 2.5, Theorem 2.7 and Theorem 2.9.
Exercises on interpretation of $H^n(\mathcal{A}, M)$ for small values of n
(i) Prove that $H^0(\mathcal{A}, M)$ is the subgroup of invariants of M, i.e. $\{m \in M : \ ma = am \ \forall a \in \mathcal{A}\}$.
(ii) Prove that $H^1(\mathcal{A}, M)$ is isomorphic with the group of outer derivations, i.e. $\mathrm{Der}(\mathcal{A}, M)/\mathrm{Inn}(\mathcal{A}, M)$, where $\mathrm{Der}(\mathcal{A}, M)$ is the abelian group of all derivations from $\mathcal{A}$ to M, and $\mathrm{Inn}(\mathcal{A}, M)$ denotes its subgroup generated by all inner derivations, i.e. derivations of the form $\mathcal{A} \ni a \mapsto ra - ar$ $(r \in M)$.
(iii) (a) Given a 2-cocycle $f : \mathcal{A}^2 \to M$, prove that the k-module $E := M \oplus \mathcal{A}$ can be equipped with an associative multiplication given by

$$(m_1, a_1) \cdot (m_2, a_2) := (m_1 a_2 + a_1 m_2 + f(a_1, a_2), a_1 a_2).$$

(b) Using (a), we get the following interesting description of $H^2(\mathcal{A}, M)$ in terms of so-called ableian extensions. By an *abelian extension* of $\mathcal{A}$ by M

we mean an associative unital k-algebra $\mathcal{B}$ which is as k-module $M \oplus \mathcal{A}$, $(m \oplus 0).a = (m.a \oplus 0)$, $a.(m \oplus 0) = (a.m \oplus 0)$ and $(m_1 \oplus 0).(m_2 \oplus 0) = 0$ in $\mathcal{B}$, where $m, m_1, m_2 \in M$, $a \in \mathcal{A}$. Two such extensions $\mathcal{B}$ and $\mathcal{B}'$ are said to be equivalent if there is a module map $\phi : \mathcal{B} \to \mathcal{B}'$ which commutes with the inclusions of $\mathcal{A}$ and M in $\mathcal{B}$, $\mathcal{B}'$. Observe that an abelian extension $\mathcal{B}$ can be identified with a split-exact sequence of the form $0 \to M \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{A} \to 0$ of associative algebras, with a splitting homomorphism $s : \mathcal{A} \to \mathcal{B}$, satisfying $i(ama') = s(a)i(m)s(a')$ for all $a, a' \in \mathcal{A}$, $m \in M$. One can define a natural 'sum' of two such extensions, say $0 \to M \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{A} \to 0$ and $0 \to M \xrightarrow{i'} \mathcal{B}' \xrightarrow{j'} \mathcal{A} \to 0$, by taking the submodule of $\mathcal{B} \oplus \mathcal{B}'$ consisiting of $(b \oplus b')$ such that $j(b) = j'(b')$. Prove that $H^2(\mathcal{A}, M)$ is isomorphic with the group of all abelian extensions of $\mathcal{A}$ by M modulo the above-mentioned equivalence.

The case $M = \mathcal{A}^* \equiv \mathrm{Hom}(\mathcal{A}, k)$ is of particular interest and the corresponding Hochschild cohomology is denoted by $H^*(\mathcal{A})$.

3 Cyclic homology and cohomology : definition

3.1 Cyclic homology

There are three equivalent but aparently different ways of defining cyclic homology and cohomology. Recall the operators b and b' on $\mathcal{A}^{n+1}$. Consider the action of the group $Z/(n+1)Z$ on $\mathcal{A}^{n+1}$ given by $t \equiv t_n : \mathcal{A}^{n+1} \to \mathcal{A}^{n+1}$ where

$$t_n(a_0, a_1, ..., a_n) := (-1)^n(a_n, a_0, ..., a_{n-1}).$$

Let $N := 1 + t_n + t_n^2 + ... + t_n^n$ (called the *norm operator*).

Lemma 3.1 *We have*

$$(1-t)b' = b(1-t), \qquad b^N = Nb.$$

Proof :
It is simple to check that $d_i t_n = -t_{n-1} d_{i-1}$ for $0 < i \leq n$ and $d_0 t_n = (-1)^n d_n$. From this, it follows that $(-1)^i d_i t = t((-1)^{i-1} d_{i-1})$, hence $(b - d_0)t = tb'$. Thus $bt = tb' + (-1)^n d_n$, so $(1-t)b' = b(1-t)$. Moreover, one has

$$d_i t^j = (-1)^j t^j d_{i-j}, \quad i \geq j,$$

$$d_i t^j = (-1)^{n-j+1} t^{j-1} d_{n+1+i-j}, \quad i < j.$$

Thus, we have

$$\begin{aligned} & b'N \\ = & \left(\sum_{i=0}^{n-1} (-1)^i d_i\right)\left(\sum_{j=0}^{n} t^j\right) \\ = & \sum_{0 \leq i \leq j \leq n-1} (-1)^{i-j} t^j d_{i-j} + \sum_{0 \leq i < j \leq n-1} (-1)^{n+1+i-j} t^{j-1} d_{n+1+i-j}. \end{aligned}$$

We complete the proof of the Lemma by observing that for $0 \leq k \leq n$, the coefficient of $(-1)^k d_k$ in the above expression for $b'N$ is $\sum_{0 \leq j \leq n-1-k} t^j + \sum_{n-k \leq j-1 \leq n-1} t^{j-1} = N$.

This lemma shows in particular that the image of $(1-t)$ is invariant under the Hochschild boundary operator b. Following the original approach of

Connes, we consider $C_n^\lambda \equiv C_n^\lambda(\mathcal{A}) := \mathcal{A}^{n+1}/(1-t)$, and the complex (called *Connes' complex*) (C_*^λ, b). Its homology is denoted by $H_*^\lambda(\mathcal{A})$. On the other hand, the formulae $(1-t)b' = b(1-t)$ and $b'N = Nb$ can be re-written as $b(1-t) - (1-t)b' = 0$, $-b'N + Nb = 0$, which allows us to construct a bi-complex $CC \equiv (CC_{pq})_{p,q\geq 0}$ where $CC_{pq} = \mathcal{A}^{p+1}$ and the horizontal and vertical differentials on CC_{pq} are given by the following :

$$d^h = 1-t \ \text{ if } q \text{ is odd}, \quad N \text{ if } q \text{ is even},$$

$$d^v = b \text{ if } \ q \text{ is even} \quad , -b' \text{ if } \ q \text{ is odd}.$$

Note that the relations between N, b, b' and t as above enable us to verify the conditions in the definition of a bi-complex.

Definition 3.2 *The above bi-complex is called the cyclic bi-complex and its homology, i.e.* $H_*(\mathrm{Tot}(CC))$ *is called the cyclic homology of* $\mathcal{A}$*. It is denoted by* $HC_*(\mathcal{A})$*.*

Note that this definition of cyclic homology is the most general one, since this is valid for the arbitrary characteristic and also in the non-unital case. However, we now relate it to the Connes' complex. Let $p : \mathrm{Tot}(CC) \to C^\lambda$ be the map, which is given on $\oplus_{p+q=n} CC_{pq}$ by the composition of projection onto $CC_{n0} \cong \mathcal{A}^{n+1}$ and the quotient map $\mathcal{A}^{n+1} \to \mathcal{A}^{n+1}/(1-t) = C_n^\lambda$. This is clearly a morphism of complexes, hence induces a homomorphism p_* at the level of homology groups.

Theorem 3.3 *If k has an isomorphic copy of Q (i.e. k has zero characteristic) the above map p_* is an isomorphism, i.e.* $H_*^\lambda(\mathcal{A}) \cong HC_*(\mathcal{A})$.

Proof :
Fix any $n \geq 0$. We claim that $H_q(CC_{n,*}) = 0$ for all $q > 0$. To this end, let us construct a homotopy $h \equiv h_q : CC_{n,q} \cong \mathcal{A}^{n+1} \to CC_{(n,q+1)} \cong \mathcal{A}^{n+1}$ given by

$$h_q = \frac{1}{n+1} \mathrm{id}_{\mathcal{A}^{n+1}} \ \text{ if } q \text{ is odd},$$

$$h_q = -\frac{1}{n+1} \sum_{i=1}^{n} i t^i \ \text{ if } \ q \text{ is even}.$$

It can easily be verified that $hd^h + d^h h = \mathrm{id}$, so the claim is proved. Now, the theorem follows by applying Theorem 1.6, once we observe that $H_0(C_{n,*})$ is nothing but H_n^λ.

Next we give yet another description of the cyclic homology. Write the complex $T_n := \mathrm{Tot}(CC)$ as a direct sum $T_n = A_n \oplus A'_n$, with $A'_n = C_{n1}$ (the second column of CC) and A_n being the rest. Recall that A'_n is contractible, with the homotopy given by the map

$$s(a_1, ..., a_n) := 1 \otimes a_1 \otimes ... \otimes a_n.$$

Using the notation of Theorem 1.8, with $\alpha = b, \beta = 1-t, \gamma = N, \delta = -b'$ and $h = s$, we conclude that the homology of $\mathrm{Tot}(CC)$ is isomorphic with that of $A_*, b+B$, where $B := (1-t)sN$. Applying this procedure repeatedly, we get a bi-complex $\mathcal{B} \equiv \mathcal{B}(\mathcal{A})$, whose homology is same as the cyclic homology, such that $\mathcal{B}_{pq} = \mathcal{A}^{p-q+1}$ if $p \geq q$ and 0 otherwise, with the horizontal differentials given by B and the vertical ones by b. This is called the $b - B$ bi-complex. Clearly, the total complex $A_* = \mathrm{Tot}(\mathcal{B})$ is given by

$$A_n = \mathcal{A}^{n+1} \oplus \mathcal{A}^{n-1} ...$$

(upto $\mathcal{A}^1$ or $\mathcal{A}^2$ depending on n is even or odd). We also need the following useful identification of this complex. Let C_* be a complex whose n-th module is same as that of the total complex of CC, i.e. $C_n = \mathcal{A}^{n+1} \oplus \mathcal{A}^n \oplus ...\mathcal{A}$, but with the new differential $b+B$. Also, consider the module $k[x^2]$ (polynomials in square of an indeterminate x) as a complex with odd modules being 0 and the even ones isomorphic with k, and with the zero differential.
Exercise : Prove that $(A_*, b+B)$ is isomorphic as a chain complex with the tensor product complex $k[x^2] \otimes C_*$.

