"Chern-Weil" in noncomm

Set-up:

We shall see how some canonical ~~cy~~ cyclic cocycles can be produced using 'connection' on projective finitely gen. modules; which includes in particular the classical construction of Chern character.

Fix unital alg $\mathcal{A}$.

Defn Given a cycle $(\Omega, P, \int)$ on $\mathcal{A}$, and a finitely gen proj. module $\mathcal{E}$ over $\mathcal{A}$, a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ is called a connection if

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes dP(x)$$

Lemma

a) For $e \in \mathrm{End}_A(\mathcal{E})$, idempotent and connection ∇ on $\mathcal{E}$, $\xi \mapsto (e \otimes 1)\nabla\xi$ also gives a connection on $e\mathcal{E}$.

b) Any finitely gen. proj A-module admits a connection.

c) Any connection ∇ on $\mathcal{E}$ extends to a unique lin. map on $\tilde{\mathcal{E}} = \mathcal{E} \otimes_A \Omega^{\cdot}$, denoted by ∇ again, satisfying

$$\nabla(\xi \otimes \omega) = (\nabla\xi)\omega + \xi \otimes d\omega,$$

$\forall \xi \in \mathcal{E}$, $\omega \in \Omega^{\cdot}$

Pf is quite standard, immitate the classical ~~construction~~ case.

We now construct a cycle over the alg. $\mathrm{End}_A(\mathcal{E})$. First consider the graded alg $\mathrm{End}_{\Omega^\bullet}(\tilde{\mathcal{E}})$

($\tilde{\mathcal{E}} = \mathcal{E} \otimes_A \Omega^\bullet$, so grading comes from $\Omega^\bullet$),

where $T \in \mathrm{End}_{\Omega^\bullet}(\tilde{\mathcal{E}})$ is of degree K if $T(\mathcal{E} \otimes_A \Omega^j)$

$$\subseteq \mathcal{E} \otimes_A \Omega^{j+K} \quad \forall j.$$

Define $\delta(T) = \nabla T - (-1)^K T \nabla$

for T of degree K,

δ is a graded derivation.

Moreover, $\tilde{\mathcal{E}}$ f.g. proj, so is a complemented submodule of the free $\mathcal{A}$-module $\Omega^{\bullet}\mathcal{A} \otimes \mathbb{C}^N$ for some N, hence $\int : \Omega^n \to \mathbb{C}$ gives a natural (graded) trace on $\tilde{\mathcal{E}}$, again denoted by $\int$.

<u>Thm</u> $\int \delta(T) = 0 \quad \forall\, T$ of degree $n-1$.

<u>Pf:</u> ~~clearly~~

Note that if we fix any connection ∇_0, then any other connection on $\mathcal{E}$ is of the form $\nabla_0 + \Gamma$, for some $\Gamma \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1)$ (ie $\nabla - \nabla_0$ is $\mathcal{A}$-linear)

Moreover, $\int([\Gamma, T]) = 0 \;\; \forall\, T$, as $\int$ is trace.

$\rightarrow$ graded commutator.

In view of this, it suffices to check $\int \delta_0(T)=0$ for $\delta_0 = [\nabla_0, \cdot\,]$ for <u>some</u> connection, say the 'free' or 'canonical' connection ∇_0 on $\mathcal{A}^k$ given by $\nabla_0(\sum_i e_i a_i) = \sum e_i \otimes dP(a_i)$

Again, replacing $\mathcal{A}$ by $\mathcal{A} \otimes M_k$ it is possible to assume $\mathcal{E} = e\mathcal{A}$.

By induction on n, we may also assume wlg that $n=1$, i.e. to show $\int \delta_0(T) = 0$ for $T = a \in e\mathcal{A}e$,

But $\delta_0(T) = e(da)e$ for such T, hence

$$\int \delta_0(T) = \int e(da)e = \int d(eae) - \int (de)a - \int a\cdot de$$

$= 0$, as $\int$ is closed graded trace on $\mathcal{A}$. ■

~~Thm~~: However, $(\mathcal{M}, \int, \delta)$, $\mathcal{M} = \mathrm{End}_{\Omega^\cdot}(\tilde{\mathcal{E}})$

is still NOT a cycle of dim n,

because $\delta^2 \neq 0$ in general.

