K theory of C^* algebras

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outline

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2. K groups of C^* algebras
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   - Definition of K_0
   - Definition of K_1
3. Properties of K groups: Functoriality,stability and homotopy invariance
4. Computational tools: Six term exact sequences
5. Trace and $K_0(A)$

Definition of $K(X)$

Let X be a compact Hausdorf space. Let Vect(X) denote the collection of isomorphism classes of complex vector bundles over X. Then

Proposition

The pair $(Vect(X), +)$ is a commutative semigroup where $+$ denotes the whitney sum of vector bundles. Define $K(X)$ to be the "Grothendick group" of $Vect(X)$.

For a commutative semigroup $(H, +)$, it's Grothendick group $(G(H), +)$ is defined by considering the set of formal differences $\{a - b : a, b \in H\}$ where $a - b = c - d$ if and only if there exists e such that $a + d + e = c + b + e$.

Serre-Swan theorem

Let $\pi : E \rightarrow X$ be a vector bundle over X. For $x \in X$, let $E_x = \pi^{-1}(x)$ be the fibre over x. A section of $\pi : E \rightarrow X$ is a map $s : X \rightarrow E$ such that $s(x) \in E_x$. Let $\Gamma(E)$ denote the set of all sections. Then $\Gamma(E)$ is a module over $C(X)$ where the module structure is given by

$$(f.s)(x) := f(x)s(x) \; f \in C(X), \; s \in \Gamma(E)$$

We denote the isomorphism class of finitely generated projective modules over $C(X)$ by $Proj_{fin}(C(X))$. Then $(Proj_{fin}(C(X)), \oplus)$ is a commutative semigroup.

Theorem (Serre,Swan)

For a vector bundle $\pi : E \to X$, $\Gamma(E)$ is a finitely generated projective module over $C(X)$. Furthermore, the map $Vect(X) \ni [E] \to [\Gamma(E)] \in Proj_{fin}(C(X))$ is a semigroup isomorphism.

Hence $K(X)$ is the Grothendick group of the commutative semigroup $(Proj_{fin}(C(X)), \oplus)$ which depends only on the algebra $C(X)$. We will now replace the algebra $C(X)$ by non-commutative C^* algebras.

$C^\star$ algebras

All the algebras that we consider will be algebras over $\mathbb{C}$.

Definition

A C^* algebra is a Banach algebra A together with an involution $* : A \rightarrow A$ such that

1. *The map $* : A \rightarrow A$ is an antilinear involution and $(ab)^* = b^* a^*$.*
2. *For $a \in A$, we have the identiy*

$$\| aa^* \| = \| a \|^2 \ (C^* \text{ identity})$$

Examples

- Let X be a compact Hausdorff space. Then $C(X)$ is a commutative C^* algebra where the involution $*$ is given by the complex conjugation. In fact all unital commutative C^* algebras arise this way(**Gelfand-Naimark**)

contd $\cdots$

- Let H be a Hilbert space and $B(H)$ denote the set of bounded operators on H. Then $B(H)$ is a C^* algebra where $*$ denotes the Hilbert space adjoint. In fact any C^* algebra can be embedded as a $*$ subalgebra of $B(H)$ for some H (**Gelfand-Naimark**)

Definition

Let A be a $C^\star$ algebra. An element $a \in A$ is said to be

- **normal** *if* $aa^* = a^*a$.
- **selfadjoint** *if* $a^* = a$.
- **projection** *if* $a = a^* = a^2$.
- **unitary** *if* $aa^* = a^*a = 1$.

Continuous functional calculus

Let A be a unital C^* algebra. Let $a \in A$. The spectrum of a is defined as

$$\sigma(a) := \{\lambda \in \mathbb{C} : \ a - \lambda \ \text{ is invertible}\}$$

Theorem

Let A be a unital C^* algebra and let $a \in A$. Then

- *The spectrum $\sigma(a)$ is compact.*
- *If a is normal then there exists a unique $*$ algebra homomorphism $\Phi : C(\sigma(a)) \to A$ such that $\Phi(z) = a$.*

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Non-unital algebras and unitisation

Non-unital algebras can be considered by considering it's smallest unitisation which we now explain. This is analogous to considering the one point compactification of a locally compact hausdorff space. Let A be a C^* algebra unital or not. Define $A^+ := A \oplus \mathbb{C}$. Then A^+ is an unital $*$ algebra where the multiplication and the involution $*$ are defined as

$$(a,\lambda)(b.\mu) := (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a,\lambda)^* := (a^*, \bar{\lambda})$$

Then one can show that there exists a unique C^* norm $\|\quad\|$ on A^+ such that A^+ is a C^* algebra. The map $\epsilon : A^+ \to \mathbb{C}$ defined by $\epsilon(a,\lambda) = \lambda$ is a $*$ algebra homomorphism and gives the following exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

Definition of K_0

- Note that $A^n := \underbrace{A \oplus A \oplus \cdots A}_{ntimes}$ is a $A - M_n(A)$ bimodule. Hence for an idempotent $p \in M_n(A)$, the left A module $A^n p$ is a finitely generated projective A module. In fact all finitely generated projective A modules arise this way.
- It is not difficult to see that the modules $A^n p$ and $A^m q$ are isomorphic if and only if there exists matrices u and v such that $uv = p$ and $vu = q$.