Now we define an important operator on the cyclic complex (equivalently, on $\mathcal{B}$ or on C^λ). Let $CC^{(2)}$ denote the bi-complex obtained by taking the first two columns of CC.

Lemma 3.4 *The tolal complex of $CC^{(2)}$ has the same homology as the homology of the first column, which is same as $HH_*(\mathcal{A})$.*

Proof is left as an exercise using Theorem 1.8, and noting that the second column is acyclic.

Now, consider a short exact sequence of bi-complexes

$$0 \to CC^{(2)} \xrightarrow{i} CC \xrightarrow{j} CC[0, 2],$$

where i is the inclusion map, $CC[0,2]_{pq} := CC_{p,q-2} = \mathcal{A}^{p+1}$ for $q \geq 2$, with the same differentials as CC (i.e. $CC[0,2]$ is indexed by $p \geq 0, q \geq 2$), and j is the map which sends $CC_{p,q}$ to 0 if $q = 0, 1$ and acts as the identity on $\mathcal{A}^{p+1} = CC_{p,q-2} = CC[0,2]_{p,q}$ for $q \geq 2$. Clearly, this induces a long exact sequence in homology, and since the homolgy $H_n(\mathrm{Tot}(CC[0,2]))$ is obviosly isomorphic (**exercise** : prove it) with $H_{n-2}(\mathrm{Tot}(CC)) = HC_n(\mathcal{A})$ for $n \geq 2$, we have the following exact sequence (called *Connes' long exact sequence*) :

$$... \to HH_n(\mathcal{A}) \xrightarrow{I} HC_n(\mathcal{A}) \xrightarrow{S} HC_{n-2}(\mathcal{A}) \xrightarrow{\partial} HH_{n-1}(\mathcal{A}) \xrightarrow{I} ...$$

where I and S are respectivlely the maps i_* and j_* induced at the level of homology by the morphisms i and j. Moreover, in the next lemma we verify that the connecting map ∂ in the above long exact sequence is indeed the operator B defined earlier. However, before that, let us quickly see some immediate but very important implications of this long exact sequence.

Theorem 3.5 *The theorems 2.5, 2.7 and 2.9 hold by replacing the Hochschild homology by cyclic homology. Thus, in particular, γ_*^g is identity on HC_*. More generally, whenever a homomorphsim $\phi : \mathcal{A} \to \mathcal{B}$ induces isomorphism at the level of HH_*, it does so at the level of HC_* too.*

The proof is straightforward application of the Connes' long exact sequence, and we leave it as an **Exercise**.

Lemma 3.6 *If we identify HC_* with the homology of $\mathrm{Tot}(\mathcal{B})$, (i.e. if we are in the $b - B$ picture of cyclic homology) then the connecting map ∂ is given by B, i.e. $\partial([x]) = [B(x_1)]$, where $[x]$ denotes the homology class of $x = (x_1, x_2, ...) \in \mathrm{Tot}(\mathcal{B})_{n-2} = \mathcal{A}^{n-1} \oplus \mathcal{A}^{n-3} \oplus ...$ and $[B(x_1)]$ is the class of $B(x_1) \in \mathcal{A}^n$ in the first column of $\mathcal{B}$, which is the Hochschild complex.*

Proof :
It follows from the observations that the map i in the $b - B$ picture is the inclusion of the first column in the total complex of $\mathcal{B}$, that the map j sends $(x, y, z, ...) \in \mathrm{Tot}(\mathcal{B})_n$ to $(y, z, ...) \in \mathrm{Tot}(\mathcal{B})_{n-2}$ and the definition of the connecting map. Indeed, given $(y, z, ...) \in \mathrm{Tot}(\mathcal{B})_{n-2}$, we take any $x \in \mathcal{A}^{n+1}$ so that $j(x, y, ...) = (y, z, ...)$ and then observe that the differential d of $\mathrm{Tot}(\mathcal{B})$ is $b+B$, so that $d(x, y, ...) = (b(x)+B(y), b(y)+B(z), ...)$. However, the class of $b(x)$ is 0 in the homology of the first column, so $[B(y)] = [b(x) + B(y)]$, which completes the proof.

Let us now describe explicitly the map S in the Connes' complex C^λ.

Lemma 3.7 *Let $b^{(2)} := \sum_{0\le i<j\le n}(-1)^{i+j}d_i d_j : \mathcal{A}^{n+1} \to \mathcal{A}^{n-1}$, and assume that k is of characterisric zero. Then the map S on $H_n^\lambda(\mathcal{A}) \cong HC_n(\mathcal{A})$ is given by*

$$S(\overline{x}) = -\frac{1}{n(n-1)}\overline{b^{(2)}(x)},$$

where $x \in \mathcal{A}^{n+1}$ and $\overline{x}$ denotes the image of x under the quotient map $\mathcal{A}^{n+1} \to \mathcal{A}^{n+1}/\mathrm{Im}(1-t)$.

Sketch of proof :
We break the proof into several exercises, each of which can be done by elementary and routine algebraic manipulation. Let $\beta = \sum_{i=0}^{n}(-1)^i i d_i : C_n \to C_{n-1}$. Verify the following :
(i) $b\beta + \beta b = b^{(2)}$.
(ii) $bb^{(2)} = b^{(2)}b$.
(iii) $(1-t)h = I - \frac{1}{n+1}N$, where $h : \mathcal{A}^{n+1} \to \mathcal{A}^n$ given by $-\frac{1}{n+1}\sum_{i=0}^{n} it^i$.
(iv) $b'hb = -frac1n\beta b + bhb$ on $C_n \equiv \mathcal{A}^{n+1}$ (use the identity $d_{n-1}t^i = (-1)^i t^i d_{n-1-i}$).
Now, given $x \in \mathcal{A}^{n+1}$ such that $\overline{x} := x + \mathrm{Im}(1-t$ is a cycle in C^λ, i.e. $b(x) \in \mathrm{Im}(1-t_{n-1})$, it can be verified easily that $\alpha := (x = x_0, x_1, x_2, ...)$ is a cycle in $\mathrm{Tot}(CC)_n$, where $x_k \in \mathcal{A}^{n+1-k}$ is given by :

$$x_{2k+1} = -hb(x_{2k}), \quad x_{2k+2} = -\frac{1}{n-1}b'(x_{2k+1}), k = 0, 1, ...$$

For example, we have

$$\begin{aligned} &b(x) + (1-t)(x_1) \\ &\quad = \ b(x) - (1-t)h(b(x)) \\ &\quad = \ \frac{1}{n+1}N(b(x)) \ \ (\text{using} \ \ (iii)) = 0, \end{aligned}$$

since $b(x) \in \mathrm{Im}(1-t)$ and $N(1-t) = 0$.

Thus, $S(\alpha) = (x_2, x_3, ...) \in \mathrm{Tot}(CC)_{n-2}$, and the homology class of $\overline{x_2} = x_2 + \mathrm{Im}(1-t)$ in H_{n-2}^λ is same as that of

$$-\frac{1}{n-1}b'(x_1) = -\frac{1}{n-1}b'hb(x) \sim -\frac{1}{n(n-1)}\beta b(x) \sim -\frac{1}{n(n-1)}b^{(2)}(x),$$

since $bhb(x) \sim 0$ and $b\beta(x) \sim 0$, where $a \sim b$ means a and b are homologous, i.e. have the same homology class.

Exercise
Prove that $HC_{2n}(k) \cong k$, $HC_{2n+1}(k) \cong 0$ for all $n = 0, 1, ...$ Solve this explicitly in all three descriptions of HC; in particular, observe that a generator of $HC_{2n}(k)$ in the CC bicomplex picture can be taken to be the following

$$((-1)^n 2(n-1)!, ..., -6, 2, -1, 1) \in k^{2n} \oplus k^{2n-1} ... \oplus k.$$

Exercise : normalization: Using the normalization of Hochshild homology and the role of the Z_n-action in the definition of cyclic homology, argue carefully that in the definition of cyclic homology the algebra $\mathcal{A}$ can be replaced by $\mathcal{A}/k$ without loss of generality. That is, in particular, one can consider the *normalized bicomplex* $\bar{CC}$ defined by replacing $\mathcal{A}$ by $\mathcal{A}/k$.

3.2 Cyclic cohomology

We can 'dualize' the cyclic homology theory to obtain the theory of cyclic cohomology. Take $CC^{pq}(\mathcal{A}) = \mathrm{Hom}_k(\mathcal{A}^{p+1}, k) \equiv C^p$ for $p, q \geq 0$, with differential maps of the bicomplex of the cyclic homology theory replaced by their duals, i.e. we have the following bicomplex :

We define *cyclic cohomology* of $\mathcal{A}$ as the cohomology of the total complex of the bicomplex CC^*, i.e. $HC^*(\mathcal{A}) := H^*(\mathrm{Tot}(CC^*))$. In analogy with the case of cyclic homology, we can consider Connes' complex $C_\lambda^n := \mathrm{Ker}(1 - t_n^*)$. It is easy to see that C_λ^n consists of all multilinear functionals $\phi : \mathcal{A}^{n+1} \to k$ satisfying $\phi(a_0, a_1, ..., a_n) = (-1)^n \phi(a_n, a_0, a_1, ..., a_{n-1})$, and the coboundary map b^* is given by,

$$b^*\phi(a_0, ..., a_{n+1}) = \sum_{i=0}^{n} (-1)^i \phi(a_0, ..., a_{i-1}, a_i a_{i+1}, a_{i+2}, ..., a_{n+1}) + (-1)^{n+1} \phi(a_{n+1} a_0, a_1, ..., a_n).$$

Such functionals are called cyclic cocycles.

Exercise Prove that when k is of characteristic zero, the cohomology of the cochain complex (C_λ^*, b^*) is isomorphic with $HC^*(\mathcal{A})$.

Exercise : Prove that the natural pairing $< \cdot, \cdot >$: $\mathcal{A}^{n+1} \times \mathrm{Hom}(\mathcal{A}^{n+1}, k) \to k$ induces a pairing between $HC_*(\mathcal{A})$ and $HC^*(\mathcal{A})$, also denoted by $< \cdot, \cdot >$.

Exercise : Obtain an analogue of the $b-B$ bicomplex for cyclic cohomology, and prove that this gives an equivalent definition of cyclic cohomology. Moreover, derive the long exact sequence connecting Hochschild and cyclic cohomology groups, as done in the context of cyclic homology earlier. Prove also the cohomological analogue of Theorem 3.5.