We have:

Thm: (i) $\theta \equiv \nabla^2 : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ is $\Omega^\cdot$-linear endomorphism.

(ii) Moreover, $\langle [\mathcal{E}], [\tau] \rangle = \frac{1}{m!} \int \left(\frac{\theta}{2\pi i}\right)^m$

where $n = 2m$ (so n even),

& $[\mathcal{E}] \in K_0(A)$, τ = character

of $(\Omega, d, \int)$ on A.

Pf: (i) trivial.

(ii) Verbatim generalization of the classical proof: [illegible]

Index theorems :

We shall generalize the classical index theorem for elliptic pseudodiff ops to our noncomm set up.

Def^n: Fredholm module:

Given an alg $\mathcal{A}$ over $\mathbb{C}$, an (odd) Fredholm module on $\mathcal{A}$ is given by the data $(\pi, \mathcal{H}, F)$ where

- $\mathcal{H}$ Hil space (separable)
- $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ homomorphism
- $F \in \mathcal{B}(\mathcal{H})$ st $F^2 = 1$, $F = F^*$,

$[F, \pi(a)]$ ~~[illegible]~~ $\in \mathcal{K}(\mathcal{H})$ (cpt) $\forall a \in \mathcal{A}$

If, moreover, $\mathcal{H}$ has a grading given by ε, st $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, ~~then we s~~ and $\pi(a)\varepsilon = \varepsilon\pi(a)$, $F\varepsilon = -\varepsilon F$, we say the Fred mod is even.

If the (even/odd) Fredholm module ~~is~~ satisfies the cond^n

$$[F, \pi(a)] \in \mathcal{L}^p(\mathcal{H}) \text{ for}$$

some $p > 0$, we say that the module is p-summable.

Typical example:

Comes from classical Riemannian Spin manifolds. Let D be the Dirac op on L^2(spinor bundle), Then $F = \mathrm{Sgn}(D)$, or $F = D(1+D^2)^{-1/2}$ satisfy our def^n ... in fact these are p-summable for $p =$ dim of the manifold.

Ind-3

There's a ~~related~~ version with unbounded operator, called 'unbdd Fredholm module' or 'Spectral triple', which directly generalizes the Dirac op. of spin geometry.

This is given by ~~a self adj~~ similar data, but F replaced by an unbdd self-adj op. D satisfying $[D, \pi(a)] \in \mathcal{B}(H)$ and ~~[illegible]~~ $D\varepsilon = -\varepsilon D$. (even case)

The p-summability cond^n can also be formulated in the unbdd set-up.

In fact, there is a 'Gelfand Thm' due to Connes, saying that a spectral triple on ~~the comm alg~~ a comm alg satisfying some more reasonable cond^{n}s must come from a spin structure.

Let us from now on fix an even, n-summable (n integer ≥ 0), Fredholm module on A, and also assume wlg that A is unital. We will identify A with $\pi(A) \subset B(H)$, so just write a for $\pi(a)$. We remark that the odd case can be treated similarly.

Chern character of the Fredholm module (π, H, ε, F):

The strategy: $\forall\, m \geq n$, m even, construct a cyclic cocycle $\tau_m \in HC^m(A)$, using the Fredholm module,

- show the S-invariance, hence get a cocycle τ in $HC^{even}_{per}(A)$.
- Show that the pairing $\langle[\tau], [e]\rangle$ = Index of some op.

Lemma 3: $p \geq 1$. $P, Q \in \mathcal{B}(H)$ st $1 - PQ$, $1 - QP \in \mathcal{L}^p(H)$. Then P is Fredholm and

$$\text{Index}(P) = \text{Tr}\left[(1-QP)^n\right] - \text{Tr}\left[(1-PQ)^n\right]$$

$\forall n \geq p$

Pf: $QP, PQ = 1 \mod K(H)$

$\Rightarrow P$ Fredholm.

Since cpt op's have only 0 as the possible accumulation pt in the spectrum, it's clear that 1 is an isolated pt in $K := \{1\} \cup \sigma(1-PQ) \cup \sigma(1-QP)$.