Hence the semigroup $(Proj_{fin}(A), \oplus)$ can be expressed in terms of equivalence classes of idempotent matrices over A. In C^* algebras one can replace idempotents by projections.

contd $\cdots$

Let us introduce some notations: For a C^* algebra A let

$$\begin{aligned} P(A) &:= \{p \in A : \text{ p is a projection}\} \\ U(A) &:= \{u \in A : \text{ u is a unitary}\} \\ M_n(A) &:= \text{ the algebra of } n \times n \text{ matrices over } A \\ P_\infty(A) &:= \bigcup P(M_n(A)) (\text{disjoint union}). \end{aligned}$$

Two projections p and q in a C^* algebra are said to be **Murray-von Neumann** equivalent if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* = q$. One can check that **Murray-von Neumann** equivalence is an equivalence relation.

Example

Let p, q be projections in $M_n(\mathbb{C})$. Then p is Murray-von Neumann equivalent to q if and only if $Trace(p) = Trace(q)$

contd $\cdots$

Define an equivalence relation $\sim$ on $P_\infty(A)$ as follows:
The projections $p \in P_n(A)$ and $q \in P_m(A)$ are equivalent if the projections $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$ are Murray-von Neumann equivalent in $M_N(A)$ for some N. Then we have the following.

Proposition

The operation $\oplus$ defined as $[p] \oplus [q] := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is well defined on $P_\infty(A)/\sim$. Also $(P_\infty(A)/\sim, \oplus)$ is a commutative semigroup.

We define

$$\tilde{K}_0(A) := \text{ the Grothendick group of the semigroup } (P_\infty(A)/\sim, \oplus).$$

Example

$\tilde{K}_0(\mathbb{C}) := \mathbb{Z}$

contd $\cdots$

The following are easy consequences of the definition.

- For projections $p, q \in P_\infty(A)$, $[p] = [q]$ in $\tilde{K}_0(A)$ if and only if there exists r such that $\begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}$ and $\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}$ are Murray-von Neumann equivalent.
- $\tilde{K}_0$ is a covariant functor from the category of C^* algebras to the category of abelian groups.

Let A be a C^* algebra. Then we have the following exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$$

Define

$$K_0(A) := \ Ker \ \tilde{K}_0(\epsilon)$$

For unital algebras one can show that $K_0(A)$ is naturally isomorphic to $\tilde{K}_0(A)$.

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Definition of K_1

Let A be a unital C^* algebra. Define $U_n(A) := U(M_n(A))$. We embedd $U_n(A) \hookrightarrow U_{n+1}(A)$ by $u \to \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$. Let

$$U_n^0(A) := \{u \in U_n(A) : \text{u is connected to 1 in } U_n(A)\}$$

The unitary group $U_n(A)$ is a topological group and hence $U_n^0(A)$ is a normal subgroup of $U_n(A)$. Also $U_n^0(A) \subset U_{n+1}^0(A)$.
Hence we get a directed system of groups $U_n(A)/U_n^0(A) \to U_{n+1}(A)/U_{n+1}^0(A)$. Define

$$\tilde{K}_1(A) := \lim_{n \to \infty} \frac{U_n(A)}{U_n^0(A)}$$

contd $\cdots$

- The unitary group $U_n(\mathbb{C})$ is path-connected. Hence $\tilde{K}_1(\mathbb{C}) := 0$.
- For a unitary $u \in M_n(A)$, in $\tilde{K}_1(A)$ one has [u]$= \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$.
- The group operation on $\tilde{K}_1(A)$ is $[u].[v] = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$.
- The group $\tilde{K}_1(A)$ is abelian.

For a C^* algebra A (unital or not), define $K_1(A) := \tilde{K}_1(A^+)$. It is easy to show that for a unital C^* algebra A, $K_1(A) = \tilde{K}_1(A)$.