Exercise : If k is a field and $\mathcal{A}$ is finite dimensional as k-vector space, prove that $HC^*(\mathcal{A}) \cong \mathrm{Hom}(HC_*, k)$ (note : the conclusion of this exercise can actually be derived under much weaker hypothesis, for example by applying the *Universal Coefficient Theorem*, but we shall not discuss it here).

Exercise : By considering $k = Z$ and $\mathcal{A} = Q$ as a k-algebra, compute $HC_*(\mathcal{A})$ and $HC^*(\mathcal{A})$, and observe that the conclusion of the previous exercise fails in this case, i.e. $HC^*(\mathcal{A})$ is not isomorphic with $\mathrm{Hom}(HC_*(\mathcal{A}), k)$.

Exercise: Prove that, by considering the normalized version of all the complexes or bicomplexes involved, any cyclic cocycle τ is cohomologous to a *normalized* one, say $\bar{\tau}$, i.e. $\bar{\tau}(a_0, a_1, ..., a_n) = 0$ if any of the a_i is 1 (for $n \geq 1$). Of course, for $n = 0$, normalization does not put any additional restriction. 0-dimensional cyclic cocycles are just traces, i.e. functionals τ on $\mathcal{A}$ satisfying $\tau(ab) = \tau(ba)$.

We shall now give an alternative and convenient description (due to Connes) of cyclic cocycles. Before this, let us show how cyclic cocycles arise naturally in the classical differential geometry and topology. Consider a compact oriented smooth manifold M of dimension m, and let $\mathcal{A}$ denote the $\mathbb{C}$-algebra of smooth complex-valued functions on M. Let C be a closed oriented submanifold of dimension $n \leq m$, and consider the following $n+1$-linear functional on $\mathcal{A}$:

$$\phi(f_0, f_1, ..., f_n) := \int_C f_0 df_1 \wedge df_1 \wedge ... \wedge df_n,$$

where $\wedge$ denotes the exterior product of differential forms. It is an interesting **exercise** to verify using the well-known properties of the exterior product that ϕ is indeed a cyclic cocycle (of dimension n) in $C^*_\lambda(\mathcal{A})$. In fact, if we shall soon see that this is a generic construction of cyclic cocycles. In order to do so, we observe that the commutativity of the algebra of smooth functions do not play a role in the proof that ϕ is cyclic cocycle; but the algebraic structure of the exterior algebra of forms and the fact that the integral of closed forms over C is zero are essential. This gives a clue how to extend these ideas to the general situation. For this, we need a suitable abstraction of the algebra of forms, called *differential graded algebra*, which is defined below.

Definition 3.8 *Let S be an abelian semigroup with identity (denoted by 0). By an S-graded algebra(or just graded algebra, if S is understood from the context) over k we mean a k-algebra $\mathcal{B}$ which has a direct sum decomposition $\mathcal{B} = \bigoplus_{i \in S} \mathcal{B}_i$ into k-submodules $\mathcal{B}_i$ satisfying $\mathcal{B}_i \mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ for all $i, j \in S$. If $\mathcal{B}$ is unital, we also assume that the identity of $\mathcal{B}$ belongs to $\mathcal{B}_0$. Morphisms of graded algebras $\mathcal{B}$ and $\mathcal{C}$, say, are algebra homomorphisms $\phi : \mathcal{B} \to \mathcal{C}$ which respects the grading, i.e. the direct sum decomposition, in the sense that $\phi(\mathcal{B}_i) \subseteq \mathcal{C}_i \ \forall i$.*

An S-graded (right)-module over an S-graded algebra $\mathcal{A}$ is a (right)- $\mathcal{A}$-module E, which has a direct sum decomposition $E = \bigoplus_{i \in S} E_i$, where E_i are k-modules and $E_i \mathcal{A}_j \subseteq E_{i+j}$ for all $i, j \in S$.

An element e of E_i ($a \in \mathcal{A}_i$ respectively) is called homogeneous of degree i of E ($\mathcal{A}$), i is called the degree of e (a), written $\deg(e) = i$ ($\deg(a) = i$).

Let us assume, for the rest of our discussion, that S in the above definition is one of the following : subgroups of Z, Z_n, Z_+. In case $S = Z_+$, we extend the grading to Z by setting $\mathcal{A}_i = (0)$ for $i < 0$. We also consider the *trivial grading* on algebra and modules, i.e. view any algebra or module to be graded by the trivial group (0), taking $\mathcal{A}_0 = \mathcal{A}$ ($E_0 = E$). We shall also consider an S-graded $\mathcal{A}$-module $E = \oplus_i E_i$, such that each E_i is a module over an algebra $\mathcal{A}$ without grading, to be a graded module over a graded algebra by viewing the algebra $\mathcal{A}$ to be S-graded with $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_i = (0)$ for $i \neq 0$.

Definition 3.9 *A differential S-graded algebra is a pair $(\mathcal{B}, d)$ where $\mathcal{B}$ is an S-graded algebra and $d : \mathcal{B} \to \mathcal{B}$ is a k-module map satisfying the following :*
(i)$d(\mathcal{B}_i) \subset \mathcal{B}_{i+1}$,
(ii) $d^2 \equiv d \circ d = 0$,
(iii)$d(ab) = d(a)b + (-1)^{\deg(a)} a d(b)$ for homogeneous elements $a, b \in \mathcal{B}$.
The map d is called a differential of degree 1 (or just differential).

It is perhaps clear that the above definition captures the algebraic structure of the exterior algebra of differential forms, except that we do not need the commutativity of the function algebra and want to work with more general noncommutative algebras. However, to construct cyclic cocyles, we shall also need an analogue of the integral w.r.t. volume form, which leads to the following definition.

Definition 3.10 *A cycle of degree n over a k-algebra $\mathcal{A}$ is a quadruplet $(\Omega, d, \int, \rho)$, where*
(i) $\Omega \equiv \oplus k = 0^\infty \Omega^k$ is a differential graded k-algebra (with Z_n-grading), d being the differential map;
(ii) $\rho : \mathcal{A} \to \Omega^0$ is a homomorphism of k-algebras;
(iii) $\int : \Omega^n \to k$ is a linear functional satisfying
(a) $\int(\omega_2\omega_1) = (-1)^{\deg(\omega_1)\deg(\omega_2)} \int(\omega_1\omega_2)$ for all homogeneous elements ω_1, ω_2 of Ω (graded trace property),
(b) $\int(d(\omega)) = 0$ for all $\omega \in \Omega^{n-1}$ (closedness).
Given a cycle over $\mathcal{A}$ as above, the following $n+1$-linear functional ϕ is called the character of the cycle:

$$\phi(a_0, a_1, ..., a_n) := \int (\rho(a_0)d\rho(a_1)...d\rho(a_n)), \quad a_i \in \mathcal{A}.$$

We now claim that ϕ is a cyclic cocycle.

Theorem 3.11 *The character ϕ of a cycle $(\Omega, d, \int, \rho)$ over $\mathcal{A}$ is a cyclic cocycle.*

Proof :
Let $a_0, ..., a_{n+1} \in \mathcal{A}$ and denote $\rho(a_i)$ by b_i. We have

$$\begin{aligned} &\phi(a_0, a_1, ..., a_n) \\ &= \int((b_0 db_1...db_{n-1})db_n) \\ &= (-)^{n-1} \int(db_n b_0 db_1...db_{n-1}) \text{ by graded trace property)} \\ &= (-1)^{n-1} \int(d(b_n b_0)db_1...db_{n-1}) + (-)^n \int(b_n db_0 db_1...db_{n-1}) \\ &= (-1)^n \phi(a_n, a_0, ..., a_{n-1}), \end{aligned}$$

since $\int(b(b_n b_0)db_1...db_{n-1}) = \int(d(b_n b_0 b_1...b_{n-1}) = 0$ by the closedness condition. So, it is enough to show that ϕ is a Hochschild cocycle, i.e. $b^*\phi = 0$. To this end, observe that

$$\begin{aligned} &\int((b_0 db_1...db_k)(b_{k+1}...db_{n+1})) \\ &= \int(b_0 db_1...db_{k-1}(d(b_k b_{k+1}) - b_k db_{k+1})db_{k+2}...db_{n+1}) \\ &= \int(b_0 db_1...d(b_k b_{k+1})db_{k+2}...db_{n+1}) - \int(b_0 db_1...db_{k-1} b_k db_{k+1}...db_{n+1}) \end{aligned}$$

$$
\begin{aligned}
&= \int (b_0 db_1 ... d(b_k b_{k+1}) db_{k+2} ... db_{n+1}) - \int (b_0 db_1 ... d(b_{k-1} b_k) db_{k+1} ... db_{n+1}) \\
&+ \int (b_0 db_1 ... db_{k-2} b_{k-1} db_k ... db_{n+1}) \\
&\quad \cdots \\
&= \sum_{j=0}^{k} (-1)^{k-j} \int (b_0 db_1 ... db_{j-1} d(b_j b_{j+1}) db_{j+2} ... db_{n+1}) \\
&= \sum_{j=0}^{k} (-1)^{k-j} \phi(a_0, a_1, ..., a_{j-1}, a_j a_{j+1}, a_{j+2}, ..., a_{n+1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int ((b_{k+1} ... db_{n+1})(b_0 db_1 ... db_k)) \\
&= \sum_{j=0}^{n-k-1} (-1)^{n-k-j} \phi(a_{k+1}, .., a_{k+1+j} a_{k+1+j+1}, ..., a_{n+1}, a_0, ..., a_k) \\
&+ \phi(a_{k+1}, ..., a_n, a_{n+1} a_0, a_1, ..., a_k) \\
&= (-1)^{n(k+1)} \{ \sum_{j=0}^{n-k-1} (-1)^{n-k-j} \phi(a_0, a_1, ..., a_{k+1+j} a_{k+1+j+1}, a_{k+1+j+2}, ..., a_{n+1}) \\
&+ \phi(a_{n+1} a_0, a_1, ..., a_n) \}.
\end{aligned}
$$

However, by the graded trace property,

$$
\int ((b_0 db_1 ... db_k)(b_{k+1} ... db_{n+1})) = (-1)^{k(n-k)} \int ((b_{k+1} ... db_{n+1})(b_0 db_1 ... db_k)),
$$

from which we get

$$
\begin{aligned}
&\sum_{j=0}^{k} (-1)^{k-j} \phi(a_0, a_1, ..., a_{j-1}, a_j a_{j+1}, a_{j+2}, ..., a_{n+1}) \\
&= (-1)^{k(n-k)+n(k+1)} \{ \sum_{j=0}^{n-k-1} (-1)^{n-k-j} \phi(a_0, a_1, ..., a_{k+1+j} a_{k+1+j+1}, a_{k+1+j+2}, ..., a_{n+1}) \\
&+ \phi(a_{n+1} a_0, a_1, ..., a_n) \},
\end{aligned}
$$

i.e.

$$
\sum_{j=0}^{k} (-1)^{j} \phi(a_0, a_1, ..., a_{j-1}, a_j a_{j+1}, a_{j+2}, ..., a_{n+1})
$$

$$= (-1)^{k+n(k+1)}\{\sum_{j=0}^{n-k-1}(-1)^{n-k-j}\phi(a_0,a_1,...,a_{k+1+j}a_{k+1+j+1},a_{k+1+j+2},...,a_{n+1})$$
$$+ \phi(a_{n+1}a_0,a_1,...,a_n)\}$$
$$= -(-1)^{n+k+k(n-k)+n(k+1)}\{\sum_{j=0}^{n-k-1}(-1)^{1+k+j}\phi(a_0,a_1,...,a_{k+1+j}a_{k+1+j+1},a_{k+1+j+2},...,a_{n+1})$$
$$+ (-1)^{n+1}\phi(a_{n+1}a_0,a_1,...,a_n)\}$$
$$= -\{\sum_{j=k+1}^{n}(-1)^j\phi(a_0,a_1,...,a_ja_{j+1},a_{j+2},...,a_{n+1})+(-1)^{n+1}\phi(a_{n+1}a_0,a_1,...,a_n)\},$$

which proves that ϕ is indeed a cocycle for the Hochschild coboundary operator, thus completing the proof.