Ind-~~6~~

Taking a contour γ with ~~contro~~ $1 \in \text{int}(\gamma)$ but $K - \{1\} \subseteq \text{Ext}(\gamma)$, we have, by setting $e = \frac{1}{2\pi i} \int_\gamma (\lambda - (1 - QP))^{-1} d\lambda$,

$$f = \frac{1}{2\pi i} \int_\gamma (\lambda - (1 - PQ))^{-1} d\lambda, \therefore$$

- $e = e^2$, $f = f^2$ (by Spectral Thm)
- $E_1 \equiv \text{Ran}(e)$, $F_1 \equiv \text{Ran}(f)$ are finite dim ($\because$ $(1-PQ)$, $(1-QP)$ are cpt, so eigenspaces for nonzero eigenvalues are finite dimensional)
- ~~So~~ Since ~~=~~ $Q(\lambda - PQ) = (\lambda - QP) Q$ we have, $\forall \lambda \notin K$:

$$(\lambda - (1 - QP))^{-1} Q = Q(\lambda - (1 - PQ))^{-1}$$

i.e. $Qf = eQ$ and $Pe = fP$.

Write $E_2 = \ker e$, $F_2 = \ker f$ and observe

- $P(E_1) \subseteq F_1$, $P(E_2) \subseteq F_2$, $Q(F_1) \subseteq E_1$, $Q(F_2) \subseteq E_2$,

i.e. $PQ(F_1) \subset F_1$, $PQ(F_2) \subset F_2$

$QP(E_1) \subset E_1$, $QP(E_2) \subset E_2$.

- Moreover, $PQ\big|_{F_2}$ & $QP\big|_{E_2}$ are invertible by construction;

Now, $\mathrm{Index}(P) = \dim \ker(QP) - \dim \ker(PQ)$

$= \dim E_1 - \dim F_1$ (clearly)

But $\sigma(1_{E_1} - Q_1P_1) = \sigma(1_{F_1} - P_1Q_1) = \{1\}$ by construction, so ~~$1_{E_1} - Q_1P_1$~~

$\mathrm{tr}\,(1_{E_1} - Q_1P_1)^n = \dim E_1$,

$\mathrm{tr}\,(1_{F_1} - P_1Q_1)^n = \dim F_1$

However, we have $\mathrm{tr}\,(1-QP)^n - \mathrm{tr}\,(1-PQ)^n$
$= \dim$ (eigenspace at 1 of $(1-QP)^n$) $-$ same for $(1-PQ)^n$.

End-8

The last formula comes from the fact that $\sigma(1-QP)$ & $\sigma(1-PQ)$ are identical (counting mult.) except possibly for the eigenvalue 1, and hence the same holds for $(1-QP)^n$ & $(1-PQ)^n$.

They are trace-class, so

$\sum$ eigenvalues of $(1-QP)^n$

$- \sum$ " " $(1-PQ)^n$

$=$ dim (eigensp. of $(1-QP)^n$ corr to 1)

$-$ dim (" " $(1-PQ)^n$ " " 1).

$\blacksquare$

Now, we construct a canonical cycle from a given (even) Fred. module $(\pi, \mathcal{H}, F, \varepsilon)$ over $\mathcal{A}$. Assume it is n-summable.

Set $d_F T = i[F, T]$ (graded commutator) $\equiv \begin{cases} i(FT - TF) & \text{if } T \text{ is even} \\ i(FT + TF) & \text{if } T \text{ is odd.} \end{cases}$

Here the grading on T comes from that of $\mathcal{H}$: T is even if $T\varepsilon = \varepsilon T$, and odd if $T\varepsilon = -\varepsilon T$.

Now, for $j \in \mathbb{N}$, let Ω^j denote the linear span of the operators of the form $a_0 \, d_F(a_1) \cdots d_F(a_j)$, $a_i \in A$

We observe:

<u>Thm</u>: $d_F^2 = 0$, $d_F(T_1 T_2) = (d_F T_1) T_2 + (-1)^{\deg(T_1)} T_1 d_F(T_2)$

$d_F \Omega^j \subseteq \Omega^{j+1}$; $\Omega^k \subseteq \mathcal{L}^{\frac{n+1}{k}}(\mathcal{H})$.

