Lie groupoids and their emanations

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1. Introduction

The concept of Lie groupoid was introduced by Ehresmann in the 1950s. Ehresmann was one of the founders of fibre bundle theory, of connection theory, and of foliation theory and these must have been in his mind when he found the concept of Lie groupoid.

- the holonomy groupoid of a foliation;
- the frame groupoid of a vector bundle, and the gauge groupoid of a principal bundle;
- the holonomy groupoid of a connection.

What Ehresmann did not see was the notion of Lie algebroid and the concept of a Lie theory for Lie groupoids.

- the tangent distribution of a foliation;
- a certain algebra of differential operators on a vector bundle, and the Atiyah sequence of a principal bundle;
- the Atiyah sequence of the reduced principal bundle for a connection.

2. The Lie algebroid of a Lie groupoid

Let $G \rightrightarrows M$ be a Lie groupoid. The Lie algebra of a Lie group consists of the right-invariant vector fields on the group (or the tangent space at the identity). In the groupoid case, for $g \in G$ the right translation is $\alpha^{-1}(\beta g) \to \alpha^{-1}(\alpha g)$; it is a map of source-fibres. So for a vector field $\mathscr{X}$ on G to be right-invariant it must be tangent to the α-fibres; that is, $T(\alpha)(\mathscr{X}) = 0$.

Define a vector field $\mathscr{X}$ on G to be *right-invariant* if $T(\alpha)(\mathscr{X}) = 0$ and $(R_g)_*(\mathscr{X}) = \mathscr{X}$ for all $g \in G$. Then the bracket of right-invariant vector fields is right-invariant.

A right-invariant vector field $\mathscr{X}$ is determined by its values on the identity elements, since $\mathscr{X}(g) = T(R_g)(\mathscr{X}(1_{\beta g}))$ for all g. Take the α-vertical bundle $T^\alpha G$ and restrict it to the manifold of identity elements. This gives a vector bundle $AG \to M$ with fibre $T_{1_m}(\alpha^{-1}(m))$ over m. Given a section $X \in \Gamma AG$, define a vector field on G by $\overrightarrow{X}(g) = T(R_g)(X(\beta g))$. This is right-invariant and we have a bijective correspondence between sections of AG and right-invariant vector fields.

3. The Lie algebroid of a Lie groupoid, p2

So the bracket of right-invariant vector fields can be transferred to the sections of AG, so that $[\overrightarrow{X}, \overrightarrow{Y}] = \overrightarrow{[X, Y]}$. This bracket is automatically skew-symmetric and obeys Jacobi. What is $[X, fY]$ where f is a smooth function on M? Well, $\overrightarrow{fY} = (f \circ \beta)\overrightarrow{Y}$. So expanding out by the Leibniz rule, we have

$$\overrightarrow{[X, fY]} = [\overrightarrow{X}, (f \circ \beta)\overrightarrow{Y}] = (f \circ \beta)[\overrightarrow{X}, \overrightarrow{Y}] + \overrightarrow{X}(f \circ \beta)\,\overrightarrow{Y}.$$

Now, right-invariant vector fields are projectable under β. (Think of $\beta(hg) = \beta(h)$.) Denote the β-projection of $\overrightarrow{X}$ by $a(X)$. Then the above becomes

$$\overrightarrow{[X, fY]} = (f \circ \beta)[\overrightarrow{X}, \overrightarrow{Y}] + (a(X)(f) \circ \beta)\,\overrightarrow{Y},$$

so $[X, fY] = f[X, Y] + a(X)(f)Y$. This is the *Leibniz rule for a Lie algebroid*. The map a is the *anchor of AG*. It is the restriction of $T(\beta)\colon\ TG \to TM$ to AG.

4. Lie algebroids

Definition: A *Lie algebroid* is a vector bundle A on a base M together with a bracket on ΓA which makes ΓA an $\mathbb{R}$-Lie algebra, and a map $a\colon A \to TM$, such that $[X, fY] = f[X, Y] + a(X)(f)Y$ for all $X, Y \in \Gamma A$ and $f \in C^\infty(M)$. ▲

It follows quickly that $a[X, Y] = [aX, aY]$ for all $X, Y \in \Gamma A$.

Simple examples: • For G a Lie group, the Lie algebroid AG is the Lie algebra $\mathfrak{g}$.