Recall that a from a k-algebra $\mathcal{A}$ to an $\mathcal{A}-\mathcal{A}$ bimodule M is a k-linear map $\delta:\mathcal{A}\to M$ satisfying the Leibnitz rule, i.e. $\delta(ab)=\delta(a)b+a\delta(b)$ for all $a,b\in\mathcal{A}$. If $\mathcal{A}$ is unital (that is convention, unless otherwise specified in a particular context), it is easy to see that $\delta(1)=0$. Moreover, note that the differential map d of the differential graded algebra Ω in the Definition 3.10, is a derivation when restricted to Ω^0, and $d\circ\rho:\mathcal{A}\to\Omega^0$ is a derivation from $\mathcal{A}$ to $\rho(1)\Omega^0\rho(1)$, viewed as an $\mathcal{A}$-$\mathcal{A}$ bimodule by $a\rho(1)\omega_0\rho(1)b:=\rho(a)\omega_0\rho(b)$ for $a,b\in\mathcal{A}$, $\omega_0\in\Omega^0$. Moreover, if $\rho(1)=1$, observe that the character ϕ is in fact a normalized cocycle, i.e. it satisfies $\phi(a_0,a_1,...,a_n)=0$ whenever at least one of the a_i is 1 ($n\geq 1$).

We want to show that any normalized cocycle does arise from a cycle over $\mathcal{A}$ in the way described by Theorem 3.11. For this, we need to construct a universal bimodule, which is a very abstract analogue of the module of one-forms on a manifold. Consider the k-module $E=(\mathcal{A}\otimes_k\mathcal{A})/(\mathcal{A}\otimes 1)$, denoting the class $(a_0\otimes a_1)+(\mathcal{A}\otimes 1)$ by $[a_0\otimes a_1]$, and define a bimodule structure of E by setting

$$a.[a_0\otimes a_1]:=[aa_0\otimes a_1],\quad [a_0\otimes a_1]a:=[a_0\otimes a_1a]-[a_0a_1\otimes a].$$

Exercise : Verify that the above is well-defined and defines an $\mathcal{A}$-$\mathcal{A}$ bimodule structure on E.

We shall denote the above bimodule by $\Omega^1_{\rm univ}(\mathcal{A})$. Define a k-linear map $\delta:\mathcal{A}\to\Omega^1_{\rm univ}(\mathcal{A})$ by $\delta(a)=[1\otimes a]$. It is easy to verify the derivation property (**Exercise**). This derivation is universal in the following sense :

Theorem 3.12 *Consider a category with objects being the pairs (M,β), where M is an $\mathcal{A}$-$\mathcal{A}$ bimodule and $\beta:\mathcal{A}\to M$ is a derivation; and the* *morphisms from (M, β) to (N, η) being the bimodule maps $\psi : M \to N$ satisfying $\psi \circ \beta = \eta$. Then the pair $(\Omega^1_{\rm univ}(\mathcal{A}), \delta)$ is a universal (initial) object in this category.*

Proof :
We have to show that given any $\mathcal{A}$-$\mathcal{A}$ bimodule M and a derivation $\beta : \mathcal{A} \to M$, we can find a unique bimodule map $\psi : \Omega^1_{\rm univ}(\mathcal{A}) \to M$ such that $\psi \circ \delta = \beta$. Let us define $\psi_0 : \mathcal{A} \otimes_k \mathcal{A} \to M$ by $\psi_0([a_0 \otimes a_1] := a_0\beta(a_1)$, and observe that $\psi_0(\mathcal{A} \otimes 1) = (0)$ since $\beta(1) = 0$. Thus ψ_0 induces a map, say ψ, from $\Omega^1_{\rm univ}(\mathcal{A})$ to M given by $\psi([a_0 \otimes a_1]) = a_0\beta(a_1)$. It is now quite straightforward to verify that ψ is indeed a bimodule map, by using the definition of the bimodule structure of $\Omega^1_{\rm univ}(\mathcal{A})$ and the derivation property of β. The uniqueness of ψ follows from the fact that elements of the form $[a_0 \otimes a_1]$ are total in $\Omega^1_{\rm univ}(\mathcal{A})$.

We call $\Omega^1_{\rm univ}(\mathcal{A})$ the *bimodule of universal one-forms on* $\mathcal{A}$ and the δ the *universal derivation on* $\mathcal{A}$. By taking repeated tensor product over $\mathcal{A}$, we can define the *universal forms* of higher orders, i.e. set $\Omega^n_{\rm univ}(\mathcal{A}) := \Omega^1_{\rm univ}(\mathcal{A}) \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} \Omega^1_{\rm univ}(\mathcal{A})$($n$ copies) for any $n \geq 1$, with $\Omega^0_{\rm univ}(\mathcal{A}) := \mathcal{A}$. Before we proceed further, we want to identify $\Omega^n_{{\rm univ}(\mathcal{A})}$ as a k-module with a quotient of $\mathcal{A} \otimes_k ... \otimes_k \mathcal{A}$ (n copies). This is done in the following set of exercises with hints.

Exercise

In these exercises, $\mathcal{E}$ denotes the k-submodule of $\mathcal{A}^3 := \mathcal{A} \otimes_k \mathcal{A} \otimes_k \mathcal{A}$ generated by elements of the form $a_0 \otimes a_1 \otimes a_2$ with a_1 or a_2 equal to 1, and denote by $[a_0 \otimes a_1 \otimes a_2]_{\mathcal{E}}$ the class of $a_0 \otimes a_1 \otimes a_2$ in the quotient module $\mathcal{A}^3/\mathcal{E}$. Recall also that $[a_0 \otimes a_1]$ denotes the class of $a_0 \otimes a_1$ in the quotient $\Omega^1_{\rm univ}(\mathcal{A}) = (\mathcal{A} \otimes_k \mathcal{A})/\mathcal{D}$ where $\mathcal{D} = (\mathcal{A} \otimes 1)$.

(i) Observe that $\Omega^2_{\rm univ}(\mathcal{A})$ is generated as k-module by elements of the form $a_0\delta(a_1)\delta(a_2) \equiv [a_0 \otimes a_1] \otimes_{\mathcal{A}} [1 \otimes a_2]$.

(ii) Define $\Phi_0 : \mathcal{A}^3 \to \Omega^2_{\rm univ}(\mathcal{A})$ by $\Phi_0(a_0 \otimes a_1 \otimes a_2) := a_0\delta(a_1)\delta(a_2)$, and show that $\Phi_0(\mathcal{E}) = (0)$, so that Φ_0 induces a map (indeed a k-module map), say Φ, from $\mathcal{A}^3/\mathcal{E}$ to $\Omega^2_{\rm univ}(\mathcal{A})$.

(iii) Set $\Psi_0 : \Omega^1_{\rm univ}(\mathcal{A}) \times \Omega^1)\mathrm{univ}(\mathcal{A}) \to \mathcal{A}^3/\mathcal{E}$ by $\Psi_0([a_0 \otimes a_1], [b_0 \otimes b_1]) = [a_0 \otimes a_1 b_0 \otimes b_1]_{\mathcal{E}} - [a_0 a_1 \otimes b_0 \otimes b_1]_{\mathcal{E}}$ (and extending k-linearly). Observe that this is well-defined and moreover, $\Psi_0([a_0 \otimes a_1]a, [b_0 \otimes b_1]) = \Psi_0([a_0 \otimes a_1], a[b_0 \otimes b_1])$ for all $a, a_i, b_i \in \mathcal{A}$ (use :$[a_0 \otimes a_1]a \equiv [a_0 \otimes a_1 a] - [a_0 a_1 \otimes a]$, $a[b_0 \otimes b_1] \equiv [ab_0 \otimes b_1]$). Thus, Ψ_0 induces a k-module map $\Psi : \Omega^1_{\rm univ}(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_{\rm univ}(\mathcal{A}) \to \mathcal{A}^3/\mathcal{E}$, denoted by Ψ.

(iv) Verify that Ψ and Φ are inverses of each other, hence $\Omega^2_{\rm univ}(\mathcal{A}) \cong \mathcal{A}^3/\mathcal{E}$ as k-modules (**Caution :** the right $\mathcal{A}$- module structures are different!).
(v) Similarly, prove that $\Omega^n_{\rm univ}(\mathcal{A})$ is isomorphic as k-modules with $\mathcal{A}^{n+1}/\mathcal{E}^n$, where $\mathcal{A}^{n+1}$ is tensor product of $n+1$ copies of $\mathcal{A}$ (as k-module), and $\mathcal{E}^n$ denotes the k-submodule generated by elements of the form $a_0 \otimes a_1 \otimes ...a_n$, with at least one of the a_i $(i = 1, ..., n)$ is 1 We denote by $[a_0 \otimes ...a_n]^n_{\mathcal{E}}$ or just by $[a_0 \otimes ... \otimes a_n]$ the image of $a_0 \otimes ...a_n$ i the quotient module.