Pf: For homogeneous $T \in B(H)$,

$$d_F^2(T) = F\left(FT - (-1)^{\deg(T)} TF\right) - (-1)^{\deg(T)+1}\left(FT - (-1)^{\deg T} TF\right)F,$$

$$(\because \deg F = 1)$$

$$= F^2 T - \cancel{(-1)^{\deg T} FTF} + \cancel{(-1)^{\deg T} FTF} - TF^2 = 0, \because F^2 = 1.$$

The other assertions also follow in a straightforward manner. ∎

Thm: For $T \in B(H)$ st $[F,T] \in \mathcal{L}^1(H)$ define $Tr_s(T) = \frac{1}{2} Tr(\varepsilon F [F,T])$ (graded). Then:

(i) if T is odd, $Tr_s(T) = 0$

(ii) if $T \in \mathcal{L}^1(H)$, $Tr_s(T) = Tr(\varepsilon T)$,

(iii) $[F, \Omega^n] \subseteq \mathcal{L}^1(H)$.

PTO

Ind-4

Pf (i) $F[F,T]$ odd when T is odd, so $\varepsilon F[F,T] = -F[F,T]\varepsilon$

so $\mathrm{Tr}(\varepsilon F[F,T])$

$= \mathrm{Tr}(F[F,T]\varepsilon) = -\mathrm{Tr}(\varepsilon F[F,T])$ (by trace property) $\Rightarrow \mathrm{Tr}(\varepsilon F[F,T]) = 0$

(ii) Enough to check for T even. For such T, $\frac{1}{2}\mathrm{Tr}(\varepsilon F[F,T])$

$= \frac{1}{2}\mathrm{Tr}(\varepsilon(T - FTF))$

If T is $\mathcal{L}^1$, $\mathrm{Tr}(\varepsilon FTF)$

$= \mathrm{Tr}(F\varepsilon FT)$

$= -\mathrm{Tr}(\varepsilon F^2 T) = -\mathrm{Tr}(\varepsilon T)$

which proves (ii)

(iii) ~~left as exercise~~

(iii) exercise.

Thm: $Tr_s|_{\Omega^n}$ is a d closed graded trace.

Pf: $d_F^2=0 \Rightarrow Tr_s(d_F\omega)$

$$=\frac{1}{2}Tr\Big(\varepsilon F \underbrace{[F,[F,\omega]]}_{=0}\Big)=0$$

$\forall\, \omega\in\Omega^{n-1}$.

Now, for $\omega_1\in\Omega^{n_1}$, $\omega_2\in\Omega^{n-n_1=m}$

$$Tr(\varepsilon F d_F(\omega_1\omega_2))$$

$$= Tr(\varepsilon F\, d_F(\omega_1)\cdot\omega_2) + (-1)^{n_1} Tr(\varepsilon F \omega_1 d_F(\omega_2))$$

$$= Tr(\varepsilon F\, d_F(\omega_1)\cdot\omega_2) + (-1)^{n_1} Tr(\varepsilon F d_F(\omega_2)\omega_1)$$

$$= Tr(\varepsilon F\omega_2 d_F(\omega_1))$$

(∵ εF commute with $d_F(\omega_i)$, $i=1,2$)

$$= (-1)^{n_1} Tr(\varepsilon F d_F(\omega_2\omega_1))$$

This implies the Thm.

(iii) exercise.

Prd-12

Thm: $Tr_s\big|_{\Omega^n}$ is a d closed graded trace.

Pf: $d_F^2=0 \Rightarrow Tr_s(d_F\omega)$

$$= \frac{1}{2}Tr\big(\epsilon F\,\underbrace{[F,[F,\omega]]}_{=0}\big)=0$$

$\forall\, \omega\in\Omega^{n-1}$.