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Properties of K groups: Functoriality $\cdots$

Let $\phi : A \to B$ be a C^* algebra homomorphism. Then

- The ϕ naturally extends to an algebra homomorphis $M_n(A) \to M_n(B)$ which we again denote by ϕ and given by $\phi((a_{ij})) = (\phi(a_{ij}))$.
- The map ϕ extends to a map $\phi^+ : A^+ \to B^+$ given by $\phi^+(a, \lambda) = (\phi(a), \lambda)$.

Thus the algebra homomorphism ϕ induces maps $K_*(\phi)$ at the K theory level given by

$$K_0(\phi)([p]) = [\phi^+(p)]$$
$$K_1(\phi)([u]) = [\phi^+(u)]$$

Thus K_* is a functor from the category of C^* algebras to the category of abelian groups.

Stability $\cdots$

- The map $A \ni a \rightarrow \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in M_n(A)$ induces isomorphism at the K theory level.
- More generally the map $A \ni a \rightarrow a \otimes p \in A \otimes \mathcal{K}$ (where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space H and p is a rank-one projection on H) induces isomorphism at the K theory level.

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Homotopy invariance

If $\{\phi_t\}_{t \in [0,1]} : A \to B$ is a continuous family of C^* algebra homomorphisms(i.e. for every $a \in A$, the map $t \to \phi_t(a)$ is continuous) then $K_*(\phi_t) = K_*(\phi_0)$.

We give a simple application of this. For a C^* algebra A define $CA := \{f : [0,1] \to A :$ f is continuous and $f(0) = 0\}$.

Proposition

The K groups of CA are trivial.

Proof. Define $\phi_t : CA \to CA$ as $\phi_t(f)(x) := f(tx)$. Then $\phi_1 := id$ and $\phi_0 = 0$.

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Six term exact sequence in K theory

An important computational tool in K theory is the six term exact sequence. For an exact sequence

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

one has the following six term exact sequence

$$\begin{tikzcd}
K_0(J) \arrow[r, "K_0(\phi)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(A/J) \arrow[d, "\sigma"'] \\
K_1(A/J) \arrow[u, "\partial"] & K_1(A) \arrow[l, "K_1(\pi)"] & K_1(J) \arrow[l, "K_1(\phi)"]
\end{tikzcd}$$

The map ∂ is called the index map and σ is called the exponential map.

contd $\cdots$

The construction of the six term exact sequence involves the following.

- Half exactness of K groups.
- The construction of the index map.
- Bott periodicity and the construction of the exponential map.

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Construction of the index map

The index map ∂ is defined as follows:

- For a C^* algebra A, $U^0(A) := \{e^{ia_1}e^{ia_2}\cdots e^{ia_n} : a_i \text{ is selfadjoint}\}$.
- For a unitary $u \in M_n(B^+)$, the unitary $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix} \in U_n^0(B^+)$.
- Let V be a lift of $\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$ in $M_{2n}(A^+)$.
- Define $\partial([u]) := [Vp_nV^*] - [p_n]$ where $p_n := \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}$.

Then ∂ is well defined,functorial and makes the following diagram exact.

contd ⋯

$$\begin{tikzcd}
K_0(J) \arrow[r, "K_0(\phi)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(A/J) \\
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\end{tikzcd}$$

As a corollary we calculate $K_0(SA)$ where SA is the suspension of A defined as $SA := \{f : [0,1] \to A : f$ is continuous and $f(0) = f(1) = 0\}$.

Corollary

For a C^* algebra A, $K_0(SA)$ is "naturally" isomorphic to $K_1(A)$.

Proof. Follows from the exact sequence and of the fact that CA has trivial K theory

$$0 \longrightarrow SA \longrightarrow CA \xrightarrow{f \to f(1)} A \longrightarrow 0$$

The construction of the "exponential" map involves Bott periodicity.

Bott periodicity

Let $\Omega A := \{f : \mathbb{T} \to \mathbb{C} : \text{f is continuous}\}$. Then ΩA is a unital C^* algebra. Then we can write $(SA)^+$ and $M_n((SA)^+)$ as

$$\begin{aligned}(SA)^+ &:= \{f \in \Omega A : \ f(1) \in \mathbb{C}\} \\ M_n(SA^+) &:= \{f \in \Omega(M_n(A)) : \ f(1) \in M_n(\mathbb{C})\}\end{aligned}$$

The Bott map $\beta_A : K_0(A) \to K_1(SA)$ is defined as

$$\beta([p]) := [zp + 1 - p]$$

Theorem

The Bott map $\beta_A : K_0(A) \to K_1(SA)$ is well defined and is an isomorphism of abelian groups. Moreover the map β_A is functorial.