Thus we have a differential Z_+-graded algebra $\Omega_{\rm univ}(\mathcal{A}) \equiv \bigoplus_{n \geq 0} \Omega^n_{\rm univ}(\mathcal{A})$, where the algebra multiplication is inherited from the structure of tensor algebra, i.e. $\xi \cdot \eta = \xi \otimes_{\mathcal{A}} \eta$, for $\xi, \eta \in_{\rm univ} (\mathcal{A})$, and the differential is given by $d([a_0 \otimes a_1 \otimes ... \otimes a_n] = [1 \otimes a_0 \otimes a_1 ... \otimes a_n]$.

Theorem 3.13 *Any normalized cyclic cocycle τ on $\mathcal{A}$ is the character of some cycle over $\mathcal{A}$.*

Proof :
Let $\phi : \mathcal{A}^{n+1} \to k$ be a normalized cyclic cocycle. Then we consider $\Omega = \Omega_{\rm univ}(\mathcal{A})$ and d as mentioned before, $\rho : \mathcal{A} = \Omega^0_{\rm univ}(\mathcal{A}) \to \Omega^0_{\rm univ}(\mathcal{A})$ to be the identity homomorphism, and let $\int : \Omega^n_{\rm univ}(\mathcal{A}) \to k$ be given by

$$\int([a_0 \otimes a_1 \otimes ...n]) := \phi(a_0, a_1, ..., a_n).$$

Note that it is well defined because ϕ is normalized, hence $\phi(\mathcal{E}^n) = 0$. It remains to verify that $\int$ is a closed graded trace, i.e satisfies the conditions (a) and (b) of Definition 3.10. However, this follows by essentially the same calculations as in the proof of Theorem 3.11, by reversing the arguments. So, we leave this as an easy exercise.

Remark 3.14 *One can extend the above result to cyclic cocycles which are not necessarily normalized. For this, one should slightly weaken the definition of a bimodule, by not demanding that (even when $\mathcal{A}$ is unital) the left or right action by the identity of $\mathcal{A}$ is necessarily the identity on the bimodule. One can obtain an analogue of the universal bimodule of forms in this framework too, and then by more or less similar arguments as above, can prove a generalization of Theorem 3.13. We refer [1] for the details of this construction.*

We conclude this subsection by citing few more interesting explicit examples of cyclic cocycles, arising in group theory, algebra and geometry (taken from [2]).

Example 1

Let G be a group and $c : G^{n+1} \equiv G \times ... \times G(n \text{ copies }) \to \mathbb{C}$ satisfying
(a) $c(gg_0, ...gg_n) = c(g_0, ...g_n)$ for all $g, g_i \in G$,
(ii) $\sum_{j=0}^{n}(-1)^j c(g_0, ..., g_{j-1}, g_{j+1}, ..., g_{n+1}) = 0$ for all $(g_0, ..., g_{n+1}) \in G^{n+2}$.
Such a function c is called a *group cocycle*, and is very important in the theory of cohomology of groups. we can recast it as a cyclic cocycle as follows. Consider the group algebra over $\mathbb{C}$, i.e. $\mathcal{A} := \mathbb{C}G$, which is defined to be the set of all $\mathbb{C}$-valued finitely supported functions on G with the multiplication given by convolution, i.e. $(f_1 \cdot f_2)(g) := \sum_{h \in G} f_1(h) f_2(h^{-1}g)$. We denote by χ_g the function which is 1 at g and 0 elsewhere. Clearly, $\{\chi_g, g \in G\}$ is a total set of $\mathcal{A}$. We define $\phi_c : \mathcal{A}^{\otimes^{n+1}} \to \mathbb{C}$ by $\phi_c(\chi_{g_0} \otimes \chi_{g_1} \otimes ... \otimes \chi_{g_n}) = c(1_G, g_1, g_1 g_2, ..., (g_1 g_2 ... g_n))$, if $g_0 g_1 ... g_n = 1_G$, and $\phi_c(\chi_{g_0} \otimes ... \otimes \chi_{g_n}) = 0$ otherwise. Here 1_G denotes the identity of G. We leave it as an **Exercise** to prove that ϕ_c is indeed a cyclic cocycle.

Example 2

Let $\mathcal{A}$ be a unital algebra, $\delta_1, \delta_2 : \mathcal{A} \to \mathcal{A}$ be two derivations (i.e. $\delta_i(ab) = \delta_i(a)b + a\delta_i(b)$) which commute, i.e. $\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1$. Let $\tau : \mathcal{A} \to k$ be such that $\tau(ab) = \tau(ba)$ and $\tau \circ \delta_i = \tau$ for $i = 1, 2$.
Exercise : Prove that the following defines a 2-dimensional cyclic cocycle on $\mathcal{A}$:

$$\phi(a_0, a_1, a_2) := \tau\left(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2)\right).$$

3.3 Cyclic modules and mixed complexes

We briefly discuss here an abstraction of cyclic homology and cohomology to the set-up of the so-called cyclic modules, to be defined below.

Definition 3.15 *A cyclic module $C \equiv (C_n)_{n \geq 0}$ is a simiplicial k-module (c.f. Definition 1.11), with the degeneracy and face maps denoted by d_i, s_j respectively, endowed with an action t_n on C_n of the group $Z_{n+1} := Z/(n+1)Z$ for all n, satisfying*

$$t_n^{n+1} = \text{id}; \tag{.1}$$

$$d_i t_n = -t_{n-1} d_{i-1}, \quad s_i t_n = -t_{n+1} s_{i-1} \quad \text{for } 1 \leq i \leq n; \tag{.2}$$

$$d_0 t_n = (-1)^n d_n, \quad s_0 t_n = (-1)^n t_{n+1}^2 s_n. \tag{.3}$$

A morphism between two cyclic modules is a morphism of the corresponding simplicial modules which commutes with the action of Z_n for each n.

Given two cyclic modules C and C', we can make the product $C \times C'$ (in the sense of product of simplicial modules defined earlier) a cyclic module by taking the Z_n-action on $(C \times C')_n = C_n \otimes C'_n$ to be $(-1)^n t_n \otimes t^p rime_n$, where t_n and t'_n denote the Z_n-action on C_n and C'_n respectively.
Exercise : Observe that the concrete cyclic complex $\mathcal{A}^{n+1}$ considered earlier in the context of cyclic homology is indeed a special case of the cyclic module defined above.

It is straightforward to extend almost all the constructions and results obtained so far for cyclic homology to the set-up of cyclic modules. For example, one can associate a bicomplex (called *cyclic bicomplex for the cyclic module*) $CC = (CC_{pq})$ by taking $CC_{pq} = C_p$ $CC \equiv (CC_{pq})_{p,q \geq 0}$ where $CC_{pq} = \mathcal{A}^{p+1}$ and the horizontal and vertical differentials are given by the following :

$$d^h = 1 - t \text{ if } q \text{ is odd}, \quad N \text{ if } q \text{ is even},$$

$$d^v = b \text{ if } q \text{ is even} \quad , -b' \text{ if } q \text{ is odd},$$

where $b = \sum_{i=0}^{n} (-1)^i d_i$, $b' = \sum_{i=0}^{n-1} d_i$, $N = \sum_{i=0}^{n} t_n^i$. As in the concrete case of cyclic homology of an algebra $\mathcal{A}$, we can prove that the above defines a bicomplex and we call the homology of the corresponding total complex *cyclic homology of the cyclic module* C, denoted by $HC_*(C)$. We also have analogue of the $b - B$ bicomplex as well as the Connes' complex here : define $B = (1 - t_{n+1})t_{n+1}s_n : C_n \to C_{n+1}$ and $C_n^\lambda = C_n / \text{Im}(1 - t_n)$. We leave it as an easy **Exercise** to prove the equivalence of CC and $b - B$ bicomplexes in general, and also their equivalence with the Connes' complex in characteristic zero case.

We now want to point out that some of the basic properties of cyclic homology do not really need the full cyclic bicomplex; just an analogue of $b - B$ bicomplex is enough. To make this statement more precise, we define below a *mixed complex*, which is a weaker notion that cyclic module.

Definition 3.16 *A mixed complex is given by a family $C \equiv (C_n)_{n \geq 0}$ of k-modules, equipped with chain maps $b : C_n \to C_{n-1}$ $(n \geq 1)$, $B : C_n \to C_{n+1}$ $(n \geq 0)$ satisfying*

$$b^2 = B^2 = bB + Bb = 0.$$

Given a mixed complex C, we consider a bi-complex $\mathcal{B} \equiv \mathcal{B}(C)$, such that $\mathcal{B}_{pq} = C_{p-q}$ if $p \geq q$ and 0 otherwise, with the horizontal differentials given by B and the vertical ones by b. The homology of $\mathrm{Tot}(\mathcal{B})$ is called the cyclic homology of the mixed complex C, denoted by $HC_*(C)$. Moreover, the homology $H_*(C)$ of the chain complex (C_*, b) is called the ordinary homology of C and we shall also use the notation $HH_*(C)$ for this ordinary homology group.

We should perhaps remark here that the motivation for choosing the notation $HH_*(C)$ is the fact that when $C = C(\mathcal{A}) = (\mathcal{A}^{n+1})$, i.e. the complex for constructing cyclic homology of an algebra $\mathcal{A}$, the homology of (C, b) is nothing but the Hochschild homology $HH_*(\mathcal{A})$. Observe also that $\mathrm{Tot}(\mathcal{B}(C))$ is isomorphic with the tensor product complex $k[x^2] \otimes C$, as seen before in the concrete case $C = C(\mathcal{A})$. This observation will be needed later to define coproduct in cyclic homology.

The interesting point about the abstract definition of mixed complex is that we can have the Connes' long exact sequence as obtained for cyclic homology, which is the described in the exercises below.

Exercise:
Consider the following sequence of chain complexes :

$$0 \to (C, b) \xrightarrow{i} \mathrm{Tot}(\mathcal{B}(C)) \xrightarrow{j} \mathrm{Tot}(\mathcal{B}(C)[0, 2]),$$

where i is given by the inclusion of C as the first column of $\mathcal{B}(C)$, and j is given by deleting the first column of $\mathcal{B}(C)$, so that the above is a short exact sequence of chain complexes. Show the the induced long exact sequence has the form :

$$... \to HC_{n-1}(C) \xrightarrow{B} HH_n(C) \xrightarrow{I} HC_n(C) \xrightarrow{S} HC_{n-2}(C) \to ...$$

Here, I and S denote the maps induced by i and j respectively at the level of the long exact sequence, and B is the map appearing in the definition of the mixed complex C.