Now, for $\omega_1\in\Omega^{n_1}$, $\omega_2\in\Omega^{n-n_1 = m}$,

$$Tr(\epsilon F d_F(\omega_1\omega_2))$$

$$= Tr(\epsilon F\, d_F(\omega_1)\cdot\omega_2) + (-1)^{n_1}Tr(\epsilon F\omega_1 d_F(\omega_2))$$

$$= Tr(\epsilon F\,\omega_2 d_F(\omega_1)) + (-1)^{n_1}Tr(\epsilon F d_F(\omega_2)\,\omega_1)$$

$$= (-1)^{n_1}Tr(\epsilon F d_F(\omega_2\omega_1))$$

($\because$ ϵF commute with $d_F(\omega_i)$, $i=1,2$)

This implies the Thm.

This allows us to define the cycle $(\Omega_{d_F}, \int)$

where $\int \omega := (2\pi i)^m \cdot m! \; \mathrm{Tr}_s(\omega)$, $\omega \in \Omega^n$ $(n = 2m)$

we then have:

<u>Thm</u>: Let $(\mathcal{H}, F)$ be a p-summable, even Fredholm module over $\mathcal{A}$. $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

ε grading, $F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$,

Then for any $n = 2m > p$,

denote by τ_n the character of the cycle mentioned above, i.e.

$$\tau_n(a_0, \dots, a_n) = c_n \, \mathrm{Tr}_s\big(a_0 [F, a_1] \cdots [F, a_n]\big)$$

$c_n = (2\pi i)^m \cdot m!$

Then for any +ve integer r & $e \in M_r(\mathcal{A})$ we have

$$\langle [e], [\tau_n] \rangle = \mathrm{Index}(F_e^+),$$

where $F_e^+ = \tilde{P} \,|\, e\tilde{\mathcal{H}}^+$,

Ind-14

Note that here $\tilde{P} = P \otimes I_{\mathbb{C}^q}$

$: \underbrace{\mathcal{H}^{+} \otimes \mathbb{C}^q}_{\tilde{H}^{+}} \to \underbrace{\mathcal{H}^{-} \otimes \mathbb{C}^q}_{\tilde{H}^{-}}$

Moreover, $[\tilde{\tau}_{n+2}] = [S \tilde{\tau}_n]$ in $HC^{n+2}(A)$

and $\tilde{\tau}_n$ is also 'homotopy invariant'

in the following sense:

if $(\mathcal{H}, F_t)$, $t \in [0,1]$ is a family of (p-summable) Fred. mod. over $\mathcal{A}$, st $t \mapsto F_t$ is op-norm continuous, Then

$[\tilde{\tau}_n^t] \equiv [\tilde{\tau}_n(\mathcal{H}, F_t)]$ is indep. of t.

We denote $[\tilde{\tau}_n]$ by $Ch_n(\mathcal{H}, F)$.

Also since $[\tilde{\tau}_{n+2}] = [\tilde{\tau}_n]$, we actually have a class $Ch^{ev}(\mathcal{H}, F) \in HC^{ev}(A)$

We only give the proof of the index formula, leaving the other assertions as (slightly difficult) exercise.

Pf of the index formula:

Wlg assume $a=1$, i.e. $e \in A$ idempotent. Let $\mathcal{H}_1 = e\mathcal{H}^+$, $\mathcal{H}_2 = eH^-$

& $P' = P|_{H_1 \to H_2}$, $Q' = Q|_{H_2 \to H_1}$,

i.e. $P' = eP|_{e\mathcal{H}^+}$. $Q' = eQ|_{eH^-}$.

Thus, $1_{H_1} - Q'P' = (e - eFeFe)|_{H_1}$

$1_{H_2} - P'Q' = (e - eFeFe)|_{H_2}$

But $e - eFeFe = -e[F,e]^2 e$, using $F^2 = 1$, & by assumption, $[F,e] \in \mathcal{L}^{n+1}(\mathcal{H})$

so $1 - Q'P', 1 - P'Q' \in \mathcal{L}^{\frac{n+1}{2}} \subseteq \mathcal{L}^{n+1}$

So, by a previously proved formula, we get:

$$\mathrm{Index}(P') = \mathrm{Tr}\left((1-\alpha' P')^{m+1}\right) - \mathrm{Tr}\left((1-P'\alpha')^{m+1}\right)$$

$$= \mathrm{Tr}\left(\varepsilon\,(e - eFeFe)^{m+1}\right) \text{ as } \varepsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