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$$(SA)^+ := \{f \in \Omega A : \ f(1) \in \mathbb{C}\}$$
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Construction of the exponential map

The exponential map is defined through the following steps

- For an exact sequence $0 \to I \to A \to B \to 0$ applying the suspension functor gives the exact sequence $0 \to SI \to SA \to SB \to 0$
- Consider the index map $\partial : K_1(SB) \to K_0(SI)$.
- Since $K_0(B)$ is naturally isomorphic to $K_1(SB)$ and $K_1(I)$ is naturally isomorphic to $K_0(I)$, we can consider ∂ as a map from $K_0(B) \to K_1(I)$ which we declare it to be the exponential map σ.
- The map σ is given explicity on projections as follows. For a projection $p \in M_n(B)$ let a be a self-adjoint lift. Then $\sigma([p]) = [exp(2\pi ia)]$.

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- The map σ is given explicity on projections as follows. For a projection $p \in M_n(B)$ let a be a self-adjoint lift. Then $\sigma([p]) = [exp(2\pi ia)]$.

contd $\cdots$

With the above definition of σ we have the following six term exact sequence.

$$\begin{tikzcd}
K_0(J) \arrow[r, "K_0(\phi)"] & K_0(A) \arrow[r, "K_0(\pi)"] & K_0(B) \arrow[d, "\sigma"'] \\
K_1(B) \arrow[u, "\partial"] & K_1(A) \arrow[l, "K_1(\pi)"] & K_1(J) \arrow[l, "K_1(\phi)"]
\end{tikzcd}$$

A simple consequence of the six term sequence is that both the functors K_0 and K_1 takes split exact sequences to split exact sequences.

Crossed product of C^* algebras

Let $\alpha : G \to Aut(A)$ be a homomorphism. We will construct a C^* algebra $A \rtimes_\alpha G$ in which the action is "inner". We will restrict ourselves to the case where A is unital and G is discrete. First consider the algebraic crossed product

$$A \rtimes_\alpha^{alg} G := \{\sum_g a_g g : \ a_g \in A\}$$

We will write a formal sum $\sum_g a_g g$ as $\sum_g a_g U_g$. Then $A \rtimes_\alpha^{alg} G$ is a $*$ algebra where the multiplication and the involution are defined so that $U_g a U_g^* := \alpha_g(a)$. Hence one has

$$\begin{aligned} aU_g bU_h &: = a\alpha_g(b)U_{gh} \\ (aU_g)^* &: = \alpha_{g^{-1}}(a^*)U_{g^{-1}} \end{aligned}$$

contd ⋯

Let (A, G, α) be a C^* dynamical system.

Definition

A **covariant representation** *of (A, G, α) on a Hilbert space H is a pair (π, U) where π is a unital representation of A on H and U a unitary representation of G on H such that $U_g \pi(a) U_g^* = \pi(\alpha_g(a))$.*

It is not difficult to show that

1. If (π, U) is a covariant representation of (A, G, α) on H then it gives a $*$ representation of $A \rtimes_{\alpha}^{alg} G$ on H which we denote by $\pi \rtimes_{\alpha} U$.
2. For $a \in A \rtimes_{alpha}^{alg} G$, $\|a\| := Sup_{(\pi, U)} \|\pi \rtimes_{\alpha} U(a)\|$ is finite.
3. $\| \ \|$ is a norm on $A \rtimes_{\alpha}^{alg} G$. Then $A \rtimes_{\alpha} G$ is defined to be the completion of $A \rtimes_{\alpha}^{alg} G$ in this norm.

Pimsner-Voiculescu exact sequence

The PV six term sequence is another important computational tool for computing K groups. We have the following proposition

Theorem

Let A be a C^*-algebra and let τ be an action of $\mathbb{Z}$ on A. Then there is a six-term exact sequence

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-K_0(\tau)"] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes_\tau \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\tau \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "K_1(\iota)"] & K_1(A) \arrow[l, "1-K_1(\tau)"]
\end{tikzcd}$$

The point of the theorem is that the vertical maps exist.

An example of K theory-cyclic cohomology pairing

Definition (Trace)

A trace on an algebra A is a linear map $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for every $a, b \in A$.

Let τ be a trace on a C^* algebra A. Then note the following

- The trace τ extends to a trace τ_n on the algebra $M_n(A)$ given by $\tau(a_{ij}) = \sum_{i=1}^{n} \tau(a_{ii})$.
- The maps τ_n's are consistent in the sense that $\tau_{n+1}(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) = \tau_n(a)$. Thus we will denote τ_n simply by τ.
- If the projections p and q in $M_n(A)$ are Murray-von Neumann equivalent then $\tau(p) = \tau(q)$.
- Hence τ descends to a homomorphism $\tau_\star : K_0(A) \to \mathbb{C}$ such that $\tau_\star([p]) = \tau(p)$.

This is the simplest instance of the K theory- cyclic cohomology pairing.

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