

Poisson groupoids and double structures

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1. General cotangent groupoids

The source and target maps $T^*G \rightarrow A^*G$ are

$$\langle \tilde{\alpha}(\Phi), X \rangle = \langle \Phi, T(L_g)(X - T(1)(aX)) \rangle, \quad \langle \tilde{\beta}(\Phi), Y \rangle = \langle \Phi, T(R_g)(Y) \rangle,$$

where $\Phi \in T_g^*G$ and $Y \in A_{\beta g}G$, $X \in A_{\alpha g}G$. Here we have used $X - T(1)(aX) = -T(i)(X)$.

The multiplication in T^*G is given by

$$\langle \Phi \bullet \Psi, X \bullet Y \rangle = \langle \Phi, X \rangle + \langle \Psi, Y \rangle.$$

To define the identity $\tilde{1}_\varphi \in T_{1_m}^*G$ corresponding to $\varphi \in A_m^*G$, note that every element ξ of $T_{1_m}G$ can be written uniquely in the form $T(1)(x) + X$ where $x = T(\alpha)(X) \in T_mM$ and $X \in A_mG$. We can therefore define

$$\langle \tilde{1}_\varphi, T(1)(x) + X \rangle = \langle \varphi, X \rangle.$$

The inverses are

$$\langle \Phi^{-1}, X^{-1} \rangle = -\langle \Phi, X \rangle.$$

2. Coisotropic submanifolds

Let $C \subseteq M$ be a closed embedded submanifold of a Poisson manifold M . Then C is a *coisotropic submanifold of M* if it satisfies one and hence all of the following equivalent conditions:

- the Poisson anchor $\pi^\#: T^*M \rightarrow TM$, when restricted to $(TC)^\circ$, goes into TC ;
- whenever $u \in C^\infty(M)$ vanishes on C , the vector field X_u , restricted to C , is tangent to C ;
- the subset of $C^\infty(M)$ consisting of functions which vanish on C is a Lie subalgebra of $C^\infty(M)$.

3. Lie subalgebroids

Definition: Let A be a Lie algebroid on M and let $M' \subseteq M$ be a closed embedded submanifold. A *Lie subalgebroid of A over M'* is a vector subbundle $A' \rightarrow M'$ of $A|_{M'} \rightarrow M'$ such that:

- ▶ the anchor $a: A \rightarrow TM$ restricts to $A' \rightarrow TM'$;
- ▶ if $X, Y \in \Gamma A$ have $X|_{M'}, Y|_{M'} \in \Gamma A'$ then $[X, Y]|_{M'} \in \Gamma A'$ also;
- ▶ if $X, Y \in \Gamma A$ have $X|_{M'} = 0$ and $Y|_{M'} \in \Gamma A'$, then $[X, Y]|_{M'} = 0$.

4. Symplectic groupoids

Original definition of Weinstein (1986):

A Lie groupoid $G \rightrightarrows M$ with a symplectic structure on G is a *symplectic groupoid* if the graph of multiplication $\{(gh, g, h) \mid \alpha g = \beta h\}$ is a Lagrangian submanifold of $\overline{G} \times G \times G$.

Here \overline{G} is G with the opposite symplectic structure.

A submanifold $L \subseteq S$ of a symplectic manifold S is *Lagrangian* if it is coisotropic and $\dim L = \frac{1}{2} \dim S$.

A Poisson groupoid is a symplectic groupoid iff the Poisson structure is non-degenerate.

Examples: $\overline{M} \times M$ and $\Pi(M)$ for M a symplectic manifold;
 $T^*G \rightrightarrows A^*G$ for $G \rightrightarrows M$ any Lie groupoid.

5. Lie bialgebroids

Let (A, q, M) be a vector bundle equipped with Lie algebroid structures both on A itself and on the dual bundle (A^*, q_*, M) . Then these two structures constitute a *Lie bialgebroid* structure on A if, for all $X, Y \in \Gamma A$,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y].$$

Cohomology of A with trivial coefficients:

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{r=1}^{k+1} (-1)^{r+1} a(X_r)(\omega(X_1, \dots, \hat{X}_r, \dots, X_{k+1})) \\ &\quad + \sum_{r < s} (-1)^{r+s} \omega([X_r, X_s], X_1, \dots, \hat{X}_r, \dots, \hat{X}_s, \dots, X_{k+1}) \end{aligned}$$

Cohomology of A^* with trivial coefficients:

$$\begin{aligned} d_*\xi(\varphi_1, \dots, \varphi_{k+1}) &= \sum_{r=1}^{k+1} (-1)^{r+1} a_*(\varphi_r)(\xi(\varphi_1, \dots, \hat{\varphi}_r, \dots, \varphi_{k+1})) \\ &\quad + \sum_{r < s} (-1)^{r+s} \xi([\varphi_r, \varphi_s], \varphi_1, \dots, \hat{\varphi}_r, \dots, \hat{\varphi}_s, \dots, \varphi_{k+1}) \end{aligned}$$

6. References

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