Let us show how mixed complexes naturally arise even in the context of cyclic homology of algebras. Let $\mathcal{A}$, $\mathcal{B}$ be two algebras, and let $C = C(\mathcal{A})$, $C' = C(\mathcal{B})$ be the associated cyclic complex, which are cyclic modules. The tensor product complex $C \otimes C'$, defined by $(C \otimes C')_n = \oplus_{k+l=n} C_k \otimes C'_l$, (this is *different* from the product $C \times C'$, which is a cyclic module) is *not* a cyclic module. However, it is a mixed complex (**Exercise** : what are the chain maps for $C \otimes C'$ in terms of the chain maps of C and C' ?). Moreover, the complex $C_n(\mathcal{A} \otimes \mathcal{A}') = (\mathcal{A} \otimes \mathcal{A}')^{\otimes^{n+1}}$ is isomorphic with $(C \times C')_n = \mathcal{A}^{\otimes^{n+1}} \otimes \mathcal{A}'^{\otimes^{n+1}}$. We shall often identify $C_n(\mathcal{A} \otimes \mathcal{A}')$ with $(C \times C')_n$. The following theorem due to Eilenberg and Zilber (which we quote without proof) tells that at the level of cyclic homology, the product and tensor product complexes are not really different.

Theorem 3.17 *The cyclic homology groups of $C \times C'$ and $C \otimes C'$ are isomorphic, hence $HC(C \times C') \cong HC(C \otimes C')$.*

We refer to [3] as references therein for a proof. However, we briefly describe the isomorphism map. Recall the shuffle product defined in the context of simplicial modules. In the present case, the map sh : $\mathrm{Tot}(C \otimes C') \to \mathrm{Tot}(C \times C')$ is given by $\mathrm{sh} \equiv \mathrm{sh}_n = \oplus_{p,q:\ p+q=n} \mathrm{sh}_{pq}$, where $\mathrm{sh}_{pq} : C_p \otimes C'_q \to C_{p+q}(\mathcal{A} \otimes \mathcal{A}') \cong (C \times C')_{p+q}$,

$$\begin{aligned}
&\mathrm{sh}_{pq}((a_0 \otimes a_1 \otimes ... \otimes a_p) \otimes (a'_0 \otimes ... \otimes a'_q)) \\
&= \sum_\sigma \mathrm{sgn}(\sigma)(a_0 \otimes a'_0) \otimes (a_{\sigma(1)} \otimes 1) \otimes ... \otimes (a_{\sigma(p)} \otimes 1) \otimes (1 \otimes a'_{\sigma(p+1)}) \otimes ... \otimes (1 \otimes a'_{\sigma(p+q)}),
\end{aligned}$$

where the sum is over all permutations σ of $\{1, 2, ..., p+q\}$ satisfying $\sigma(1) < ... < \sigma(p)$, $\sigma(p+1) < ... < \sigma(p+q)$. Similarly, one can define *cyclic shuffle map*, sh$'$, as follows. Let Σ_{pq} denote the set of all permutations σ of $\{1, ..., p+q\}$ which are obtained by first performing some cyclic permutation on $\{1, ..., p\}$ and $\{p+1, ..., p+q\}$, and then shuffling these two cyclically permuted sets, and also ensuring that 1 appears before $p+1$ in σ. Then we define $\mathring{s}h' \equiv \mathrm{sh}'_n = \oplus_{p,q:\ p+q=n} \mathrm{sh}'_{pq}$, where $\mathrm{sh}'_{pq} : C_p \otimes C'_q \to (C \otimes C')_{p+q+2}$, given by

$$\begin{aligned}
&\mathrm{sh}'_{pq}((a_1 \otimes a_2 \otimes ... \otimes a_{p+1}) \otimes (a'_1 \otimes ... \otimes a'_{q+1})) \\
&= \sum_{\sigma \in \Sigma_{pq}} \mathrm{sgn}(\sigma)(1 \otimes 1) \otimes (a_{\sigma(1)} \otimes 1) \otimes ... \otimes (a_{\sigma(p)} \otimes 1) \otimes (1 \otimes a'_{\sigma(p+1)}) \otimes ... \otimes (1 \otimes a'_{\sigma(p+q)}).
\end{aligned}$$

Using sh and sh$'$, we define Sh from $\mathrm{Tot}(\mathcal{B}(C \otimes C'))_n = (C \otimes C')_n \oplus (C \otimes C')_{n-2} \oplus ...$ to $\mathrm{Tot}(\mathcal{B}((C \times C')))_n$ by setting $\mathrm{Sh}\,(\xi_n \oplus \xi_{n-2} \oplus ...) = (\mathrm{sh}(\xi_n) + \mathrm{sh}'(\xi_{n-2})) \oplus (\mathrm{sh}(\xi_{n-2}) + \mathrm{sh}'(\xi_{n-4})) \oplus ...)$. The Eilenberg-Zilber Theorem states that Sh is an isomorphism.

The above theory can be further generalized by considering a more abstract categorical formulation of cyclic module, namely *cyclic category.* However, we shall not go into such abstraction in these notes, and let us refer to [3] and [1] for discussion on this topic.

3.4 Some variations : periodic and continuous

We assume that the reader is familiar with the concept of inductive and projective limits of groups and modules. Consider the sequence of k-modules $G_n^e := HC^{2n}(\mathcal{A})$, and $\pi_{m,n} \equiv S^{n-m} : G_m^e \to G_n$ $(m \leq n)$, where S is the periodicity operator. Clearly, $(G_n^e, \pi_{m,n})$ is an inductive system of k-modules, and we call the limit of this system the *even periodic cyclic cohomology of* $\mathcal{A}$, given by,

$$HC^0_{\text{per}}(\mathcal{A}) \equiv HC^{\text{ev}}_{\text{per}}(\mathcal{A}) :=\stackrel{\lim}{\rightarrow} HC^{2n}(\mathcal{A}).$$

A similar definition can be given for the odd periodic cyclic homology group, denoted by $HC^1_{\text{per}}(\mathcal{A})$ or $HC^{\text{odd}}_{\text{per}}(\mathcal{A})$, by considering the inductive system $(G_n^o := HC^{2n+1}(\mathcal{A}), \pi_{m,n} := S^{n-m})$.

The definition of periodic cyclic homology groups requires the projective limit and is slightly more technical; we omit its definition and refer [3] to the interested reader. An alternative definition of both periodic cyclic cohomology and homology can be given in terms of a suitable bicomplex which is similar to the CC bicomplex but infinite at both horizontal ends (this is called periodic bicomplex). Again, we refer [3] for details.

So far we have considered the cyclic homology and cohomology theories from a purely algebraic point of view. However, for applications to topology and geometry (including noncommutative geometry) often it is necessary to encode the topological and/or geometric structure of the $\mathcal{A}$, if any, in the definition of the homology/cohomology. This is slightly easier in the cohomological context. For example, if $\mathcal{A}$ is a Banach algebra, it is natural to consider only those cyclic cocycles ϕ on $\mathcal{A}^n$ which are bounded in the sense that $|\phi(a_0, ..., a_n)| \leq C\|a_0\|...\|a_n\|$ for all $a_i \in \mathcal{A}$, where $\|\cdot\|$ denotes the Banach norm on $\mathcal{A}$. It can easily be shown that the Hochschild coboundary map is continuous w.r.t. the natural topology on the space of bounded cocycles, thus we get a cochain complex and cohomology groups. This construction goes over verbatim to the case of locally convex algebras, which include, for example, $C^\infty(M)$. It is remarkable that the continupus cohomology of a locally convex algebra can differ from the algebraic cyclic cohomology. However, we do not go into much details of these topics here. Let us also mention that an analogous formulation of continuous cyclic homology for locally convex algebras can be done by using the notion of Grothendieck's projective tensor product, the discussion of which is beyond the scope of these notes.

4 Product and coproduct on cyclic homology and cohomology

Let $\mathcal{A}$ and $\mathcal{A}'$ be two (unital) k-algebras and let $C = C(\mathcal{A})$ and $C' = C(\mathcal{A}')$, be the corresponding Hochschild complex. We shall use the same notation b, B, j, S etc. for the maps on both the complexes C, C', as well as the associated bicomplexes, total complexes of bicomplexes and at the homology level. Recall that $\mathrm{Tot}(\mathcal{B}(C)) \cong k[x^2] \otimes C$, $\mathrm{Tot}(\mathcal{B}(C')) \cong k[y^2] \otimes C'$, and $\mathrm{Tot}(\mathcal{B}(C \otimes C')) \cong k[z^2] \otimes (C \otimes C')$. Moreover, the map $j : \mathrm{Tot}(\mathcal{B}(C)) \cong k[x^2] \otimes C \to \mathrm{Tot}(\mathcal{B}(C))[2]$ is given by $j(x^{2n} \otimes c) = x^{2n-2} \otimes c$ (with $c \in C$), for $n \geq 1$, and 0 otherwise. A similar definition of j on $\mathrm{Tot}(\mathcal{B}(C'))$ is given.

Exercise : Show that $\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')) \cong k[x^2, y^2] \otimes (C \otimes C')$, where $k[x^2, y^2]$ is the subalgebra in $k[x, y]$ (polynomial algebra in two indeterminates) generated by x^2, y^2, viewed as a chain complex.

Now, define the map $\Delta_0 : k[z^2] \to k[x^2, y^2]$ by

$$\Delta_0(z^{2n}) = \sum_{p+q=n} x^{2p} y^{2q}.$$

Then, define $\Delta = (\Delta_0 \otimes \mathrm{id}_{C \otimes C'}$ from $\mathrm{Tot}(\mathcal{B}(C \otimes C')) \cong k[z^2] \otimes (C \otimes C')$ to $k[x^2, y^2] \otimes (C \otimes C') \cong \mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C'))$.

Clearly, Δ_0 and hence Δ is one-to-one. We identify its image in lemma that follows.

Lemma 4.1 $\mathrm{Im}(\Delta) = \mathrm{Ker}(j \otimes \mathrm{id} - \mathrm{id} \otimes j) : \mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')) \to (\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')))\,[2]$.