• For M a manifold, let $G = M \times M$ be the pair groupoid. A vector field on G is of the form $\mathscr{X}(n, m) = (Y(n, m), Z(n, m))$ where $Y, Z\colon M \times M \to TM$ are "vector fields on M depending on two variables". The source is $\alpha(n, m) = m$ so $\mathscr{X}$ is α-vertical iff $Z = 0$. Now right-translation by (m, p) sends (n, m) to (n, p) so $(Y, 0)$ will be right-invariant iff $Y(n, m) = Y(n, p)$. So a right-invariant vector field on $M \times M$ is just a vector field on M and $A(M \times M) = TM$.

5. Lie algebroids, p2

- Let $\Pi(M)$ be the fundamental groupoid of M. Then the smooth structure of M can be lifted to $\Pi(M)$ and $\Pi(M)$ is a Lie groupoid on M, and the projection $\Pi(M) \to M \times M$ induces an isomorphism $A\Pi(M) \to TM$.

6. Foliations

The simplest example of a foliation is the kernel pair of a surjective submersion $q\colon M \to Q$ (assume the fibres are connected). Write $G = M \times_Q M$ for the equivalence relation. Then G is a Lie groupoid and the Lie algebroid is $T^q M = \ker T(q)$, the vertical tangent bundle.

For a general foliation $\mathscr{F}$ (without singularities), the equivalence relation is not generally a Lie groupoid. In the 1960s Pradines tried to rectify this by considering a weaker notion of smooth structure on $\mathscr{F} \subseteq M \times M$.

It is more effective to consider instead the monodromy groupoid $\mathscr{M}on(\mathscr{F})$ of $\mathscr{F}$. As a set, this is the union of the universal covers of the leaves of $\mathscr{F}$, with a smooth structure defined in terms of charts adapted to $\mathscr{F}$. This is always a Lie groupoid and the projection $\mathscr{M}on(\mathscr{F}) \to M \times M$ induces an isomorphism $A(\mathscr{M}on(\mathscr{F})) \to \Delta$ where Δ is the distribution tangent to $\mathscr{F}$.

7. Technical example

For G a Lie group and M a manifold, define a Lie groupoid structure on $\Omega = M \times G \times M$ by

$$\beta(n, g, m) = n, \quad \alpha(n, g, m) = m, \quad (p, h, n)(n, g, m) = (p, hg, m).$$

This is sometimes called a *trival groupoid* since it corresponds to the trivial principal bundle $M \times G$.

Ω is also the groupoid pullback of $G \rightrightarrows \{\cdot\}$ to M.

The Lie algebroid is $TM \oplus (M \times \mathfrak{g})$ with anchor the projection to TM and bracket

$$[X \oplus U,\ Y \oplus V] = [X, Y] \oplus \{X(V) - Y(U) + [U, V]\},$$

for X, Y vector fields on M and U, V maps $M \to \mathfrak{g}$. The proof is an extension of the proof for the case of $M \times M$.

This example helps with the next one.

8. Action groupoids

Let G be a Lie group. There are two maps $T^*G \to \mathfrak{g}^*$, the left and right translations to the identity. Let $\alpha(\theta_g) = \theta \circ T(L_g)$ and $\beta(\theta_g) = \theta \circ T(R_g)$. So if $\psi_h \cdot \varphi_g$ is defined, we have $\psi \circ T(L_h) = \varphi \circ T(R_g)$ and we define $\psi_h \cdot \varphi_g = \psi \circ T(R_{g^{-1}})$. Then $T^*G \rightrightarrows \mathfrak{g}^*$ is a Lie groupoid.

Trivialize $T^*G \cong G \times \mathfrak{g}^*$ by left translations. Then the groupoid structure is

$$\beta(g, \varphi) = \varphi \circ \mathrm{Ad}_{g^{-1}}, \quad \alpha(g, \varphi) = \varphi, \quad (h, \psi)(g, \varphi) = (hg, \varphi).$$

This is an example of an *action groupoid*. Given any action of a Lie group G on a manifold M, define a Lie groupoid on $G \times M$ by

$$\beta(g, m) = gm, \quad \alpha(g, m) = m, \quad (h, n)(g, m) = (hg, m).$$

Denote this $G \lessdot M$.