But $\langle [e], [\tau_{2m}] \rangle = (-1)^{m+1}\, \mathrm{Tr}\left(\varepsilon F [F,e]^{2m+1}\right)$

So we've to show:

$$(*) \quad \boxed{\mathrm{Tr}\left(\varepsilon F [F,e]^{2m+1}\right) = (-1)^{m+1}\, \mathrm{Tr}\left(\varepsilon (e - eFeFe)^{m+1}\right)}$$

Using the identities $[F,e] = [F,e^2] = e[F,e] + [F,e]e$, $F^2 = 1$, $\varepsilon F = -F\varepsilon$, $e[F,e]^2 = [F,e]^2 e$, and $F[F,e]^{2m+1} = -[F,e]^{2m+1} F$ ($\because$ F is odd, e even), we have:

$$\mathrm{Tr}\left(\varepsilon (e - eFeFe)^{m+1}\right)$$

$$= \mathrm{Tr}\left(\varepsilon \cdot (e[F,e]^2 e)^{m+1}\right) \cdot (-1)^{m+1}$$

$$= (-1)^{m+1}\, \mathrm{Tr}\left(\varepsilon e [F,e]^{2m+2}\right)$$

(Dnd-17)

$$= (-1)^{m+1} \mathrm{Tr}(\varepsilon e F [F,e]^{2m+1})$$

$$- (-1)^{m+1} \mathrm{Tr}(\varepsilon e [F,e]^{2m+1} F)$$

$$= 2(-1)^{m} \mathrm{Tr}(\varepsilon e [F,e]^{2m+1} F)$$

$$= 2 \cdot (-1)^{m} \mathrm{Tr}(F \varepsilon e [F,e]^{2m+1})$$

$$= 2(-1)^{m+1} \mathrm{Tr}(\varepsilon F e [F,e]^{2m+1})$$

On the other hand,

$$\mathrm{Tr}(\varepsilon F [F,e]^{2m+1})$$

$$= \mathrm{Tr}(\varepsilon F [F,e] \ [F,e]^{2m})$$

$$= \mathrm{Tr}(\varepsilon F e [F,e] [F,e]^{2m})$$

$$+ \mathrm{Tr}(\varepsilon F [F,e] e [F,e]^{2m}) \quad \left(\begin{aligned} &[F,e] \\ &= e[F,e] \\ &+ [F,e] e \end{aligned} \right)$$

$$= \mathrm{Tr}(\varepsilon F e [F,e]^{2m+1})$$

$$+ \mathrm{Tr}(e [F,e]^{2m} \varepsilon F [F,e])$$

$$- \cancel{\mathrm{Tr}(\varepsilon F e [F,e]^{2m+1}) + \mathrm{Tr}(\varepsilon e F [F,e]}^{2m+1})$$

$$= \mathrm{Tr}\left(\varepsilon F e\,[F,e]^{2m+1}\right)$$

$$+ \mathrm{Tr}\left(e\,[F,e]^{2m+1}\varepsilon F\right)$$

$$= 2\,\mathrm{Tr}\left(\varepsilon F e\,[F,e]^{2m+1}\right) \quad \left(\because\ \varepsilon F\,[F,e] = [F,e]\,\varepsilon F\right)$$

$\blacksquare$

We have a similar result in the odd case, which we state without proof:

Thm For odd p-summable Fred mod. $(\mathcal{H}, F)$ over A, we have

$$\langle [u], [\tau_{2m+1}] \rangle = \mathrm{Ind}\left(\left(\frac{1+\tilde{F}}{2}\right) u \left(\frac{1+\tilde{F}}{2}\right)\right)$$

$u \in U(M_n(A))$, say, $\tilde{F} = F \otimes I_{\mathbb{C}^n}$,

$$\tau_{2m+1}(a_0, \ldots, a_{2m+1}) = \mathrm{const.}\ \mathrm{Tr}\left(a_0 [F, a_1] \cdots [F, a_{2m+1}]\right)$$