Proof :
We begin with a simple observation : for two chain complexes A and A', one has $\mathrm{Tot}(A \otimes A')[p] \cong \mathrm{Tot}(A[p] \otimes a')$, in a canonical way. Using this we note that $j : \mathrm{Tot}(\mathcal{B}(C)) \cong k[x^2] \otimes C \to \mathrm{Tot}(\mathcal{B}(C'))[2] \cong k[x^2][2] \otimes C$ is given by $j = l \otimes \mathrm{id}_C$, where $l : k[x^2] \to k[x^2][2]$ is the map which sends x^{2n} to x^{2n-2} for $n \geq 1$ and 0 for $n = 0$. Similar observation can be made

for $\mathcal{B}(C')$, and thus $j_2 := j \otimes \mathrm{id} - \mathrm{id} \otimes j$ is given by $(l \otimes \mathrm{id}_{k[y^2]} - \mathrm{id}_{k[x^2]} \otimes l) \otimes \mathrm{id}_(C \otimes C')$ from $\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')) \cong k[x^2, y^2] \otimes (C \otimes C') \cong k[x^2] \otimes k[y^2] \otimes (C \otimes C')$ to $(k[x^2] \otimes k[y^2])[2] \otimes (C \otimes C')$. In view of this identification, it now suffices to prove that $\mathrm{Im}(\Delta_0) = \mathrm{Ker}(l \otimes \mathrm{id} - \mathrm{id} \otimes l)$ on $k[x^2, y^2]$. One way inclusion is obvious; so let us prove the other way. Let $P \equiv \sum_{p,q=0}^m c_{p,q} x^{2p} y^{2q} \in \mathrm{Ker}(l \otimes \mathrm{id} - \mathrm{id} \otimes l) \subseteq k[x^2, y^2]$, where m is nonnegative integer and $c_{p,q} \in k$. Then we have $\sum_{p,q} c_{p,q}(x^{2p-2}y^{2q} - x^{2p}y^{2q-2}) = 0$ as polynomial, so that $c_{p-1,q+1} = c_{p,q}$ for all p, q, where we have set $c_{p,q} = 0$ for (p, q) outside the set $\{0, 1, ..., m\} \times \{0, 1, ..., m\}$. Inductively, this implies that $c_{p-k,q+k} = c_{p,q}$ for all p, q, k, hence $c_{p,q}$ depends only on the value of $p + q$, say, $c_{p,q} = c'_{p+q}$. Thus, $P = \sum_j c'_j (\sum_{p+q=j} x^{2p} \otimes y^{2q}) = \Delta_0(\sum_j c'_j z^j)$, which completes the proof.

Thus, we have a short exact sequence

$$0 \to \mathrm{Tot}(\mathcal{B}(C \otimes C')) \overset{\Delta}{\to} \mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')) \overset{S_2}{\to} (\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C')))\,[2] \to 0.$$

This induces a long exact sequence connecting the homology groups :

$$\cdots \to H_n\left(\mathrm{Tot}(\mathcal{B}(C \otimes C'))\right) \overset{\prod}{\to} H_n\left(\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C'))\right)$$
$$\overset{S\otimes \mathrm{id} - \mathrm{id} \otimes S}{\to} H_{n-2}\left(\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C'))\right) \overset{\partial}{\to} H_{n-1}\left(\mathrm{Tot}(\mathcal{B}(C \otimes C'))\right) \to \cdots$$

, where the map induced by Δ has been denoted by $\prod$ and it has been observed that the map induced by j is nothing but S, the periodicity operator. The map ∂ is the connecting homomorphism. Moreover, by the Eilenberg-Zilber's Theorem 3.17 $H_n\left(\mathrm{Tot}(\mathcal{B}(C \otimes C'))\right) \cong H_n(\mathcal{A} \otimes \mathcal{A}')$ (**Exercise** : argue clearly why it follows from Eilenberg-Zlber), and when k is a *field* (we actually need weaker hypothesis than this) we have by Theorem 1.7 that

$$H_n\left(\mathrm{Tot}(\mathcal{B}(C)) \otimes \mathrm{Tot}(\mathcal{B}(C'))\right) \cong \bigoplus_{p+q=n} H_p(\mathrm{Tot}(\mathcal{B}(C))) \otimes H_q(\mathrm{Tot}(\mathcal{B}(C'))) = \bigoplus_{p+q=n} HC_p(\mathcal{A}) \otimes HC_q(\mathcal{A}').$$

Combining all these identifications, we can view $\prod$ as a map from $HC_n(\mathcal{A} \otimes \mathcal{A}')$ to $\bigoplus_{p+q=n} HC_p(\mathcal{A}) \otimes HC_q(\mathcal{A}')$. The map ∂, in view of the isomorphisms mentioned above, can be thought of as a map from $\bigoplus_{p+q=n-2} HC_p(\mathcal{A}) \otimes HC_q(\mathcal{A}')$ to $HC_{n-1}(\mathcal{A} \otimes \mathcal{A}')$. We denote ∂ restricted to the direct summand $HC_p(\mathcal{A}) \otimes HC_q(\mathcal{A}')$ by $* \equiv *_{pq}$, which is a map from $HC_p(\mathcal{A}) \otimes HC_q(\mathcal{A}')$ to $HC_{p+q+1}(\mathcal{A} \otimes \mathcal{A}')$. In fact, this map can be explicitly written down in terms of the shuffle map sh_{pq} discussed earlier. It is a somewhat tedious and perhaps challenging **Exercise** to verify that $*$ sends the homology class of an element of $\mathrm{Tot}(\mathcal{B}(C))_p \otimes \mathrm{Tot}(\mathcal{B}(C'))_q$, say $(x_p \otimes x_{p-2}, ...) \otimes (y_q, y_{q-2}, ...)$ (where $x_i \in C_i, y_j \in C'_j$) to the homology class of the element $(\mathrm{sh}_{p+1,q}(B(x_p) \otimes y_q), \mathrm{sh}_{p+1,q-2}(B(x_p) \otimes y_{q-2}), ...) \in \mathrm{Tot}(\mathcal{B}(C \times C'))_{p+q+1}$. Those who are not interested to accept this challenge can just see the proof done in details in [3].

Definition 4.2 *The map* $*$ *and* $\prod$ *are called the product and coproduct (respectively) in cyclic homology.*

By dualizing we get a map $\# : HC^p(\mathcal{A}) \otimes HC^q(\mathcal{A}') \to HC^{p+q}(\mathcal{A} \otimes \mathcal{A}')$. For convenience, we write $\#(\phi, \psi)$ as $\phi\#\psi$, and call it the *cup product*. Similarly, one gets a *coproduct for cyclic cohomology* by dualizing $*$. The details of the dualization argument for obtaining the cup product $\#$ are left as an exercise given below.

Exercise : Consider the dual (cochain)-complexes C^*, C'^*, and carry out the details of the arguments for constructing the coproduct $\prod$ in the cohomological set-up to get a dual map $\prod$. Observe that under the conditions which ensure $HC^n(\mathcal{C}) \cong \mathrm{Hom}(HC_n(\mathcal{C}), k)$ for $\mathcal{C} = \mathcal{A}, cla', \mathcal{A} \otimes \mathcal{A}'$, the map $\#$ is given by $< \phi\#\psi, x > = < \phi \otimes \psi, \pi_{p,q} \circ \prod(x) >$, for $\psi \in HC^p(\mathcal{A}), \psi \in HC^q(\mathcal{A}')$, $x \in HC_{p+q}(\mathcal{A} \otimes \mathcal{A}')$ where $\pi_{p,q} : \bigoplus_{i,j:i+j=p+q} HC^i(\mathcal{A}) \otimes HC^j(\mathcal{A}') \to HC^p(\mathcal{A}) \otimes HC^q(\mathcal{A}')$ denotes the projection map.

Exercise : In this (perhaps slightly tedious, but recommended for a good understanding of the product) exercise, we sketch an interesting alternative description of the cup product $\#$. Without loss of generality we restrict our discussion to normalized cocycles. Recall that any normalized cyclic cocycle (m dimensional) ϕ on $\mathcal{A}$ is the character of a cycle of degree m the form $(\Omega = \Omega_{\mathrm{univ}}(\mathcal{A}), d \equiv d_{\mathcal{A}}, \int, \rho)$. Denote $\int$ (only this dependeds on ϕ) by $\hat{\phi}$. Similarly, let $(\Omega' \equiv \Omega_{\mathrm{univ}}(\mathcal{A}'), d_{\mathcal{A}'}, \hat{\psi}, \rho')$ be the canonical cycle od degree n over $\mathcal{A}'$ whose character is an n-dimensional cocycle ψ. Prove the following :
(i) The map $d = d_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}'} + \mathrm{id}_{\mathcal{A}} \otimes d_{\mathcal{A}'} : \mathcal{A} \otimes \mathcal{A}' \to \Omega^1_{\mathrm{univ}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{univ}}(\mathcal{A}')$ is a derivation into the $\mathcal{A} \otimes \mathcal{A}'$ bimodule $\Omega^1_{\mathrm{univ}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{univ}}(\mathcal{A}')$.
(ii) Using the universal property of $\Omega^1_{\mathrm{univ}}(\mathcal{A} \otimes \mathcal{A}')$, prove that there exists a natural bimodule morphism from $_{\mathrm{univ}}(\mathcal{A} \otimes \mathcal{A}')$ to $\Omega \otimes \Omega'$, given by $\pi = \oplus_{n \geq 0}(\eta \otimes ...\eta)$, where $\eta : \Omega^1_{\mathrm{univ}}(\mathcal{A} \otimes \mathcal{A}') \to \Omega^1_{\mathrm{univ}}(\mathcal{A}) \otimes \Omega^1_{\mathrm{univ}}(\mathcal{A}')$ obtained from d; and define $\hat{\tau} := (\hat{\phi} \otimes \hat{\psi}) \circ \pi : \Omega^{m+n}_{\mathrm{univ}}(\mathcal{A} \otimes \mathcal{A}') \to k$. Here it is necessary to extend the definition of $\hat{\phi}$ and $\hat{\psi}$ on the whole of $\Omega_{\mathrm{univ}}(\mathcal{A})$ and $\Omega_{\mathrm{univ}}(\mathcal{A}')$

(respectively) by defining them to be 0 on $\Omega^k_{\rm univ}(\mathcal{A})$ (or $\Omega^k_{\rm univ}(\mathcal{A}')$) for $k \neq m$ (resp. $k \neq n$).
(iii) Prove that the cyclic cocycle given by $\hat{\tau}$, say τ, which is given by $\tau(c_0, c_1, ...c_{m+n}) := \hat{\tau}(c_0 d(c_1)...d(c_{m+n}))$ for $c_i \in \mathcal{A} \otimes \mathcal{A}'$, coincides with (possibly upto some constant multiple) the cup product $\phi \# \psi$.