9. Action groupoids, p2

Take $v \in \mathfrak{g}$ and $m \in M$. Then $G \to M$, $g \mapsto gm$, differentiates to $\mathfrak{g} \to T_m M$. Write $(v^\dagger)(m)$ for the image of v under this. Then $v^\dagger$ is a vector field on M and $\mathfrak{g} \to \mathscr{V}(M)$, $v \mapsto v^\dagger$, is the *infinitesimal action of* $\mathfrak{g}$ *on* M.

To calculate the Lie algebroid of $G \lessdot M$, map it into $M \times G \times M$ by $(g, m) \mapsto (gm, g, m)$. Then the Lie algebroid is $M \times \mathfrak{g}$ with anchor $a(m, v) = v^\dagger(m)$ and bracket

$$[V, W] = a(V)(W) - a(W)(V) + [V, W]^\bullet.$$

Here V, W are maps $M \to \mathfrak{g}$ and $[V, W]^\bullet$ is the 'pointwise bracket' as maps $M \to \mathfrak{g}$.

This construction can be carried out with any infinitesimal action $\mathfrak{g} \to \mathscr{V}(M)$. An infinitesimal action does not necessarily integrate to a global action $G \times M \to M$. There are cases where a global action does not exist but a Lie groupoid Ω with $A\Omega = \mathfrak{g} \lessdot M$ exists. That is, there may be a global groupoid even if there is not a global action.

10. Cotangent bundles of Poisson manifolds

A *Poisson structure* on a manifold M is a Lie bracket on $C^\infty(M)$ such that $\{f, gh\} = g\{f, h\} + \{f, g\}h$ for all $f, g, h \in C^\infty(M)$.

From the bracket of functions, define a map $T^*M \to TM$ denoted $\#$. First, every $f \in C^\infty(M)$ defines a vector field X_f by $X_f(g) = \{f, g\}$. Next, define $\#(df) = X_f$ and extend so that $\#(g\,df) = g\,X_f$.

The image of $\#$ is a foliation with singularities on M, the *characteristic foliation*, and each leaf has a symplectic structure.

$\#$ is the anchor of a Lie algebroid structure on T^*M, with bracket defined by $[df, dg] = d\{f, g\}$ and extended by the Leibniz rule.

If Σ is a Lie groupoid with $A\Sigma = T^*M$ then, under mild conditions, it has a symplectic structure which is compatible with the groupoid structure; in particular $\alpha\colon \Sigma \to M$ is a surjective submersive Poisson map.

11. Frame groupoids of vector bundles

Consider a vector bundle $E \to M$. The *frame groupoid* $\Phi(E)$ of E consists of all isomorphisms $\xi\colon E_m \to E_n$ between the fibres of E. The source of ξ is m, the target n, and composition is the composition of maps.

Sections of the Lie algebroid $A\Phi(E)$ are differential operators $D\colon \Gamma E \to \Gamma E$ of order $\leqslant 1$, for which there is a vector field X on M such that $D(f\mu) = fD(\mu) + X(f)\mu$ for all $\mu \in \Gamma E$, $f \in C^\infty(M)$. The bracket is $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ and the anchor is $D \mapsto X$.

A *connection in* E is a map $\nabla\colon \mathscr{V}(M) \times \Gamma E \to \Gamma E$ such that $\nabla_{fX}(\mu) = f\nabla_X(\mu)$ and $\nabla_X(f\mu) = f\nabla_X(\mu) + X(f)\mu$ identically. That is, a connection is a right-inverse to the anchor $A\Phi(E) \to TM$.

12. Frame groupoids, p2

Suppose E has a metric $\langle\ ,\ \rangle$. Define $\Phi_\sigma(E)$ to be the subgroupoid of isometries $\xi\colon E_m \to E_n$. Then $\Gamma A\Phi_\sigma(E)$ consists of those D for which $\langle D(\mu), \nu\rangle + \langle \mu, D(\nu)\rangle = \langle \mu, \nu\rangle$, where $\mu, \nu \in \Gamma E$.

So a metric connection ∇ in E is a right-inverse to the anchor $A\Phi_\sigma(E) \to TM$.

Corresponding results hold for any additional structure σ on E which is given by tensor fields: a complex structure, an orientation, Lie algebra structure, . . .