5 Pairing with K theory; Chern character

Let $\mathcal{A}^\infty = \bigcup_{r=1}^\infty M_r(\mathcal{A})$, and let e be an idempotent in $\mathcal{A}$, say $e \in \mathcal{B} := M_r(\mathcal{A})$. Let $\mathcal{B}^m$ denote tensor product of m copies of $\mathcal{B}$. For $n \geq 1$, define

$$C_n(e) := (y_n, z_n, y_{n-1}, ..., z_1, y_1) \in {\rm Tot}_{2n}(CC(\mathcal{B})) \equiv \mathcal{B}^{2n+1} \oplus \mathcal{B}^{2n} \oplus ... \oplus \mathcal{B},$$

where $y_k = (-1)^k \frac{(2k)!}{k!} e^{\otimes^{2k+1}}$ and $z_k = (-1)^{k-1} \frac{(2k)!}{2k!} e^{\otimes^{2k}}$.

Lemma 5.1 *$C_n(e)$ is a cycle.*

Proof :
Using $e^2 = e$, we have

$$t(e^{\otimes^{2k}}) = -e^{\otimes^{2k}}, \quad t(e^{\otimes^{2k+1})=e^{\otimes^{2k+1}}},$$

$$b(e^{\otimes^{2k}}) = 0, \quad b(e^{\otimes^{2k+1}}) = e^{\otimes^{2k}}.$$

So, $b(2e^{\otimes^{2k+1}}) = (1-t)(e^{\otimes^{2k}})$. Using this, we get

$$\begin{aligned} & b(y_k) \\ = \;& (-1)^k \frac{(2k)!}{k!} b(e^{\otimes^{2k+1}}) = (-1) \frac{(2k)!}{2(k!)} b(2e^{\otimes^{2k+1}}) \\ = \;& -(-1)^{k-1} \frac{(2k)!}{2(k!)} (1-t)(e^{\otimes^{2k}}) = -(1-t)(z_k). \end{aligned}$$

Similarly, $b'(z_k) = N(y_{k-1})$ follows, thereby completing the proof. Recall the generalized trace map ${\rm Tr} : \mathcal{B}^m \to \mathcal{A}^m$, and that it induces an isomorphism of cyclic homology. Define ${\rm ch}_{0,n}(e) := [{\rm Tr}(C_n(e))]$, where $[\cdot]$ denotes the cyclic homology class. It is a map from the set of idempotents of $\mathcal{A}^\infty$ to $HC_{2n}(\mathcal{A})$. We make a few elementary observations in the exercises listed below.

Exercise: Prove the following:
(i) For $e_1 \in M_r(\mathcal{A})$, $e_2 \in M_l(\mathcal{A})$, we have

$$\mathrm{ch}_{0,n}(e_1 \oplus e_2) = \mathrm{ch}_{0,n}(e_1) + \mathrm{ch}_{0,n}(e_2).$$

In particular, $\mathrm{ch}_{0,n}(e \oplus 0) = \mathrm{ch}_{0,n}(e)$.
(ii) For every invertible g in $\mathcal{A}^\infty$, we have

$$\mathrm{ch}_{0,n}(\gamma^g(e)) = \mathrm{ch}_{0,n}(e).$$

(Hint : use the cyclic cohomological version of Theorem 2.9, i.e. the fact that γ_g induces identity map on HC^*.)

Now, recall the definition of the $K_0(\mathcal{A})$, which is the Grothendieck group of the abelian semigroup $V(\mathcal{A})$, which is the set of idempotents *clp* in $\mathcal{A}^\infty$ modulo the following equivalence relation $\sim$(where $p \in M_k(\mathcal{A})$, $q \in M_l(\mathcal{A})$, idempotents):

$$p \sim q \text{ if } \exists\ r \geq 0,\ u, v \in M_{k+l+r} \text{ s.t. } uv = p \oplus 0_{l+r}, \quad vu = q \oplus 0_{k+r}.$$

Moreover, recall from the lectures on K-theory that $p \sim q$ if and only if we can find sufficiently large s such that $p \oplus 0_{l+s}$ and $q \oplus 0_{k+s}$ are related by conjugation by an invertible element of $\mathcal{A}$. For example, we can choose $g = \begin{pmatrix} v & 1 - vu \\ uv - 1 & u \end{pmatrix}$ such that $g\ \mathrm{diag}(p, 0)\ g^{-1} = \mathrm{diag}(q, 0)$. It is clear from the exercises (i) and (ii) above that $\mathrm{ch}_{0,n}$ respects this equivalence relation, i.e. $\mathrm{ch}_{0,n}(e_1) = \mathrm{ch}_{0,n}(e_2)$ whenever $e_1 \sim e_2$. So, we obtain a well-defined map, again denoted by $\mathrm{ch}_{0,n}$ from $V(\mathcal{A})$ to $HC_{2n}(\mathcal{A})$. Since the semigroup operation on $V(\mathcal{A})$ is direct sum, which is respected by $\mathrm{ch}_{0,n}$ by exercise (i), we have a homomorphism from the semigroup $V(\mathcal{A})$ to the abeian group $HC_{2n}(\mathcal{A})$, hence a canonical group homomorphism (again denoted by $\mathrm{ch}_{0,n}$ from $K_0(\mathcal{A})$ to $HC_{2n}(\mathcal{A})$, defined by $\mathrm{ch}_{0,n}([e]) = [\mathrm{ch}_{0,n}(e)]$, where $[\cdot]$ on the left hand side denotes the K-theory class of e.

Definition 5.2 *The above map* $\mathrm{ch}_{0,n} : K_0(\mathcal{A}) \to HC_{2n}(\mathcal{A})$ *which sends* $[e] \in K_0(\mathcal{A})$ *to* $[\mathrm{ch}_{0,n}(e)]$, *is called the Chern character in cyclic homology. By using the canonical pairing* $< \cdot, \cdot >$ *between the cyclic homology* HC_* *and the cyclic cohomology* HC^*, *we also get a pairing, called the Chern-Connes pairing, between* $HC^{2n}(\mathcal{A})$ *and* $K_0(\mathcal{A})$, *given by* $< [e], [\tau] > := < \mathrm{ch}_{0,n}([e]), [\tau] >$, *for* $[e] \in K_0(\mathcal{A}),\ [\tau] \in HC^{2n}(\mathcal{A})$.

Exercise : Prove that $S \circ \mathrm{ch}_{0,n} = \mathrm{ch}_{0,n-1}$. and hence prove that there is a canonical pairing between $HC^0_{\mathrm{per}}(\mathcal{A})$ and $K_0(\mathcal{A})$.
Exercise : Recall the projection $p_* : HC_* \to H^\lambda_*$, which gives the isomorphism between the CC picture and Connes' picture of cyclic homology. Prove that the

$$p_* \circ \mathrm{ch}_{0,n}([e]) = \frac{(2n)!}{n!}\mathrm{Tr}(e^{\otimes^{2n+1}}).$$

Exercise : Consider the functional $\mathrm{tr} : M_r(k) \to k$ as a 0-dimensional cyclic cocycle, and give an explicit expression for the cup product $[\tau\#\mathrm{tr}] \in HC^n(M_r(\mathcal{A}))$, for $[\tau] \in HC^n(\mathcal{A})$.

Exercise : Using the above exercise, show that the pairing between $[e] \in K_0(\mathcal{A})$ and $[\tau] \in HC^n(\mathcal{A})$ is same (upto a constant multiple; find the constant) as $\tau\#\mathrm{tr}(e, e, ..., e)$, where $e \in M_r(\mathcal{A})$ and τ is chosen to be normalized.

We shall conclude by giving a brief sketch (through exercises) of how to define a similar pairing between the K_1 group and $HC^{2n+1}(\mathcal{A})$. It is slightly delicate in this case to define it through a map from $K_1(\mathcal{A})$ to $HC_{2n+1}(\mathcal{A})$ and then use the pairing between cyclic homology and cohomology. Such a homology-valued 'Chern character' can indeed be defined, but one has to be careful about the range; it can be made sense of, for example, the so-called *negative cyclic homology*, which we haven't introduced. Moreover, unlike K_0, there are more than one versions of the K_1, namely algebraic and topological (C^*) definitions of K_1 are different, which also adds to the technicalities of defining the Chern character in this case. We refer the reader to [3] for more details in this direction. Nevertheless, let us give a definition of the pairing between $HC^{2n+1}(\mathcal{A})$ and $K_1(\mathcal{A})$, in the exercises below.

Exercise : pairing between $K_1(\mathcal{A})$ and $HC^{2n+1}(\mathcal{A})$
For an invertible $u \in GL_r(\mathcal{A})$ (invertible element in $M_r(\mathcal{A})$ and a normalized n-dimensional cyclic cocycle τ on $\mathcal{A}$, define

$$< [u], [\tau] >:= c_n(\tau\#\mathrm{tr}_r)(u^-, u, u^{-1}, ..., u),$$

where c_n is a numerical constant to be determined later and tr_r denotes the trace on $M_r(k)$. Prove, following the line of arguments sketched below (taken from [1]), that this is a well-defined k-bilinear map.
(i) Without loss of generality consider $r = 1$.
(ii) Let $\chi : GL_1(\mathcal{A}) \to k$ be defined by $\chi(u) := \tau(u^{-1}, u, ..., u^{-1}, u)$. Observe

that

$$\chi(uv) = (\tau\#\mathrm{tr}_2)(U^{-1}, U, ..., U^{-1}, U), \quad \chi(u)+\chi(v) = (\tau\#\mathrm{tr}_2)(V^{-1}, V, ..., V^{-1}, V),$$

where $U, V \in M_2(k)$ are given by $U = \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$.
(iii) Define, for $t \in [-2\pi, 2\pi]$,

$$U_t = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix}.$$

Prove that $t \mapsto U_t$ is differentiable with $\frac{d}{dt}U_t = -U_t^{-1}U_tU_t^{-1}$.
(iv) Using (iii), show that $\frac{d}{dt}(\tau\#\mathrm{tr}_2)(U_t^{-1}, U_t, ..., U_t^{-1}, U_t) = 0$ for all t. (Hint : the algebraic properties of τ, which is a normalized cyclic cocycle, will be crucially needed for this proof; better to do for $n = 2, 3$ first and then generalize.)
(v) Hence observe that $\chi(uv) = \chi(u) + \chi(v)$ for all $u, v \in GL_1(\mathcal{A})$.
(vi) Complete the rest of the proof by using (v) and some algebraic calculations.
(vi) Finally, determine c_n so that $< [u], [S\tau] >=< [u], [\tau] >$ for all τ and u, hence the pairing naturally extends to a pairing between $K_1(\mathcal{A})$ and $HC^1_{\mathrm{per}}(\mathcal{A})$.

References

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