Definition: A Lie algebroid A on M is *transitive* if the anchor is surjective. A *connection in* a transitive Lie algebroid A is a right-inverse $\gamma\colon TM \to A$ to the anchor. ▲

Since $A \to TM$ is surjective, it has a kernel, denote it L. There is an exact sequence $L \rightarrowtail A \twoheadrightarrow TM$. In general a connection γ does not preserve the brackets and curvature is the failure to do so: the curvature of γ is $R_\gamma\colon TM \oplus TM \to L$ defined by

$$R_\gamma(X, Y) = \gamma[X, Y] - [\gamma(X), \gamma(Y)].$$

13. Frame groupoids, p3

Suppose now that we have a connection γ in a vector bundle E and that its curvature takes values in a subbundle $L' \subseteq L$. By simple algebra there is a Lie subalgebroid $A' \subseteq A\Phi(E)$ which is transitive, $L' \rightarrowtail A' \twoheadrightarrow TM$, such that γ takes values in A'.

A subalgebra of the Lie algebra of a Lie group integrates to a subgroup. In the same way, the transitive Lie subalgebroid $A' \subseteq A\Phi(E)$ integrates to a subgroupoid $\Omega' \subseteq \Phi(E)$. This Ω' is the *holonomy groupoid of* γ. It can also be constructed via the holonomy of the connection γ.

These results extend to connections in principal bundles. Consider a principal bundle $P(M, G, \pi)$. The group action $P \times G \to P$ lifts to $TP \times G \to TP$ and the quotient manifold $\frac{TP}{G}$ is a vector bundle over M. Sections of $\frac{TP}{G}$ correspond to G-invariant vector fields on P and $\frac{TP}{G}$ is a Lie algebroid on M. The anchor is the map $\frac{TP}{G} \to TM$ induced from $T(\pi)\colon TP \to TM$. It is surjective, so $\frac{TP}{G}$ is a transitive Lie algebroid.

14. Frame groupoids, p4

The kernel of the anchor of $\frac{TP}{G}$ is the adjoint bundle $\frac{P\times\mathfrak{g}}{G}$ and

$$\frac{P \times \mathfrak{g}}{G} \rightarrowtail \frac{TP}{G} \twoheadrightarrow TM$$

is the *Atiyah sequence of* $P(M, G, \pi)$.

The Ambrose-Singer and Reduction Theorems for connections in principal bundles can be obtained as in the vector bundle case, by using the groupoid version of the second Lie theorem (integrability of subobjects).

If G and H are Lie groups with G simply-connected, then a morphism of Lie algebras $\mathfrak{g} \to \mathfrak{h}$ integrates to a morphism $G \to H$ of the Lie groups. The corresponding result for Lie groupoids gives: if a principal bundle $P(M, G, \pi)$ has a flat connection and M is simply-connected, then P is trivializable.

15. Integrability of Lie subalgebroids

In talking about the Reduction Theorem, I used the fact that if AG is the Lie algebroid of a Lie groupoid G, then a Lie subalgebroid $A' \subseteq AG$ integrates to a Lie subgroupoid, $G' \subseteq G$. This is only true as stated for transitive Lie algebroids and subalgebroids.

The proof in the transitive case is similar to the proof for Lie groups: define a distribution Δ on G by right-translating A', and consider the leaves which pass through identity elements. These leaves are the source-fibres of the required groupoid G'. If the transitivity assumptions are removed, the result (in the form stated) fails.

Remember the case of an involutive distribution Δ on a manifold M; we have $\Delta \subseteq TM = A(M \times M) = A\Pi(M)$ but in general the foliation integrating Δ will not be a subgroupoid of $M \times M$ or $\Pi(M)$.

Similar results hold for subalgebroids of any Lie algebroid (Moerdijk and Mrčun).

16. Foliations arising from Lie groupoids

A foliation always has constant rank — the dimensions of the leaves is constant.

For a Lie groupoid the map $(\beta, \alpha)\colon\ G \to M \times M$ defines an equivalence relation on M and the equivalence classes are immersed submanifolds, but their dimension may vary.

Likewise the image of the anchor $a\colon\ A \to TM$ of a Lie algebroid is closed under the bracket of vector fields, but is usually not of constant rank: it is a *foliation with singularities*.

Very recently Androulidakis and Skandalis have constructed a holonomy groupoid for foliations with singularities.

17. References

The two papers cited by author on frames 15 and 16 are

Moerdijk, I. and Mrčun, J.*, On the integrability of Lie subalgebroids, Adv. Math.,* **204***, 2006, 101–115.*

Iakovos Androulidakis and Georges Skandalis*, The holonomy groupoid of a singular foliation,* math/0612370.